

## An introduction to rank functions

PhD-course  
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The concept of a rank function, a classical topic in the context of  $C^*$ -algebras and von Neumann regular rings, is currently object of renewed interest [19, 4, 13, 6, 7]. Rank functions measure the size of modules, or more generally, of objects in well-behaved categories, and satisfy some form of additivity. They are closely related with the notion of a character studied by Crawley-Boevey [9, 11, 10], and also with the length functions from [24, 31, 32].

The character theory of a ring  $R$  controls several fundamental invariants of  $R$ , such as the Ziegler spectrum of  $R$ , or the ring morphisms from  $R$  to simple artinian rings, and it has been used to address questions related to stability theory or to representation type [30, 16, 11, 10]. Characters and rank functions also play an important role beyond module categories, e.g. in the study of functor categories or triangulated categories [6, 7].

We will start by reviewing some concepts related to purity, in particular matrix subgroups and algebraic compactness, in order to discuss some prominent classes of modules:  $\Sigma$ -pure-injective, endofinite and generic modules. We will then introduce rank functions and related notions such as characters and length functions. We will see that every character has a unique decomposition as a sum of irreducible characters, and that the indecomposable endofinite modules are in one-to-one correspondence with the irreducible characters. Then we will turn to connections with ring morphisms and universal localizations. The course will end with a lecture series by F. Marks (University of Stuttgart) on rank functions for triangulated categories.

### Tentative program.

This is an in person course which takes place in Verona, Sala Riunioni Ca' Vignal 2. It consists of two lecture series, and a reading course with seminars delivered by the participants.

- Introductory lectures:

Tuesday, 2 December 2025, 14:30-16:30  
Wednesday, 3 December 2025, 10:30-12:30  
Wednesday, 10 December 2025, 10:30-12:30  
Thursday, 11 December 2025, 10:30-12:30

- Reading course:

Wednesday, 14 January 2026, 10:30-12:30  
Thursday, 15 January 2026, 10:30-12:30  
Tuesday, 20 January 2026, 14:30-16:30  
Wednesday, 21 January 2026, 10:30-12:30  
Thursday, 22 January 2026, 10:30-12:30  
Wednesday, 28 January 2026, 10:30-12:30  
Thursday, 29 January 2026, 10:30-12:30

- Lecture series on rank functions for triangulated categories: March (dates t.b.a.).

# REFERENCES

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