Infinite Runs in Abstract Completion*

Nao Hirokawa[1], Aart Middeldorp[2], Christian Sternagel[2], and Sarah Winkler[2]

1 School of Information Science
JAIST, Nomi, Japan
hirokawa@jaist.ac.jp

2 Department of Computer Science
University of Innsbruck, Innsbruck, Austria
{aart.middeldorp|christian.sternagel|sarah.winkler}@uibk.ac.at

Abstract

Completion is one of the first and most studied techniques in term rewriting and fundamental to automated reasoning with equalities. In an earlier paper we presented a new and formalized correctness proof of abstract completion for finite runs. In this paper we extend our analysis and our formalization to infinite runs, resulting in a new proof that fair infinite runs produce complete presentations of the initial equations. We further consider ordered completion—an important extension of completion that aims to produce ground-complete presentations of the initial equations. Moreover, we revisit and extend results of Métivier concerning canonicity of rewrite systems. All proofs presented in the paper have been formalized in Isabelle/HOL.

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1 Introduction

Reasoning with equalities is pervasive in computer science and mathematics, and has consequently been one of the main research areas of automated deduction. Indeed completion as introduced by Knuth and Bendix [12] has evolved into a fundamental technique whose ideas appear throughout automated deduction whenever equalities are present. Many variants of the original calculus have since been proposed.

On a given set of input equalities, Knuth-Bendix completion can behave in three different ways: it may (1) succeed to compute a complete system in finitely many steps, (2) fail due to unorientable equalities, or (3) continuously compute approximations of a complete system without ever terminating.

As a remedy to problem (2), ordered completion was developed by Bachmair, Dershowitz, and Plaisted [5]. Ordered completion never fails, though the price to be paid is that the resulting system is in general only complete on ground terms. This is actually sufficient for many applications in theorem proving. However, it is still possible that a ground-complete system is only produced in the limit.

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▶ **Example 1.** Consider the equational system $\mathcal{E} = \{\mathsf{aba} \approx \mathsf{bab}\}$ of the three-strand positive braid monoid. Kapur and Narendran [10] proved that $\mathcal{E}$ admits no *finite* complete presentation. However, taking the Knuth-Bendix order [12] with a and b of weight 1 and $\mathsf{a} > \mathsf{b}$ in the precedence, completion produces in the limit the following infinite complete presentation of $\mathcal{E}$

$$\{\mathsf{aba} \to \mathsf{bab}\} \cup \{\mathsf{ab}^n\mathsf{ab} \to \mathsf{babba}^{n-1} \mid n \geqslant 2\}$$

which can be used to decide the validity problem for $\mathcal{E}$.[1]

Bachmair, Dershowitz, and Hsiang [4] recast completion procedures in inference systems. This style of presentation, *abstract completion*, has become the standard to describe completion procedures and *proof orders* the accompanying way to establish correctness [2, 4, 5]. In earlier work [8] we presented a new correctness proof for Knuth-Bendix completion which does not rely on proof orders and was entirely formalized in Isabelle/HOL [15] as part of the formal IsaFoR[2] library. However, our results were limited to finite runs.

In the present paper we adapt our proof techniques to show correctness of both Knuth-Bendix and ordered completion for potentially infinite runs. Though we emphasize the infinite case, all results are valid for finite runs, too, and thus also apply to completion procedures in practice. We believe that our proof techniques are better suited to formalization as they are more *local* than the original approach in multiple respects. We will point out differences in Sections 3 and 4.

Completion procedures raise the question whether their result is uniquely determined. Métivier [14] showed that indeed canonical rewrite systems are unique up to renaming, once a reduction order is fixed. We provide a new proof for a generalization of this result.

Contribution. We present new and comparatively short correctness proofs of Knuth-Bendix completion and ordered completion, as well as some results about canonicity, one of which generalizes the uniqueness result for complete systems due to Métivier [14]. All the proofs that are presented in the following, have been formalized as part of IsaFoR (version 2.30). With the exception of Knuth-Bendix completion for finite runs, to the best of our knowledge none of these techniques has been formalized in a proof assistant before. Also note that in the PDF version of this paper all corollary/lemma/theorem statements are active hyperlinks to HTML versions of our formalized proofs.

Overview. The remainder of this paper is organized as follows. In Section 2 we present required preliminaries followed by some mostly known results. Then, in Section 3, we recall the inference rules for (abstract) Knuth-Bendix completion and present our new correctness proof for infinite runs. Afterwards, in Section 4, we deal with ordered completion. Finally, we give some canonicity results that are related to normalization equivalence in Section 5, before we conclude in Section 6 with related and future work.

2 Preliminaries

We assume familiarity with the basic notions of abstract rewrite systems (ARSs), term rewrite systems (TRSs), and completion [1, 2], but shortly recapitulate terminology and notation

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[1] Burckel [6] constructed a complete rewrite system consisting of four rules with an additional symbol, which is no longer a presentation of $\mathcal{E}$ but can be also used to decide the validity problem for $\mathcal{E}$.

[2] The *Isabelle Formalization of Rewriting*: `http://cl-informatik.uibk.ac.at/isafor/`

that we use in the remainder. For an arbitrary binary relation $\rightarrow_\alpha$, we write ${}_\alpha\!\leftarrow$, $\underset{\alpha}{\leftrightarrow}$, $\rightarrow_\alpha^=$, $\rightarrow_\alpha^+$, and $\rightarrow_\alpha^*$ to denote its *inverse*, its *symmetric closure*, its *reflexive closure*, its *transitive closure*, and its *reflexive transitive closure*, respectively. We further use $\downarrow_\alpha$ as abbreviation for the relation $\rightarrow_\alpha^* \cdot {}_\alpha^*\!\leftarrow$, where from here on $\cdot$ denotes relation composition. If $a \rightarrow_\alpha b$ for no b then we say that a is a *normal form* of $\rightarrow_\alpha$ and write $a \in \mathsf{NF}(\rightarrow_\alpha)$. By $a \rightarrow_\alpha^! b$ we abbreviate $a \rightarrow_\alpha^* b \wedge b \in \mathsf{NF}(\rightarrow_\alpha)$. Such an element b is called a normal form of a. Given two binary relations $\rightarrow_\alpha$ and $\rightarrow_\beta$, we use $\rightarrow_\alpha / \rightarrow_\beta$ as shorthand for $\rightarrow_\beta^* \cdot \rightarrow_\alpha \cdot \rightarrow_\beta^*$. A *renaming* is a bijective variable substitution from $\mathcal{V}$ to $\mathcal{V}$. A term s is a *variant* of a term t if $s = t\sigma$ for some renaming σ. If $\ell \rightarrow r$ is a rewrite rule and σ is a renaming then the rewrite rule $\ell\sigma \rightarrow r\sigma$ is a variant of $\ell \rightarrow r$. A TRS is said to be *variant-free* if it does not contain rewrite rules that are variants of each other. Given terms s and t, we write $s \doteq t$ if $s\sigma = t$ and $s = t\tau$ for some substitutions σ and τ. We say that s *encompasses* t, written $s \trianglerighteq t$, whenever $s = C[t\sigma]$ for some context C and substitution σ. *Proper encompassment* is defined by $\vartriangleright = \trianglerighteq \setminus \trianglelefteq$ and known to be well-founded. Two variable-disjoint variants $\ell_1 \rightarrow r_1$ and $\ell_2 \rightarrow r_2$ of rules in $\mathcal{R}$ such that $\ell_1\mu = \ell_2|_p\mu$ with $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(\ell_2)$ and most general unifier (mgu) μ, constitute an *overlap*. An overlap that does not result from overlapping two variants of the same rule at the root, gives rise to a *critical pair* $\ell_2[r_1]_p\mu \approx r_2\mu$. A critical pair is called *prime* if all proper subterms of $\ell_1\mu$ are $\mathcal{R}$-normal forms. The set of (prime) critical pairs of a TRS $\mathcal{R}$ is denoted by $(\mathsf{PCP}(\mathcal{R}))$ $\mathsf{CP}(\mathcal{R})$. For a well-founded order $>$, we write $>_{\mathrm{mul}}$ to denote its *multiset extension* and $>_{\mathrm{lex}}$ to denote its *lexicographic extension* as defined by Baader and Nipkow [1].

We make use of the following result due to Bachmair and Dershowitz [3]. Here *quasi-commutation* of R over S means that the inclusion $S \cdot R \subseteq R \cdot (R \cup S)^*$ holds.

▶ **Lemma 2.** *Let R and S be binary relations.*

1. *If R quasi-commutes over S and R is well-founded then R / S is well-founded.*
2. *If R / S and S are well-founded then $R \cup S$ is well-founded.* ◀

▶ **Lemma 3.** *If R is a well-founded rewrite relation then $(R \cup \vartriangleright) / \trianglerighteq$ is well-founded.*

Proof. First we show the inclusion $\trianglerighteq \cdot R \subseteq R \cdot \trianglerighteq$. Suppose $s \trianglerighteq t \; R \; u$. So $s = C[t\sigma]$ for some context C and substitution σ. Because R is closed under contexts and substitutions, $s \; R \; C[u\sigma]$. Moreover, $C[u\sigma] \trianglerighteq u$. This establishes the inclusion, and we conclude that R (quasi-)commutes over $\trianglerighteq$. Because R is well-founded, it follows from Lemma 2(1) that the relation $R / \trianglerighteq$ is well-founded too. Then $R / \vartriangleright$ is well-founded since it is contained in $R / \trianglerighteq$. As $\vartriangleright$ is well-founded, it follows from Lemma 2(2) that $R \cup \vartriangleright$ is well-founded. We have $\trianglerighteq \cdot \vartriangleright \subseteq \vartriangleright$ and thus $R \cup \vartriangleright$ quasi-commutes over $\trianglerighteq$. Another application of Lemma 2(1) yields the well-foundedness of $(R \cup \vartriangleright) / \trianglerighteq$. ◀

We use the following simple confluence criterion for ARSs. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS equipped with a well-founded relation $>$ on A, and let $b \xrightarrow{a} c$ if and only if $b \rightarrow c$ and $a = b$. We say that $\mathcal{A}$ is *source decreasing* if the inclusion

$$\leftarrow a \rightarrow \quad \subseteq \xleftrightarrow{\vee a}^*$$

holds for all $a \in A$. Here $\xleftrightarrow{\vee a}^*$ denotes a conversion in which all steps are labeled with an element smaller than a. Source decreasingness is the specialization of peak-decreasingness [8] to source labeling [18, Example 6].

