

# General Ramified Recurrence is Sound for Polynomial Time

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Leivant’s ramified recurrence is one of the earliest examples of an implicit characterization of the polytime functions as a subalgebra of the primitive recursive functions. Leivant’s result, however, is originally stated and proved only for word algebras, i.e. free algebras whose constructors take at most one argument. This paper presents an extension of these results to ramified functions on any free algebras, provided the underlying terms are represented as graphs rather than trees, so that sharing of identical subterms can be exploited.

## 1 Introduction

The characterization of complexity classes by language restrictions (i.e., by *implicit* means) instead of explicit resource bounds is a major accomplishment of the area at the intersection of logic and computer science. Bellantoni, Cook [2], and Leivant [7], building on Cobham pioneering research [4], gave two (equivalent) restrictions on the definition of the primitive recursive functions, obtaining in this way exactly the functions computable in polynomial time. We will focus in this paper on Leivant’s seminal work.

There are (at least) two main ingredients in these implicit characterizations of polytime. First, when data are represented by strings, as usual in complexity, each recursive call must consume at least one symbol of the input. In this way the length of the recursive call sequence is linear in the size of the input. When *numeric* functions are considered, and numbers are thus represented in basis  $b \geq 2$ , this amounts to recursion on notation [4], where each call divides the input by  $b$ . The second main ingredient is a restriction on the recursion schema, in order to avoid nested recursions. This is the job of *tiers* [7, 11] (in the Bellantoni-Cook’s approach this would be achieved with a distinction between safe and normal arguments in a function). In Leivant’s system variables and functions are equipped with a tier, and composition must preserve tiers; crucially, in a legal recursion the tier of the recurrence parameter must be higher than the tiers of the recursive calls. It is noteworthy that *linearity* does *not* play a major role — a function can duplicate its inputs as many times as it likes<sup>1</sup>. In Leivant’s original paper [7], ramified recurrence over any free algebra is claimed to be computable in polynomial time on the height of the input, hence on its size. However, some proofs (in particular, the proof of Lemma 3.8) only go through when the involved algebras have constructors of at most unary arities. Indeed, the extended and

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<sup>1</sup>The naive restriction to primitive recursion on notation plus linearity (and no tiers) is too generous. Exponential functions would be easily definable. For example the function  $f$ , defined by linear recursion on notation as  $f(\mathbf{0}) = f(\mathbf{1}) = \mathbf{1}$  and  $f(w \cdot \mathbf{0}) = f(w \cdot \mathbf{1}) = g(f(w))$  (where  $g$  is any recursively defined function such that  $|g(x)| \geq 2|x|$ ) has superpolynomial growth caused by the application of  $g$  on the result of the recursive call.

revised [8] only refers to word-algebras. Marion [9] extends the polynomiality result to constructors with signature  $s_1 \times \cdots \times s_n \rightarrow s$  under the constraint that  $s$  appears at most once among the  $s_i$ , and it is held in the ICC community that the result holds also for *any* free algebra (see for instance Marion’s observation, reported as personal communication in Hofmann’s [5], page 38). This gap in the literature gives rise to subtle misunderstandings (which could amount to believing the contrary: in the unpublished [3] we read that Leivant “has given equational characterizations of complexity classes, but for constructors of arity greater than one, his classes exceed poly-time”). In this paper we thus fill the little gap, and prove anew that Leivant’s characterization of polytime holds for *general tiered recursion*, as part of a broader project aimed to give precise complexity content to rule based programming.

The point is that Leivant’s proofs does not go through when moving from unary to *arbitrary arity* constructors, since now the absence of linearity strikes back. Indeed, the following function on binary words (which is easily decorated with tiers, but which violates the constraint in the already cited [9])

$$f(\mathbf{0}) = f(\mathbf{1}) = \mathbf{nil} \quad f(w \cdot \mathbf{0}) = f(w \cdot \mathbf{1}) = \mathbf{tree}(f(w), f(w))$$

outputs the full binary tree, which has exponential size in the length of the input.

We believe that this is a *representation problem*, and not an intrinsic limitation of tiering. The apparent break of polytime appears because the explicit representation of data with strings forces the explicit duplication of (part of) the input. But this duplication is inessential to the computation itself — in fact, it could be avoided by just storing the intermediate result and re-using it when needed to produce the output. We thus prove that tiered recursion on *any free algebra* may be computed in polytime, once data is represented with directed acyclic graphs, and computation is performed via graph rewriting. In term graph rewriting the sharing of common subterms is explicitly represented, and a compact representation of data could be given. The result of a computation will be, in general, a DAG, where identical subterms that would be replicated several times in the string representation, are instead present only once. The time needed to print the string representation of the output is not (and *should not be*) counted in the computing time of the function.

The choice of a suitable representation for data is of course a crucial ingredient of any complexity theory account. Explicit string representation of arbitrary arity terms is simply too generous, akin to representation of numbers in base 1. Indeed, the discriminant for an acceptable encoding of data (e.g., [10]) is the fact that all acceptable encodings yield representations which have polynomially related lengths. And this rules out the explicit string representation, in view of the example above. On the other side, we think that graph representation of elements of a free algebra makes a good choice.

The present work solves an open question about a primal approach of ICC, joining the pure functional characterization of Leivant’s approach with the powerful features of graph rewriting, thus making the subrecursive restriction sound for general free algebras.