▶ **Lemma 4.** *Every source decreasing ARS is confluent.* ◀

Source-decreasingness is closely related to the *connectedness-below* criterion of Winkler and Buchberger [19]. Unlike the latter, it does not entail termination. For instance, for $\mathsf{a} > \mathsf{b}$ and $\mathsf{a} > \mathsf{c}$ the non-terminating ARS

$$\mathsf{b} \rightleftarrows \mathsf{a} \rightleftarrows \mathsf{c}$$

is source decreasing but the connectedness-below criterion does not apply.

The following definition and corresponding lemma [8] are key to the correctness results for both Knuth-Bendix completion and ordered completion.

▶ **Definition 5.** Given a TRS $\mathcal{R}$ and terms s, t, and u, we write $t \mathrel{\triangledown_s} u$ if $s \to_{\mathcal{R}}^{+} t$, $s \to_{\mathcal{R}}^{+} u$, and $t \downarrow_{\mathcal{R}} u$ or $t \leftrightarrow_{\mathsf{PCP}(\mathcal{R})} u$.

▶ **Lemma 6.** *Let $\mathcal{R}$ be a TRS. If $t \mathrel{{}_{\mathcal{R}}{\leftarrow}} s \to_{\mathcal{R}} u$ then $t \mathrel{\triangledown_s^2} u$.* ◀

3 Knuth-Bendix Completion

The original completion procedure by Knuth and Bendix [12] was presented as a concrete algorithm. Later on, Bachmair, Dershowitz, and Hsiang [4] presented an inference system for completion and showed that all *fair* implementations thereof (in particular the original procedure) are correct. Abstracting from a concrete strategy, their approach thus has the advantage to cover a variety of implementations. Below, we recall the inference system, which constitutes the basis of the results presented in this section.

▶ **Definition 7.** The inference system KB of abstract (Knuth-Bendix) completion operates on pairs $(\mathcal{E}, \mathcal{R})$ of equations $\mathcal{E}$ and rules $\mathcal{R}$ over a common signature $\mathcal{F}$. It consists of the following inference rules:

$$\begin{array}{llll}
\mathsf{deduce} & \dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}} \quad \text{if } s \mathrel{{}_{\mathcal{R}}{\leftarrow}} \cdot \to_{\mathcal{R}} t & \mathsf{compose} & \dfrac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}} \quad \text{if } t \to_{\mathcal{R}} u \\[2ex]
 & \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}} \quad \text{if } s > t & & \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}} \quad \text{if } s \to_{\mathcal{R}} u \\
\mathsf{orient} & & \mathsf{simplify} & \\
 & \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \to s\}} \quad \text{if } t > s & & \dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}} \quad \text{if } t \to_{\mathcal{R}} u \\[2ex]
\mathsf{delete} & \dfrac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}} & \mathsf{collapse} & \dfrac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}} \quad \text{if } t \xrightarrow{\rhd}_{\mathcal{R}} u
\end{array}$$

Here $>$ is a fixed reduction order on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ and the relation $t \xrightarrow[\mathcal{R}]{\rhd} u$ is defined as $t \xrightarrow[\ell \to r]{} u$ for some $\ell \to r \in \mathcal{R}$ such that $t \rhd \ell$.

Sternagel and Thiemann [16] showed that the strict encompassment condition in the $\mathsf{collapse}$ inference rule is not necessary for finite runs. For infinite runs however, it is indispensable: when omitted, the result need not be confluent [1, Example 7.2.9].

We write $(\mathcal{E}, \mathcal{R}) \vdash (\mathcal{E}', \mathcal{R}')$ if $(\mathcal{E}', \mathcal{R}')$ can be obtained from $(\mathcal{E}, \mathcal{R})$ by applying one of the inference rules of Definition 7. A *run* of Knuth-Bendix completion is an infinite sequence of the form

$$\Gamma \colon (\mathcal{E}_0, \mathcal{R}_0) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash (\mathcal{E}_2, \mathcal{R}_2) \vdash \cdots$$

with $\mathcal{R}_0 = \varnothing$. We define

$$\mathcal{E}_\infty = \bigcup_{i \geqslant 0} \mathcal{E}_i \qquad \mathcal{R}_\infty = \bigcup_{i \geqslant 0} \mathcal{R}_i \qquad \mathcal{E}_\omega = \bigcup_{i \geqslant 0} \bigcap_{j \geqslant i} \mathcal{E}_j \qquad \mathcal{R}_\omega = \bigcup_{i \geqslant 0} \bigcap_{j \geqslant i} \mathcal{R}_j$$

Equations in $\mathcal{E}_\omega$ and rules in $\mathcal{R}_\omega$ are called *persistent*. Note that any finite run with final state $(\mathcal{E}_n, \mathcal{R}_n)$ can be extended to an infinite run (e.g., by deduce steps followed by delete steps) such that $(\mathcal{E}_\omega, \mathcal{R}_\omega) = (\mathcal{E}_n, \mathcal{R}_n)$. Hence the following results, even though stated for infinite derivations, also capture the finite deductions of practical completion tools. The run Γ is called *non-failing* if $\mathcal{E}_\omega = \varnothing$, and *fair* if the inclusion

$$\mathsf{PCP}(\mathcal{R}_\omega) \;\subseteq\; \bigcup_{i \geqslant 0} \underset{\mathcal{E}_i}{\longleftrightarrow}$$

holds. Bachmair et al. [4] proved that for every non-failing fair run, the TRS $\mathcal{R}_\omega$ constitutes a complete presentation of $\mathcal{E}_0$. The remainder of this section is dedicated to establish the same result, but on a different route without encountering proof orders.

We start by showing a few properties of inference steps of completion. In the following proofs, they allow us to keep track of how equations and rules are modified during the completion process without caring about which inference rule was actually applied.

▶ **Lemma 8.** *Suppose $(\mathcal{E}, \mathcal{R}) \vdash (\mathcal{E}', \mathcal{R}')$. Then the following inclusions hold:*
1. $\mathcal{E}' \cup \mathcal{R}' \;\subseteq\; \leftrightarrow^*_{\mathcal{E} \cup \mathcal{R}}$
2. $\mathcal{E} \setminus \mathcal{E}' \;\subseteq\; (\to_{\mathcal{R}'} \cdot\, \mathcal{E}') \,\cup\, (\mathcal{E}' \cdot {}_{\mathcal{R}'}\!\leftarrow) \,\cup\, \mathcal{R}' \,\cup\, \mathcal{R}'^{-1} \,\cup =$
3. $\mathcal{R} \setminus \mathcal{R}' \;\subseteq\; (\xrightarrow{\rhd}_{\mathcal{R}'} \cdot\, \mathcal{E}') \,\cup\, (\mathcal{R}' \cdot {}_{\mathcal{R}'}\!\leftarrow)$ ◀

Together these properties reveal that inference steps do not change the conversion relation:

▶ **Corollary 9.** *If $(\mathcal{E}, \mathcal{R}) \vdash (\mathcal{E}', \mathcal{R}')$ then the relations $\overset{*}{\underset{\mathcal{E} \cup \mathcal{R}}{\longleftrightarrow}}$ and $\overset{*}{\underset{\mathcal{E}' \cup \mathcal{R}'}{\longleftrightarrow}}$ coincide.* ◀

Below, we consider the infinite non-failing run $\Gamma\colon (\mathcal{E}_0, \mathcal{R}_0) \vdash (\mathcal{E}_1, \mathcal{R}_1) \vdash (\mathcal{E}_2, \mathcal{R}_2) \vdash \cdots$. First we show that all rewrite rules are compatible with the reduction order $>$.

▶ **Lemma 10.** *The inclusions $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty \subseteq >$ hold.* ◀

Next, we verify that every equality in $\mathcal{E}_i$ can be turned into a valley in $\mathcal{R}_\infty$. Note that in contrast to the proof order approach [4] and to our previous correctness proof for finite runs [8] we reason separately about equations and rules. This more local rationale simplifies the analysis as we can use different well-founded induction arguments for the two cases, rather than synthesizing an order that covers both.

▶ **Lemma 11.** *The inclusion $\mathcal{E}_i \subseteq \downarrow_{\mathcal{R}_\infty}$ holds for all $i \geqslant 0$.*

Proof. Let $s \approx t \in \mathcal{E}_i$. By induction on $\{s, t\}$ with respect to $>_{\mathrm{mul}}$ we show $s \downarrow_{\mathcal{R}_\infty} t$. Because $\mathcal{E}_\omega = \varnothing$, $s \approx t \in \mathcal{E}_{j-1} \setminus \mathcal{E}_j$ for some $j > i$. Following Lemma 8(2), we distinguish three cases.
- If $s \approx t \in \mathcal{R}_j \cup \mathcal{R}_j^{-1} \cup =$ then the claim trivially holds.
- If $s \to_{\mathcal{R}_j} u$ and $u \approx t \in \mathcal{E}_j$ for some term u then $\{s, t\} >_{\mathrm{mul}} \{u, t\}$ and thus $u \downarrow_{\mathcal{R}_\infty} t$ by the induction hypothesis. Hence also $s \downarrow_{\mathcal{R}_\infty} t$.
- Similarly, if $s \approx u \in \mathcal{E}_j$ and $u \;{}_{\mathcal{R}_j}\!\leftarrow t$ for some term u then $\{s, t\} >_{\mathrm{mul}} \{s, u\}$ and we obtain $s \downarrow_{\mathcal{R}_\infty} t$ as in the preceding case. ◀

▶ **Corollary 12.** *The inclusion $\underset{\mathcal{E}_i}{\longrightarrow} \subseteq \overset{*}{\underset{\mathcal{R}_\infty}{\longleftrightarrow}}$ holds for all $i \geqslant 0$.* ◀

In order to show confluence of $\mathcal{R}_\omega$ we use source labeling to label steps in $\mathcal{R}_\infty$. The next lemma allows us to transform every non-persistent rule $\ell \to r$ into an $\mathcal{R}_\omega$-conversion below ℓ. In the proof we employ the following extension of the reduction order $>$ used in completion.

▶ **Definition 13.** We define $\succ \,=\, ((> \cup \rhd) \;/\; \trianglerighteq)^+$.

According to Lemma 3, $\succ$ is a well-founded order.

▶ **Lemma 14.** *The inclusion* $\xrightarrow[\mathcal{R}_\infty]{s} \subseteq \xleftrightarrow[\mathcal{R}_\omega]{\vee s}{}^*$ *holds for all terms* s.

Proof. Let $s \xrightarrow{s}_{\mathcal{R}_\infty} t$ by employing the rewrite rule $\ell \to r$. We prove $s \xleftrightarrow{\vee s}{}^*_{\mathcal{R}_\omega} t$ by induction on (ℓ, r) with respect to $\succ_{\text{lex}}$. If $\ell \to r \in \mathcal{R}_\omega$ then the claim trivially holds. Otherwise, $\ell \to r \in \mathcal{R}_{i-1} \setminus \mathcal{R}_i$ for some $i > 0$. Using Lemma 8(3), we distinguish two cases.

- Suppose $\ell \xrightarrow{\rhd}_{\ell' \to r'} u$ and $u \approx r \in \mathcal{E}_i$ for some term u and rule $\ell' \to r' \in \mathcal{R}_i$. We obtain $\ell \xrightarrow{\rhd}_{\ell' \to r'} u \downarrow_{\mathcal{R}_\infty} r$ from Lemma 11. We have $\ell \rhd \ell'$ and both $\ell > u$ and $\ell > r$. It follows that all rewrite rules $\ell'' \to r''$ employed in $\ell \xrightarrow{\rhd} u \downarrow_{\mathcal{R}_\infty} r$ satisfy $(\ell, r) \succ_{\text{lex}} (\ell'', r'')$. Moreover, all steps in $\ell \downarrow_{\mathcal{R}_\infty} r$ are labeled with a term $\leqslant \ell$. Hence we obtain $\ell \xleftrightarrow{\vee \ell}{}^*_{\mathcal{R}_\omega} r$ from the induction hypothesis.
- Suppose $\ell \to u \in \mathcal{R}_i$ and $u \leftarrow_{\ell' \to r'} r$ for some term u and rewrite rule $\ell' \to r' \in \mathcal{R}_i$. We have $(\ell, r) \succ_{\text{lex}} (\ell, u)$ and $(\ell, r) \succ_{\text{lex}} (\ell', r')$. Moreover, both steps are labeled with a term $\leqslant \ell$ and thus we obtain $\ell \xleftrightarrow{\vee \ell}{}^*_{\mathcal{R}_\omega} r$ from the induction hypothesis.

So in both cases we have $\ell \xleftrightarrow{\vee \ell}{}^*_{\mathcal{R}_\omega} r$ and thus also $s \xleftrightarrow{\vee s}{}^*_{\mathcal{R}_\omega} t$. ◀

▶ **Corollary 15.** *The relations* $\xleftrightarrow[\mathcal{R}_\infty]{*}$ *and* $\xleftrightarrow[\mathcal{R}_\omega]{*}$ *coincide.* ◀

We arrive at the main theorem of this section. Note that Bachmair's correctness proof [2] uses induction with respect to a well-founded order on conversions to directly show that any conversion of $\mathcal{E}_\infty \cup \mathcal{R}_\infty$ can be transformed into a joining sequence of $\mathcal{R}_\omega$. In contrast, we prove confluence via source decreasingness. This allows us to concentrate on *local* peaks.

▶ **Theorem 16.** *If* Γ *is fair then* $\mathcal{R}_\omega$ *is a complete presentation of* $\mathcal{E}_0$.

Proof. We have $\mathcal{E}_\omega = \varnothing$ because Γ is non-failing. The TRS $\mathcal{R}_\omega$ is terminating by Lemma 10. We show source decreasingness of labeled $\mathcal{R}_\omega$ reduction with respect to the reduction order $>$. So let $t \ {}_{\mathcal{R}_\omega}\!\xleftarrow{s} s \xrightarrow{s}_{\mathcal{R}_\omega} u$. From Lemma 6 we obtain $t \ \triangledown_s^2 \ u$. Let $v \ \triangledown_s \ w$ appear in this sequence (so $t = v$ or $w = u$). We have $s > v$, $s > w$, and

$$(v, w) \in \downarrow_{\mathcal{R}_\omega} \cup \bigcup_{i \geqslant 0} \leftrightarrow_{\mathcal{E}_i}$$

by the definition of $\triangledown_s$ and fairness of Γ.

- If $v \downarrow_{\mathcal{R}_\omega} w$ then $v \xrightarrow{\vee v}{}^*_{\mathcal{R}_\omega} \cdot {}_{\mathcal{R}_\omega}^{\ \ *}\!\xleftarrow{\vee w} w$ and thus $v \xleftrightarrow{\vee s}{}^*_{\mathcal{R}_\omega} w$.
- If $v \leftrightarrow_{\mathcal{E}_i} w$ for some $i \geqslant 0$ then $v \downarrow_{\mathcal{R}_\infty} w$ by Lemma 11. We obtain $v \xleftrightarrow{\vee s}{}^*_{\mathcal{R}_\infty} w$ as in the previous case and thus $v \xleftrightarrow{\vee s}{}^*_{\mathcal{R}_\omega} w$ by Lemma 14.

Hence $t \xleftrightarrow{\vee s}{}^*_{\mathcal{R}_\omega} u$. Confluence of $\mathcal{R}_\omega$ now follows from Lemma 4. It remains to show $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R}_\omega}$. Using Corollary 9 we obtain $\to_{\mathcal{E}_i \cup \mathcal{R}_i} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ by a straightforward induction on i. This in turn yields $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty}$. From Corollary 12 we infer $\leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty} = \leftrightarrow^*_{\mathcal{R}_\infty}$ and we conclude by an appeal to Corollary 15. ◀

▶ **Example 17.** Consider the equational system $\mathcal{E}$ and the KBO $>$ from Example 1. Let $\mathcal{P}_n$ denote the TRS $\{\mathsf{ab}^{i+1}\mathsf{ab} \to \mathsf{babba}^i \mid 1 \leqslant i \leqslant n\}$. One possible infinite completion run is the following:

$$\begin{aligned}
(\mathcal{E}, \varnothing) &\vdash^{\mathsf{orient}} (\varnothing, \{\mathsf{aba} \to \mathsf{bab}\}) && \vdash^{\mathsf{deduce}} (\{\mathsf{abbab} \approx \mathsf{babba}\}, \{\mathsf{aba} \to \mathsf{bab}\}) \\
&\vdash^{\mathsf{orient}} (\varnothing, \{\mathsf{aba} \to \mathsf{bab}\} \cup \mathcal{P}_1) && \vdash^{\mathsf{deduce}} (\{\mathsf{abbbab} \approx \mathsf{babbaa}\}, \{\mathsf{aba} \to \mathsf{bab}\} \cup \mathcal{P}_1) \\
&\vdash^{\mathsf{orient}} (\varnothing, \{\mathsf{aba} \to \mathsf{bab}\} \cup \mathcal{P}_2) && \vdash \cdots
\end{aligned}$$

If this run is continued in a fair way we subsequently construct the TRSs $\mathcal{P}_n$ and can in the limit obtain the result $\mathcal{R}_\omega = \{\mathsf{aba} \to \mathsf{bab}\} \cup \{\mathsf{ab}^{i+1}\mathsf{ab} \to \mathsf{babba}^i \mid 1 \leqslant i\}$, which is complete according to Theorem 16.

4 Ordered Completion

In this section $>$ is a fixed ground-total reduction order, i.e., for all ground terms $s, t \in \mathcal{T}(\mathcal{F})$ either $s > t$, $t > s$, or $s = t$ holds. Given a binary relation R, we write $R^\pm$ for the symmetric closure $R \cup R^{-1}$. For a set $\mathcal{E}$ of equations, an ordered rewrite step is a rewrite step using a rule from $\mathcal{E}^>$, which is the set of rewrite rules $\ell\sigma \to r\sigma$ such that $\ell \approx r \in \mathcal{E}^\pm$ and $\ell\sigma > r\sigma$.

An *extended overlap* is given by two variable-disjoint variants $\ell_1 \approx r_1$ and $\ell_2 \approx r_2$ of equations in $\mathcal{E}^\pm$ such that $\ell_1\mu = \ell_2|_p\mu$ with $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(\ell_2)$ and mgu μ. An extended overlap which satisfies $r_1\mu \not> \ell_1\mu$ and $r_2\mu \not> \ell_2\mu$ gives rise to the *extended critical pair* $\ell_2[r_1]_p\mu \approx r_2\mu$. An extended critical pair is called *prime* if all proper subterms of $\ell_1\mu$ are $\mathcal{E}^>$-normal forms. The set of extended prime critical pairs among equations in $\mathcal{E}$ is denoted $\mathsf{PCP}_>(\mathcal{E})$.

The following inference rules for ordered completion are due to Bachmair, Dershowitz, and Plaisted [5]. In order to simplify the notation, we abbreviate $\mathcal{R} \cup \mathcal{E}^>$ to $\mathcal{S}$, and use the following shorthands. We write $t \xrightarrow{\rhd}_{\mathcal{E}^>} u$ if there exist an equation $\ell \approx r \in \mathcal{E}^\pm$, a context C, and a substitution σ such that $t = C[\ell\sigma]$, $u = C[r\sigma]$, $\ell\sigma > r\sigma$, and $t \rhd \ell$. The union of $\to_\mathcal{R}$ and $\xrightarrow{\rhd}_{\mathcal{E}^>}$ is denoted by $\xrightarrow{\rhd_1}_\mathcal{S}$ and we write $\xrightarrow{\rhd_2}_\mathcal{S}$ for the union of $\xrightarrow{\rhd}_\mathcal{R}$ and $\xrightarrow{\rhd}_{\mathcal{E}^>}$.

▶ **Definition 18.** The inference system oKB of ordered completion operates on pairs $(\mathcal{E}, \mathcal{R})$ of equations $\mathcal{E}$ and rules $\mathcal{R}$ over a common signature $\mathcal{F}$. It consists of the following inference rules:

deduce	$\dfrac{\mathcal{E}, \mathcal{R}}{\mathcal{E} \cup \{s \approx t\}, \mathcal{R}}$	if $s \xleftarrow[\mathcal{R}\cup\mathcal{E}]{} \cdot \xrightarrow[\mathcal{R}\cup\mathcal{E}]{} t$	compose	$\dfrac{\mathcal{E}, \mathcal{R} \uplus \{s \to t\}}{\mathcal{E}, \mathcal{R} \cup \{s \to u\}}$	if $t \to_\mathcal{S} u$
orient	$\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{s \to t\}}$	if $s > t$	simplify	$\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{u \approx t\}, \mathcal{R}}$	if $s \xrightarrow{\rhd_1}_\mathcal{S} u$
	$\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E}, \mathcal{R} \cup \{t \to s\}}$	if $t > s$		$\dfrac{\mathcal{E} \uplus \{s \approx t\}, \mathcal{R}}{\mathcal{E} \cup \{s \approx u\}, \mathcal{R}}$	if $t \xrightarrow{\rhd_1}_\mathcal{S} u$
delete	$\dfrac{\mathcal{E} \uplus \{s \approx s\}, \mathcal{R}}{\mathcal{E}, \mathcal{R}}$		collapse	$\dfrac{\mathcal{E}, \mathcal{R} \uplus \{t \to s\}}{\mathcal{E} \cup \{u \approx s\}, \mathcal{R}}$	if $t \xrightarrow{\rhd_2}_\mathcal{S} u$

The deduce rule may be applied to any peak, though in practice it is typically limited to the addition of extended critical pairs. We write $(\mathcal{E}, \mathcal{R}) \vdash_\mathsf{o} (\mathcal{E}', \mathcal{R}')$ if $(\mathcal{E}', \mathcal{R}')$ can be reached from $(\mathcal{E}, \mathcal{R})$ by employing one of the inference rules of Definition 18. We start by stating the equivalents of Lemma 8 and Corollary 9 for ordered completion.

▶ **Lemma 19.** *Suppose $(\mathcal{E}, \mathcal{R}) \vdash_\mathsf{o} (\mathcal{E}', \mathcal{R}')$. Then the following inclusions hold:*

1. $\mathcal{E}' \cup \mathcal{R}' \subseteq \xleftrightarrow[\mathcal{E}\cup\mathcal{R}]{*}$
2. $\mathcal{E} \setminus \mathcal{E}' \subseteq (\xrightarrow[\mathcal{S}']{\rhd_1} \cdot \mathcal{E}'^\pm)^\pm \cup \mathcal{R}'^\pm \cup =$
3. $\mathcal{R} \setminus \mathcal{R}' \subseteq (\xrightarrow[\mathcal{S}']{\rhd_2} \cdot \mathcal{E}') \cup (\mathcal{R}' \cdot \xleftarrow[\mathcal{S}']{})$ ◀

▶ **Corollary 20.** *If $(\mathcal{E}, \mathcal{R}) \vdash_\mathsf{o} (\mathcal{E}', \mathcal{R}')$ then the relations $\xleftrightarrow[\mathcal{E}\cup\mathcal{R}]{*}$ and $\xleftrightarrow[\mathcal{E}'\cup\mathcal{R}']{*}$ coincide.* ◀

Below, we consider the infinite run $\Gamma\colon (\mathcal{E}_0, \mathcal{R}_0) \vdash_\mathsf{o} (\mathcal{E}_1, \mathcal{R}_1) \vdash_\mathsf{o} (\mathcal{E}_2, \mathcal{R}_2) \vdash_\mathsf{o} \cdots$.

▶ **Lemma 21.** *The inclusions* $\mathcal{R}_\omega \subseteq \mathcal{R}_\infty \subseteq >$ *and* $\mathcal{E}_\omega \subseteq \mathcal{E}_\infty$ *hold.* ◀

Unlike for Knuth-Bendix completion we do not assume Γ to be non-failing, and in general $\mathcal{E}_i \subseteq \downarrow_{\mathcal{R}_\infty}$ does not hold. So we take a different route. Given a rewrite relation $\to$ and a set S of terms, we write $t \xrightarrow{S} u$ if $t \to u$, $s \geqslant t$, and $s' \geqslant u$ for some terms $s, s' \in S$. Since both $\to$ and $\geqslant$ are closed under contexts and substitutions, we have $C[t\sigma] \xrightarrow{S'} C[u\sigma]$ whenever $t \xrightarrow{S} u$ and $S' = \{C[s\sigma] \mid s \in S\}$, for all contexts C and substitutions σ. We use this relation to show that any equation step below a term set S eventually turns into a conversion over $\mathcal{R}_\infty \cup \mathcal{E}_\omega$ that is still below S. Note that just like in Section 3 we avoid the use of a synthesized termination argument by handling equations and rules separately.

▶ **Lemma 22.** *The inclusion* $\xrightarrow[\mathcal{E}_\infty]{S} \subseteq \xleftrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\omega]{S}{}^*$ *holds for all sets* S *of terms.*

Proof. Let $t \approx u \in \mathcal{E}_\infty$. We prove

$$\xrightarrow[t \approx u]{S} \subseteq \xleftrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\omega]{S}{}^*$$

by induction on $\{t, u\}$ with respect to the well-founded order $\succ_{\mathrm{mul}}$. If $t \approx u \in \mathcal{E}_\omega^\pm$ then the claim follows trivially. Otherwise, $t \approx u \in (\mathcal{E}_{i-1} \setminus \mathcal{E}_i)^\pm$ for some $i > 0$. Using Lemma 19(2), we distinguish two subcases.

- Suppose $t \approx u \in (\xrightarrow{\rhd_1}_{\mathcal{S}_i} \cdot \mathcal{E}_i^\pm)^\pm$. There exist a term t' and an equation $v' \approx u' \in \mathcal{E}_i^\pm$ such that $\{t, u\} = \{t', u'\}$ and $t' \xrightarrow{\rhd_1}_{\mathcal{S}_i} v'$. It is sufficient to show

  $$t' \xleftrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\omega]{\{t',u'\}}{}^* v' \quad \text{and} \quad v' \xleftrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\omega]{\{t',u'\}}{}^* u'$$

  The second conversion follows from $t' > v'$ and the induction hypothesis for $v' \approx u' \in \mathcal{E}_i^\pm$, which is applicable as $\{t, u\} = \{t', u'\} \succ_{\mathrm{mul}} \{v', u'\}$. The first conversion is obtained as follows. Because of $t' \xrightarrow{\rhd_1}_{\mathcal{S}_i} v'$, we have $t' \to_{\mathcal{R}_i} v'$ or $t' \xrightarrow{\rhd}_{\mathcal{E}_i^>} v'$. If $t' \to_{\mathcal{R}_i} v'$ then this step can be labeled with $\{t', u'\}$ as $t' > v'$. Otherwise, there exist an equation $\ell \approx r \in \mathcal{E}_i^\pm$, a context C, and a substitution σ such that $t' = C[\ell\sigma]$, $v' = C[r\sigma]$, $\ell\sigma > r\sigma$, and $t' \rhd \ell$. We have $t' \succ \ell$ and $t' \succ r$ as $t' \trianglerighteq \ell\sigma > r\sigma \trianglerighteq r$. Therefore $\{t', u'\} \succ_{\mathrm{mul}} \{\ell, r\}$ holds, so

  $$\ell \xleftrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\omega]{\{\ell,r\}}{}^* r$$

  follows from the induction hypothesis. Closure under contexts and substitutions now yields $t \xleftrightarrow{\{t,u\}}{}^*_{\mathcal{R}_\infty \cup \mathcal{E}_\omega} u$.
- If $t \approx u \in \mathcal{R}_i^\pm \cup =$ then $t \xleftrightarrow[\mathcal{R}_\infty]{\{t,u\}}{}^= u$.

In both cases $t \xleftrightarrow{\{t,u\}}{}^*_{\mathcal{R}_\infty \cup \mathcal{E}_\omega} u$ holds. Since S contains upper bounds of t and u with respect to $\geqslant$, the desired inclusion follows from the closure under contexts and substitutions of $\to_{\mathcal{R}_\infty \cup \mathcal{E}_\omega}$ and $\geqslant$. ◀

Next, we show that a rewrite step that uses a rule in $\mathcal{R}_\infty$ and is below a term set S eventually turns into a conversion over persistent rules and equations that is still below S. We write $\curlyvee t$ for the set $\{u \in \mathcal{T}(\mathcal{F}, \mathcal{V}) \mid t \succ u\}$.

▶ **Lemma 23.** *The inclusion* $\xrightarrow[\mathcal{R}_\infty]{S} \subseteq \xleftrightarrow[\mathcal{R}_\omega \cup \mathcal{E}_\omega]{S}{}^*$ *holds for all sets* S *of terms.*

Proof. Let $\ell \approx r \in \mathcal{R}_\infty$. We prove

$$\xrightarrow[\ell \to r]{S} \subseteq \xleftrightarrow[\mathcal{R}_\omega \cup \mathcal{E}_\omega]{S}{}^*$$

by induction on (ℓ, r) with respect to the well-founded order $\succ_{\text{lex}}$. If $\ell \to r \in \mathcal{R}_\omega$ then the claim trivially holds. Otherwise, there is some $i > 0$ such that $\ell \to r \in \mathcal{R}_{i-1} \setminus \mathcal{R}_i$. From Lemma 22 and the induction hypothesis the inclusions

$$\xrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\infty]{T} \;\subseteq\; \xleftrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\omega]{T}{}^{*} \;\subseteq\; \xleftrightarrow[\mathcal{R}_\omega \cup \mathcal{E}_\omega]{T}{}^{*} \tag{1}$$

are obtained for every set $T \subseteq \curlyvee\ell$. Using Lemma 19, we distinguish two cases.

- Suppose $\ell \xrightarrow{\rhd_2}_{\mathcal{S}_i} u$ and $u \approx r \in \mathcal{E}_i$ for some term u. There exist an equation $\ell' \approx r' \in \mathcal{R}_\infty \cup \mathcal{E}_\infty^{\pm}$, a context C and a substitution σ such that $\ell = C[\ell'\sigma]$, $u = C[r'\sigma]$, $\ell\sigma > r\sigma$, and $\ell \rhd \ell'$. We have $\ell \succ \ell', r'$ as $\ell \rhd \ell'$ and $\ell \unrhd \ell'\sigma > r'\sigma \unrhd r'$ and thus $\ell' \xrightarrow{\{\ell'\}}_{\mathcal{R}_\infty \cup \mathcal{E}_\infty} r'$. Since $\{\ell', r'\} \subseteq \curlyvee\ell$ we obtain $\ell' \xleftrightarrow{\{\ell',r'\}}{}^{*}_{\mathcal{R}_\omega \cup \mathcal{E}_\omega} r'$ from (1). Therefore, $\ell \xleftrightarrow{\{\ell\}}{}^{*}_{\mathcal{R}_\omega \cup \mathcal{E}_\omega} u$ follows from closure under contexts and substitutions and $\ell > u$. Again from $\ell > u, r$ we obtain $u \xleftrightarrow{\curlyvee\ell}_{\mathcal{R}_\infty \cup \mathcal{E}_\infty} r$ and thus $u \xleftrightarrow{\curlyvee\ell}_{\mathcal{R}_\omega \cup \mathcal{E}_\omega} r$ follows from (1).
- Suppose $\ell \to u \in \mathcal{R}_i$ and $u \;{}_{\mathcal{S}_i}\!\leftarrow r$ for some term u. We have $r > u$ and thus $(\ell, r) \succ_{\text{lex}} (\ell, u)$. Hence we can apply the induction hypothesis to $\ell \xrightarrow[\ell \to u]{\{\ell\}} u$, yielding $\ell \xleftrightarrow{\{\ell\}}{}^{*}_{\mathcal{R}_\omega \cup \mathcal{E}_\omega} u$. From $\ell > r > u$ we obtain $u \xleftrightarrow{\curlyvee\ell}_{\mathcal{R}_\infty \cup \mathcal{E}_\infty} r$ and thus $u \xleftrightarrow{\curlyvee\ell}{}^{*}_{\mathcal{R}_\omega \cup \mathcal{E}_\omega} r$ follows by (1).

In both cases $\ell \xleftrightarrow{\{\ell\}}{}^{*}_{\mathcal{R}_\omega \cup \mathcal{E}_\omega} r$ holds. Since $\to_{\mathcal{R}_\omega \cup \mathcal{E}_\omega}$ and $\geqslant$ are closed under contexts and substitutions, the desired inclusion on steps using $\ell \to r$ follows. ◀

We can combine the previous two lemmas to obtain an inclusion in conversions over persistent equations and rules.

▶ **Corollary 24.** *The inclusion* $\xrightarrow[\mathcal{R}_\infty \cup \mathcal{E}_\infty]{S} \;\subseteq\; \xleftrightarrow[\mathcal{R}_\omega \cup \mathcal{E}_\omega]{S}{}^{*}$ *holds for all sets S of terms.* ◀

Below, we specialize this result to ground terms.

▶ **Corollary 25.** *If $s \xleftrightarrow[\mathcal{E}_\infty]{} t$ for ground terms s and t then $s \xleftrightarrow[\mathcal{S}_\omega]{\{s,t\}}{}^{*} t$.*

Proof. We obtain $s \xleftrightarrow{\{s,t\}}{}^{*}_{\mathcal{R}_\omega \cup \mathcal{E}_\omega} t$ from Corollary 24. Since $>$ is ground-total, all $\mathcal{E}_\omega$ steps in this conversion are $(\mathcal{E}_\omega^{>})^{\pm}$ steps or identities. Hence $s \xleftrightarrow[\mathcal{S}_\omega]{\{s,t\}}{}^{*} t$ as desired. ◀

The run Γ is called *fair* if the inclusion

$$\mathsf{PCP}_>(\mathcal{R}_\omega \cup \mathcal{E}_\omega) \subseteq \bigcup_{i \geqslant 0} \xleftrightarrow[\mathcal{E}_i]{}$$

holds. The following lemma links extended prime critical pairs to standard critical pairs and hence allows us to use results from Section 3 for our main correctness result (Theorem 27 below).

▶ **Lemma 26.** *For a TRS $\mathcal{R}$ and a set of equations $\mathcal{E}$ the inclusion $\xleftrightarrow[\mathsf{PCP}(\mathcal{S})]{} \subseteq \xleftrightarrow[\mathsf{PCP}_>(\mathcal{R} \cup \mathcal{E})]{} \cup \downarrow_{\mathcal{S}}$ holds on ground terms.*

Proof. Suppose $s \leftrightarrow_e t$ for ground terms s and t and a prime critical pair $e\colon \ell_2\sigma[r_1\sigma]_p \approx r_2\sigma$ generated from the overlap $\langle \ell_1 \to r_1, p, \ell_2 \to r_2 \rangle$ in $\mathcal{S}$. Let $u_i \approx v_i$ be the equation $\ell_i \approx r_i$ if $\ell_i \to r_i \in \mathcal{R}$ and the equation in $\mathcal{E}^{\pm}$ such that $\ell_i = u_i\tau_i$ and $r_i = v_i\tau_i$ for some substitution τ_i if $\ell_i \to r_i \in \mathcal{E}^{>}$. In the former case we let τ_i be the empty substitution. Since the equations $u_1 \approx v_1$ and $u_2 \approx v_2$ are assumed to be variable-disjoint, the substitution $\tau = \tau_1 \cup \tau_2$ is well-defined. We distinguish two cases.

- If $p \notin \mathcal{P}\mathsf{os}_{\mathcal{F}}(u_2)$ then $\langle u_1 \approx v_1, p, u_2 \approx v_2 \rangle$ is not an overlap and hence $s \downarrow_{\mathcal{S}} t$ by the (extended) Critical Pair Lemma [5].

- Suppose $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(u_2)$. Since $u_2|_p\tau\sigma = \ell_2|_p\sigma = \ell_1\sigma = u_1\tau\sigma$ there exist an mgu μ of $u_2|_p$ and u_1, and a substitution ρ such that $\mu\rho = \tau\sigma$. Because $u_i\mu\rho = \ell_i\sigma > r_i\sigma = v_i\mu\rho$, $v_i\mu > u_i\mu$ is impossible. Hence $e'\colon u_2\mu[v_1\mu]_p \approx v_2\mu \in \mathsf{CP}_>(\mathcal{R} \cup \mathcal{E})$ and

$$\ell_2\sigma[r_1\sigma]_p = u_2\mu\rho[v_1\mu\rho]_p = u_2\mu[v_1\mu]_p\rho \xleftrightarrow[e']{} v_2\mu\rho = r_2\sigma$$

Since e is prime, proper subterms of $\ell_2\sigma|_p = u_2\mu\rho|_p$ are irreducible with respect to $\mathcal{S}$, and hence the same holds for proper subterms of $u_2\mu$. It follows that $e' \in \mathsf{PCP}_>(\mathcal{R} \cup \mathcal{E})$ and thus $\ell_2\sigma[r_1\sigma]_p \xleftrightarrow[\mathsf{PCP}_>(\mathcal{R}\cup\mathcal{E})]{} r_2\sigma$. Hence also $s \xleftrightarrow[\mathsf{PCP}_>(\mathcal{R}\cup\mathcal{E})]{} t$. ◀

This relationship between extended critical pairs among $\mathcal{R} \cup \mathcal{E}$ and critical pairs among $\mathcal{S}$ is the final ingredient for the main result of this section. As in the proof for Knuth-Bendix completion, we establish correctness of ordered completion via source decreasingness.

▶ **Theorem 27.** *If Γ is fair then $\mathcal{S}_\omega$ is ground-complete and $\xleftrightarrow[\mathcal{E}_0]{*} \; = \; \xleftrightarrow[\mathcal{R}_\omega \cup \mathcal{E}_\omega]{*}$.*

Proof. Termination of $\mathcal{S}_\omega$ is a consequence of Lemma 21 and the definition of $\mathcal{E}_\omega^>$. Next we show that $\mathcal{S}_\omega$ is ground-confluent. To this end, we show that labeled $\mathcal{S}_\omega$ reduction is source decreasing on ground terms. So let s, t, and u be ground terms such that $t \;{}_{\mathcal{S}_\omega}\!\xleftarrow{s}\; s \xrightarrow{s}_{\mathcal{S}_\omega} u$. From Lemma 6 we obtain $t \;\nabla_s^2\; u$ (where $\mathcal{S}_\omega$ takes the place of $\mathcal{R}$ in the definition of ∇_s). Let $v \;\nabla_s\; w$ appear in this sequence (so $t = v$ or $w = u$ and both terms are ground). We have $s > v$, $s > w$, and

$$(v, w) \in {\downarrow_{\mathcal{S}_\omega}} \cup \bigcup_{i \geqslant 0} \xleftrightarrow[\mathcal{E}_i]{}$$

by the definition of ∇_s, Lemma 26, and fairness of Γ.

- If $v \downarrow_{\mathcal{S}_\omega} w$ then $v \xrightarrow{\vee\!\!\vee v}{}^*_{\mathcal{S}_\omega} \cdot {}_{\mathcal{S}_\omega}^{\;*}\!\xleftarrow{\vee\!\!\vee w} w$ and thus $v \xleftrightarrow{\vee s}{}^*_{\mathcal{S}_\omega} w$.
- If $v \leftrightarrow_{\mathcal{E}_i} w$ for some $i \geqslant 0$ then $v \xleftrightarrow{\vee s}{}^*_{\mathcal{S}_\omega} w$ by Corollary 25.

Hence $t \xleftrightarrow{\vee s}{}^*_{\mathcal{S}_\omega} u$. Confluence of the ARS that is obtained by restricting $\mathcal{S}_\omega$ to ground terms now follows from Lemma 4. It remains to show $\leftrightarrow^*_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R}_\omega \cup \mathcal{E}_\omega}$. Using Corollary 20 we obtain $\to_{\mathcal{E}_i \cup \mathcal{R}_i} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ for all i by a straightforward induction argument. This in turn yields $\leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$ and in particular $\leftrightarrow^*_{\mathcal{E}_\omega \cup \mathcal{R}_\omega} \subseteq \leftrightarrow^*_{\mathcal{E}_0}$. The reverse inclusion follows from Corollary 24 and the inclusion $\leftrightarrow^*_{\mathcal{E}_0} \subseteq \leftrightarrow^*_{\mathcal{E}_\infty \cup \mathcal{R}_\infty}$. ◀

5 Canonicity

In this section we revisit Métivier work [14], aiming at generalizing his uniqueness result for canonical TRSs and at establishing a transformation to simplify ground-complete TRSs. A key notion is normalization equivalence.

▶ **Definition 28.** Two ARSs $\mathcal{A}$ and $\mathcal{B}$ are said to be *(conversion) equivalent* if $\leftrightarrow^*_{\mathcal{A}} = \leftrightarrow^*_{\mathcal{B}}$ and *normalization equivalent* if $\to^!_{\mathcal{A}} = \to^!_{\mathcal{B}}$.

The following example shows that the two equivalence notions defined above are different.

▶ **Example 29.** Consider the following ARSs:

$$\mathcal{A}_1\colon \quad \mathsf{a} \longrightarrow \mathsf{b} \qquad\qquad \mathcal{B}_1\colon \quad \mathsf{a} \longleftarrow \mathsf{b}$$

$$\mathcal{A}_2\colon \quad \begin{tikzcd} \mathsf{a} \arrow[r] & \mathsf{b} \arrow[loop below] \end{tikzcd} \qquad\qquad \mathcal{B}_2\colon \quad \begin{tikzcd} \mathsf{a} \arrow[loop below] & \mathsf{b} \arrow[loop below] \end{tikzcd}$$

While $\mathcal{A}_1$ and $\mathcal{B}_1$ are conversion equivalent but not normalization equivalent, the ARSs $\mathcal{A}_2$ and $\mathcal{B}_2$ are normalization equivalent but not conversion equivalent.

The easy proof (by induction on the length of conversions) of the following result is omitted.

▶ **Lemma 30.** *Normalization equivalent terminating ARSs are equivalent.* ◀

Note that the termination assumption can be weakened to weak normalization. However, the present version suffices to prove the following lemma that we employ in our proof of Métivier's transformation result [14] (Theorem 37 below).

▶ **Lemma 31.** *Let $\mathcal{A}$ and $\mathcal{B}$ be ARSs such that $\mathsf{NF}(\mathcal{B}) \subseteq \mathsf{NF}(\mathcal{A})$ and either*

1. $\to_{\mathcal{B}} \subseteq \to_{\mathcal{A}}^{+}$ *or*
2. $\to_{\mathcal{B}} \subseteq \leftrightarrow_{\mathcal{A}}^{*}$ *and $\mathcal{B}$ is terminating.*

If $\mathcal{A}$ is complete then $\mathcal{B}$ is complete and normalization equivalent to $\mathcal{A}$.

Proof. We first show $\to_{\mathcal{B}}^{!} \subseteq \to_{\mathcal{A}}^{!}$. In case (a), from the inclusion $\to_{\mathcal{B}} \subseteq \to_{\mathcal{A}}^{+}$ we infer that $\mathcal{B}$ is terminating. Moreover, $\to_{\mathcal{B}}^{*} \subseteq \to_{\mathcal{A}}^{*}$ and, since $\mathsf{NF}(\mathcal{B}) \subseteq \mathsf{NF}(\mathcal{A})$, also $\to_{\mathcal{B}}^{!} \subseteq \to_{\mathcal{A}}^{!}$. For case (b), $\to_{\mathcal{B}}^{!} \subseteq \to_{\mathcal{A}}^{!}$ holds because $\to_{\mathcal{B}}^{!} \subseteq \leftrightarrow_{\mathcal{A}}^{*}$, so by confluence of $\mathcal{A}$ and $\mathsf{NF}(\mathcal{B}) \subseteq \mathsf{NF}(\mathcal{A})$ we obtain $\to_{\mathcal{B}}^{!} \subseteq \to_{\mathcal{A}}^{!}$. Next we show that the reverse inclusion $\to_{\mathcal{A}}^{!} \subseteq \to_{\mathcal{B}}^{!}$ holds in both cases. Let $a \to_{\mathcal{A}}^{!} b$. Because $\mathcal{B}$ is terminating, $a \to_{\mathcal{B}}^{!} c$ for some $c \in \mathsf{NF}(\mathcal{B})$. So $a \to_{\mathcal{A}}^{!} c$ and thus $b = c$ from the confluence of $\mathcal{A}$. It follows that $\mathcal{A}$ and $\mathcal{B}$ are normalization equivalent. It remains to show that $\mathcal{B}$ is locally confluent. This follows from the sequence of inclusions

$$ {}_{\mathcal{B}}\!\leftarrow \cdot \to_{\mathcal{B}} \;\subseteq\; \leftrightarrow_{\mathcal{A}}^{*} \;\subseteq\; \to_{\mathcal{A}}^{!} \cdot {}_{\mathcal{A}}^{!}\!\leftarrow \;\subseteq\; \to_{\mathcal{B}}^{!} \cdot {}_{\mathcal{B}}^{!}\!\leftarrow $$

where we obtain the inclusions from $\to_{\mathcal{B}} \subseteq \leftrightarrow_{\mathcal{A}}^{*}$, confluence of $\mathcal{A}$, termination of $\mathcal{A}$, and normalization equivalence of $\mathcal{A}$ and $\mathcal{B}$, respectively. ◀

In the above lemma, completeness can be weakened to semi-completeness (i.e., the combination of confluence and weak normalization), which is not true for Theorem 37 as shown by Gramlich [7]. Again, the present version suffices for our purposes.

▶ **Definition 32.** A TRS $\mathcal{R}$ is *left-reduced* if $\ell \in \mathsf{NF}(\mathcal{R} \setminus \{\ell \to r\})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$. We say that $\mathcal{R}$ is *right-reduced* if $r \in \mathsf{NF}(\mathcal{R})$ for every rewrite rule $\ell \to r$ in $\mathcal{R}$. A *reduced* TRS is left- and right-reduced. A reduced complete TRS is called *canonical*.

Theorem 37 below states that we can always eliminate redundancy in a complete TRS. This is achieved by the two-stage transformation defined below.

▶ **Definition 33.** Two TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ over the same signature $\mathcal{F}$ are called *literally similar*, denoted by $\mathcal{R}_1 \doteq \mathcal{R}_2$, if every rewrite rule in $\mathcal{R}_1$ has a variant in $\mathcal{R}_2$ and vice-versa.

Given a TRS $\mathcal{R}$, we write $\mathcal{R}/\doteq$ for a set of representatives of the equivalence classes of rules in $\mathcal{R}$ with respect to $\doteq$. In other words, $\mathcal{R}/\doteq$ is a variant-free version of $\mathcal{R}$.

▶ **Definition 34.** Given a terminating TRS $\mathcal{R}$, the TRSs $\dot{\mathcal{R}}$ and $\ddot{\mathcal{R}}$ are defined as follows:

$$\dot{\mathcal{R}} = \{\ell \to r{\downarrow_{\mathcal{R}}} \mid \ell \to r \in \mathcal{R}\}/\doteq$$
$$\ddot{\mathcal{R}} = \{\ell \to r \in \dot{\mathcal{R}} \mid \ell \in \mathsf{NF}(\dot{\mathcal{R}} \setminus \{\ell \to r\})\}$$

Here $t{\downarrow_{\mathcal{R}}}$ stands for an arbitrary but fixed normal form of t.

The TRS $\dot{\mathcal{R}}$ is obtained from $\mathcal{R}$ by first normalizing the right-hand sides and then taking representatives of variants of the resulting rules, thereby making sure that the result does not contain several variants of the same rule. To obtain $\ddot{\mathcal{R}}$ we remove the rules of $\dot{\mathcal{R}}$ whose left-hand sides are reducible with another rule of $\dot{\mathcal{R}}$.

The following example shows why the result of $\dot{\mathcal{R}}$ has to be variant-free.

▶ **Example 35.** Consider the TRS $\mathcal{R}_1$ consisting of the four rules

$$\mathsf{f}(x) \to \mathsf{a} \qquad \mathsf{f}(y) \to \mathsf{b} \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

Then the first transformation without taking representatives of rules would yield $\dot{\mathcal{R}}_1$

$$\mathsf{f}(x) \to \mathsf{c} \qquad \mathsf{f}(y) \to \mathsf{c} \qquad \mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

and the second one $\ddot{\mathcal{R}}_1$

$$\mathsf{a} \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

Note that $\ddot{\mathcal{R}}_1$ is *not* equivalent to $\mathcal{R}_1$. This is caused by the fact that the result of the first transformation was no longer variant-free.

The following result is folklore; the proof has been formalized [8].

▶ **Lemma 36.** *Two terms s and t are variants if and only if $s \doteq t$.* ◀

▶ **Theorem 37.** *If $\mathcal{R}$ is a complete TRS then $\ddot{\mathcal{R}}$ is a normalization and conversion equivalent canonical TRS.*

The proof by Métivier [14, Theorem 7] is hard to reconstruct. The proof in [17, Exercise 7.4.7] involves 13 steps with lots of redundancy. Our proof below proceeds by induction on the well-founded encompassment order $\rhd$.

Proof. Let $\mathcal{R}$ be a complete TRS. The inclusions $\ddot{\mathcal{R}} \subseteq \dot{\mathcal{R}} \subseteq \to_{\mathcal{R}}^{+}$ are obvious from the definitions. Since $\mathcal{R}$ and $\dot{\mathcal{R}}$ have the same left-hand sides, their normal forms coincide. We show that $\mathsf{NF}(\ddot{\mathcal{R}}) \subseteq \mathsf{NF}(\dot{\mathcal{R}})$. To this end we show that $\ell \notin \mathsf{NF}(\ddot{\mathcal{R}})$ whenever $\ell \to r \in \dot{\mathcal{R}}$ by induction on ℓ with respect to the well-founded order $\rhd$. If $\ell \to r \in \ddot{\mathcal{R}}$ then $\ell \notin \mathsf{NF}(\ddot{\mathcal{R}})$ holds. So suppose $\ell \to r \notin \ddot{\mathcal{R}}$. By definition of $\ddot{\mathcal{R}}$, $\ell \notin \mathsf{NF}(\dot{\mathcal{R}} \setminus \{\ell \to r\})$. So there exists a rewrite rule $\ell' \to r' \in \dot{\mathcal{R}}$ different from $\ell \to r$ such that $\ell \trianglerighteq \ell'$. We distinguish two cases.

- If $\ell \rhd \ell'$ then we obtain $\ell' \notin \mathsf{NF}(\ddot{\mathcal{R}})$ from the induction hypothesis and hence $\ell \notin \mathsf{NF}(\ddot{\mathcal{R}})$ as desired.
- If $\ell \doteq \ell'$ then by Lemma 36 there exists a renaming σ such that $\ell = \ell'\sigma$. Since $\dot{\mathcal{R}}$ is right-reduced by construction, r and r' are normal forms of $\dot{\mathcal{R}}$. The same holds for $r'\sigma$ because normal forms are closed under renaming. We have $r \;{}_{\dot{\mathcal{R}}}{\leftarrow}\; \ell = \ell'\sigma \to_{\dot{\mathcal{R}}} r'\sigma$. Since $\dot{\mathcal{R}}$ is confluent as a consequence of Lemma 31(1), $r = r'\sigma$. Hence $\ell' \to r'$ is a variant of $\ell \to r$, contradicting the assumption that TRSs are variant-free.

From Lemma 31(1) we infer that the TRSs $\dot{\mathcal{R}}$ and $\ddot{\mathcal{R}}$ are complete and normalization equivalent to $\mathcal{R}$. The TRS $\ddot{\mathcal{R}}$ is right-reduced because $\ddot{\mathcal{R}} \subseteq \dot{\mathcal{R}}$ and $\dot{\mathcal{R}}$ is right-reduced. From $\mathsf{NF}(\ddot{\mathcal{R}}) = \mathsf{NF}(\dot{\mathcal{R}})$ we easily infer that $\ddot{\mathcal{R}}$ is left-reduced. It follows that $\ddot{\mathcal{R}}$ is canonical. It remains to show that $\ddot{\mathcal{R}}$ is not only normalization equivalent but also (conversion) equivalent to $\mathcal{R}$. This is an immediate consequence of Lemma 30. ◀

For our next result we need the following technical lemma.

▶ **Lemma 38.** *Let $\mathcal{R}$ be a right-reduced TRS and let s be a reducible term which is minimal with respect to $\rhd$. If $s \to_{\mathcal{R}}^{+} t$ then $s \to t$ is a variant of a rule in $\mathcal{R}$.*

Proof. Let $\ell \to r$ be the rewrite rule that is used in the first step from s to t. So $s \unrhd \ell$. By assumption, $s \rhd \ell$ does not hold and thus $s \doteq \ell$. According to Lemma 36 there exists a renaming σ such that $s = \ell\sigma$. We have $s \to_{\mathcal{R}} r\sigma \to_{\mathcal{R}}^{*} t$. Because $\mathcal{R}$ is right-reduced, $r \in \mathsf{NF}(\mathcal{R})$. Since normal forms are closed under renaming, also $r\sigma \in \mathsf{NF}(\mathcal{R})$ and thus $r\sigma = t$. It follows that $s \to t$ is a variant of $\ell \to r$. ◀

In our formalization, the above proof is the first spot where we actually need that $\mathcal{R}$ satisfies the variable condition (more precisely, right-hand sides of rules do not introduce fresh variables). The next result is the main result of this section.

▶ **Theorem 39.** *Normalization equivalent reduced TRSs are unique up to literal similarity.*

Proof. Let $\mathcal{R}$ and $\mathcal{S}$ be normalization equivalent reduced TRSs. Suppose $\ell \to r \in \mathcal{R}$. Because $\mathcal{R}$ is right-reduced, $r \in \mathsf{NF}(\mathcal{R})$ and thus $\ell \neq r$. Hence $\ell \to_{\mathcal{S}}^{+} r$ by normalization equivalence. Because $\mathcal{R}$ is left-reduced, ℓ is a minimal (with respect to $\rhd$) $\mathcal{R}$-reducible term. Another application of normalization equivalence yields that ℓ is minimal $\mathcal{S}$-reducible. Hence $\ell \to r$ is a variant of a rule in $\mathcal{S}$ by Lemma 38. ◀

▶ **Example 40.** Consider the rewrite system $\mathcal{R}$ of combinatory logic with equality test, studied by Klop [11]:

$$\mathsf{S}xyz \to xz(yz) \qquad \mathsf{K}xy \to x \qquad \mathsf{I}x \to x \qquad \mathsf{D}xx \to \mathsf{E}$$

The rewrite system $\mathcal{R}$ is reduced, but neither terminating nor confluent. One might ask whether there is another reduced rewrite system that computes the same normal forms for every starting term. Theorem 39 shows that $\mathcal{R}$ is unique up to variable renaming.

We show that the corresponding result of Métivier [14, Theorem 8] is an easy consequence of Theorem 39. Here a TRS $\mathcal{R}$ is said to be compatible with a reduction order $>$ if $\ell > r$ for every rewrite rule $\ell \to r$ of $\mathcal{R}$.

▶ **Theorem 41.** *Let $\mathcal{R}$ and $\mathcal{S}$ be equivalent canonical TRSs. If $\mathcal{R}$ and $\mathcal{S}$ are compatible with the same reduction order then $\mathcal{R} \doteq \mathcal{S}$.*

Proof. Suppose $\mathcal{R}$ and $\mathcal{S}$ are compatible with the reduction order $>$. We show that $\to_{\mathcal{R}}^{!} \subseteq \to_{\mathcal{S}}^{!}$. Let $s \to_{\mathcal{R}}^{!} t$. We show that $t \in \mathsf{NF}(\mathcal{S})$. Let u be the unique $\mathcal{S}$-normal form of t. We have $t \to_{\mathcal{S}}^{!} u$ and thus $t \leftrightarrow_{\mathcal{R}}^{*} u$ because $\mathcal{R}$ and $\mathcal{S}$ are equivalent. Since $t \in \mathsf{NF}(\mathcal{R})$, we have $u \to_{\mathcal{R}}^{!} t$. If $t \neq u$ then both $t > u$ (as $t \to_{\mathcal{S}}^{!} u$) and $u > t$ (as $u \to_{\mathcal{R}}^{!} t$), which is impossible. Hence $t = u$ and thus $t \in \mathsf{NF}(\mathcal{S})$. Together with $s \leftrightarrow_{\mathcal{S}}^{*} t$, which follows from the equivalence of $\mathcal{R}$ and $\mathcal{S}$, we conclude that $s \to_{\mathcal{S}}^{!} t$. We obtain $\to_{\mathcal{S}}^{!} \subseteq \to_{\mathcal{R}}^{!}$ by symmetry. Hence $\mathcal{R}$ and $\mathcal{S}$ are normalization equivalent and the result follows from Theorem 39. ◀

The final result in this section is in the spirit of Theorem 37 but for ordered completion, showing that a ground-complete system can be interreduced to some extent. Let $>$ again be a ground-total reduction order.

▶ **Definition 42.** Given a ground-complete system $\mathcal{S} = \mathcal{R} \cup \mathcal{E}^{>}$, we define

$$\mathcal{R}' = \{\ell \to r \mid \ell \to r \in \dot{\mathcal{Q}} \text{ and } \ell \in \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})\}$$
$$\mathcal{E}' = \{s{\downarrow_{\mathcal{R}'}} \approx t{\downarrow_{\mathcal{R}'}} \mid s \approx t \in \mathcal{E}\} \setminus =$$

where $\mathcal{Q} = \mathcal{R} \cup (\mathcal{E}^{\pm} \cap >)$.

Here we write $t \xrightarrow{\rhd}_{\mathcal{S}} u$ if there are a rule $\ell \to r \in \mathcal{S}$, a context C, and a substitution σ such that $t = C[\ell\sigma]$, $u = C[r\sigma]$, and $t \rhd \ell$. For example, if $\mathcal{E}$ is empty and $\mathcal{R}$ consists of the single rule $\mathsf{f}(x,y) \to \mathsf{g}(x)$ we have $\mathsf{f}(y,z) \in \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$, but $\mathsf{f}(\mathsf{g}(x),y) \notin \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$ and $\mathsf{f}(x,x) \notin \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$.

▶ **Theorem 43.** *If $\mathcal{S} = \mathcal{R} \cup \mathcal{E}^{>}$ is ground-complete then $\mathcal{S}' = \mathcal{R}' \cup \mathcal{E}'^{>}$ is ground-complete and normalization and conversion equivalent on ground terms.*

Proof. We first show $\mathsf{NF}(\mathcal{S}') \subseteq \mathsf{NF}(\mathcal{S})$. For a rule $\ell \to r \in \mathcal{S}$, let $b_{\ell \to r}$ be $\bot$ if $\ell \to r \in \mathcal{Q}$ and $\top$ otherwise. We prove $\ell \notin \mathsf{NF}(\mathcal{S}')$ for every rule $\ell \to r \in \mathcal{S}$, by induction on $(\ell, b_{\ell \to r})$ with respect to the lexicographic combination of $\rhd$ and the order where $\top > \bot$.

- If $\ell \to r \in \mathcal{Q}$ two cases can be distinguished. If $\ell \notin \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$ then $\ell \rhd \ell'$ for some rule $\ell' \to r' \in \mathcal{S}$ and thus $\ell' \notin \mathsf{NF}(\mathcal{S}')$ by the induction hypothesis. Hence also $\ell \notin \mathsf{NF}(\mathcal{S}')$. If $\ell \in \mathsf{NF}(\xrightarrow{\rhd}_{\mathcal{S}})$ then, by construction of $\mathcal{R}'$, there is some rule $\ell \to r' \in \mathcal{R}'$ (modulo renaming), so $\ell \notin \mathsf{NF}(\mathcal{S}')$.
- If $\ell \to r \notin \mathcal{Q}$ then $\ell = u\sigma$ and $r = v\sigma$ for some equation $u \approx v \in \mathcal{E}^{\pm}$ and substitution σ such that $\ell > r$. We distinguish two cases. First, if $u \in \mathsf{NF}(\mathcal{R}')$ then $u = u{\downarrow_{\mathcal{R}'}}$. We have $\ell > r \geqslant v{\downarrow_{\mathcal{R}'}}\sigma$ because $\mathcal{R}' \subseteq >$ and hence $u \neq v{\downarrow_{\mathcal{R}'}}$. It follows that $u \approx v{\downarrow_{\mathcal{R}'}} \in \mathcal{E}'^{\pm}$ and thus $\ell \to v{\downarrow_{\mathcal{R}'}}\sigma \in \mathcal{E}'^{\succ}$. Hence $\ell \notin \mathsf{NF}(\mathcal{S}')$. Second, if $u \notin \mathsf{NF}(\mathcal{R}')$ then $u \notin \mathsf{NF}(\dot{\mathcal{Q}})$ since $\mathcal{R}' \subseteq \dot{\mathcal{Q}}$. So there exists a rule $\ell' \to r' \in \mathcal{Q}$ such that $u \unrhd \ell'$. Clearly $\ell \unrhd \ell'$. Since $\ell \to r \notin \mathcal{Q}$, the induction hypothesis yields $\ell' \notin \mathsf{NF}(\mathcal{S}')$. Hence also $\ell \notin \mathsf{NF}(\mathcal{S}')$.

We next establish the inclusion $\to_{\mathcal{S}'} \subseteq \leftrightarrow_{\mathcal{S}}^{*}$ on ground terms. We have $\mathcal{R}' \cup \mathcal{E}' \subseteq \leftrightarrow_{\mathcal{R} \cup \mathcal{E}}^{*}$ by construction. For ground terms s and t, a step $s \to_{\mathcal{S}'} t$ implies $s \leftrightarrow_{\mathcal{R}' \cup \mathcal{E}'} t$ and hence existence of a conversion $s \leftrightarrow_{\mathcal{R} \cup \mathcal{E}}^{*} t$. We can also obtain such a conversion where all intermediate terms are ground by replacing every variable with some ground term. Since the reduction order $>$ is ground-total, $\to_{\mathcal{R} \cup \mathcal{E}} \subseteq \leftrightarrow_{\mathcal{S}}^{=}$ holds on ground terms. Hence there is a conversion $s \leftrightarrow_{\mathcal{S}}^{*} t$.

Moreover, the system $\mathcal{S}'$ is clearly terminating as it is included in $>$. Thus the result follows from Lemma 31(2), viewing $\mathcal{S}$ and $\mathcal{S}'$ as ARSs on ground terms. ◀

We illustrate the transformation of Definition 42 on a concrete example.

▶ **Example 44.** Consider the following system with $\mathcal{R}$ consisting of one rule and $\mathcal{E}$ consisting of three equations:

$$\mathsf{s}(\mathsf{s}(x)) + \mathsf{s}(x) \to \mathsf{s}(x) + \mathsf{s}(\mathsf{s}(x)) \qquad x + \mathsf{s}(y) \approx \mathsf{s}(x+y) \qquad x + y \approx y + x$$
$$\mathsf{s}(x) + y \approx \mathsf{s}(x+y)$$

It is ground-complete for the lexicographic path order [9] with $+ > \mathsf{s}$ as precedence. We have $\mathcal{Q} = \mathcal{R} \cup \{x + \mathsf{s}(y) \to \mathsf{s}(x+y),\ \mathsf{s}(x) + y \to \mathsf{s}(x+y)\}$. Since the term $\mathsf{s}(\mathsf{s}(x)) + \mathsf{s}(x)$ is reducible by the rule $\mathsf{s}(x) + x \to x + \mathsf{s}(x) \in \mathcal{S}$ and $\mathsf{s}(\mathsf{s}(x)) + \mathsf{s}(x) \rhd \mathsf{s}(x) + x$, the rule of $\mathcal{R}$ does not remain in $\mathcal{R}'$. Hence, $\mathcal{R}' = \{x + \mathsf{s}(y) \to \mathsf{s}(x+y),\ \mathsf{s}(x) + y \to \mathsf{s}(x+y)\}$ and $\mathcal{E}' = \{x + y \approx y + x\}$.

One may wonder whether $\mathcal{R}'$ can simply be defined as $\ddot{\mathcal{Q}}$ instead of imposing a strict encompassment condition. The following example shows that this destroys reducibility.

▶ **Example 45.** Consider the following system where $\mathcal{R}$ consists of two rules and $\mathcal{E}$ consists of one equation:

$$\mathsf{f}(x,y) \to \mathsf{g}(x) \qquad \mathsf{f}(x,y) \to \mathsf{g}(y) \qquad \mathsf{g}(x) \approx \mathsf{g}(y)$$

Then $\mathcal{R} \cup \mathcal{E}^{>}$ is ground-complete if $>$ is the lexicographic path order with $\mathsf{f} > \mathsf{g}$ as precedence. We have $\mathcal{R}' = \dot{\mathcal{Q}} = \mathcal{Q} = \mathcal{R}$ and $\mathcal{E}' = \mathcal{E}$ but $\ddot{\mathcal{Q}} = \varnothing$.

Note that we obtain an equivalent ground-complete system if we add, for instance, an equation $\mathsf{g}(\mathsf{g}(x)) \approx \mathsf{g}(y)$. This shows that even systems which are simplified with respect to the procedure suggested by Theorem 43 are not unique.

6 Conclusion

We gave new and concise correctness proofs of Knuth-Bendix and ordered completion. These results specifically apply to infinite runs, a case in which the reasoning becomes more tedious as the encompassment condition for the collapse rule is essential. We also contributed new results about canonicity, related to normalization and conversion equivalence. In particular we generalized the distinguished theorem by Métivier on uniqueness of canonical rewrite systems. All our results are formalized in IsaFoR.

As future work, we want to extend our proofs and formalization to cover completeness results for both Knuth-Bendix and ordered completion [2, 5]. Furthermore, we will apply our techniques to AC completion and the decidable case of ground completion.

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