The rest of this paper is structured as follow:

- In Section 2 we define the class of the functions generated by general tiered recursion.
- In Section 3 graph rewriting is introduced and explained. Basic definition and fundamental properties are given.
- Section 4 is devoted to the main technical results of the paper: tiered recursion is realized by term graph rewriting and reduction can be performed in polynomial time.
- In Section 5 we state some conclusions and some final considerations about our work.

## 2 General Tiered Recursion

A *signature*  $\Sigma$  is a pair  $(S, \sigma)$  where  $S$  is a set of symbols and  $\sigma : S \rightarrow \mathbb{N}$  assigns to every symbol  $\mathbf{f} \in S$  an arity  $\sigma(\mathbf{f})$ . Given two signatures  $\Sigma = (S, \sigma)$  and  $\Theta = (T, \tau)$ , we write  $\Theta \sqsubseteq \Sigma$  iff  $T \subseteq S$  and  $\sigma(\mathbf{f}) = \tau(\mathbf{f})$  for every  $\mathbf{f} \in T$ . The set of *terms* over a signature  $\Sigma$  can be easily defined by induction as the smallest set  $\mathcal{T}_\Sigma$  such that:

- If  $\mathbf{f} \in S$  and  $\sigma(\mathbf{f}) = 0$ , then  $\mathbf{f}$  itself, seen as an expression, is in  $\mathcal{T}_\Sigma$ ;
- If  $\mathbf{f} \in S$ ,  $\sigma(\mathbf{f}) = n \geq 1$  and  $t_1, \dots, t_n \in \mathcal{T}_\Sigma$ , then the expression  $\mathbf{f}(t_1, \dots, t_n)$  is in  $\mathcal{T}_\Sigma$ .

Given a signature  $\Sigma = (\{\mathbf{f}_1, \dots, \mathbf{f}_t\}, \sigma)$  and natural numbers  $n, i_1, \dots, i_n, i$ , the classes of functions defined by *tiered recursion (on  $\Sigma$ ) with tiers  $(i_1, \dots, i_n) \rightarrow i$*  are the smallest collections of functions from  $\mathcal{T}_\Sigma^n$  to  $\mathcal{T}_\Sigma$  satisfying the following conditions:

- For every  $i \in \mathbb{N}$ , the identity function  $\text{id} : \mathcal{T}_\Sigma \rightarrow \mathcal{T}_\Sigma$  is tiered recursive with tiers  $i \rightarrow i$ .
- For every  $i \in \mathbb{N}$  and for every  $\mathbf{f}_j$ , the constructor function  $\mathbf{s}_{\mathbf{f}_j} : \mathcal{T}_\Sigma^{\sigma(\mathbf{f}_j)} \rightarrow \mathcal{T}_\Sigma$  is tiered recursive with tiers  $(i, \dots, i) \rightarrow i$ .
- For every  $i, i_1, \dots, i_n \in \mathbb{N}$  and for every  $1 \leq m \leq n$ , the projection function  $\mathbf{p}_{n,m} : \mathcal{T}_\Sigma^n \rightarrow \mathcal{T}_\Sigma$  is tiered recursive with tiers  $(i_1, \dots, i_n) \rightarrow i$  whenever  $i_m = i$ .
- For every  $i, i_1, \dots, i_n, j_1, \dots, j_m \in \mathbb{N}$ , if  $\mathbf{f} : \mathcal{T}_\Sigma^n \rightarrow \mathcal{T}_\Sigma$  is tiered recursive with tiers  $(i_1, \dots, i_n) \rightarrow i$  and for every  $1 \leq k \leq m$  the function  $\mathbf{g}_k : \mathcal{T}_\Sigma^{j_k} \rightarrow \mathcal{T}_\Sigma$  is tiered recursive with tiers  $(j_1, \dots, j_m) \rightarrow i_k$ , then the composition  $\text{comp}(\mathbf{f}, \mathbf{g}_1, \dots, \mathbf{g}_m) : \mathcal{T}_\Sigma^m \rightarrow \mathcal{T}_\Sigma$  (defined in the obvious way) is tiered recursive with tiers  $(j_1, \dots, j_m) \rightarrow i$ .
- For every  $i, j, i_1, \dots, i_n \in \mathbb{N}$  with  $i < j$ , if for every  $\mathbf{f}_k$  there is a function  $\mathbf{g}_k : \mathcal{T}_\Sigma^{2\sigma(\mathbf{f}_k)+n} \rightarrow \mathcal{T}_\Sigma$  which is tiered recursive with tiers

$$\begin{array}{c} \sigma(\mathbf{f}_k) \text{ times } \sigma(\mathbf{f}_k) \text{ times} \\ ( \overbrace{j, \dots, j}^{\sigma(\mathbf{f}_k) \text{ times}}, \overbrace{i, \dots, i}^{\sigma(\mathbf{f}_k) \text{ times}}, i_1, \dots, i_n ) \rightarrow i \end{array}$$

then the function  $\mathbf{f} = \text{rec}(\mathbf{g}_1, \dots, \mathbf{g}_t) : \mathcal{T}_\Sigma^{n+1} \rightarrow \mathcal{T}_\Sigma$  defined by primitive recursion as

$$\begin{array}{l} \mathbf{f}(\mathbf{f}_i(t_1, \dots, t_{\sigma(\mathbf{f}_i)}), u_1, \dots, u_n) = \\ \mathbf{g}_i(t_1, \dots, t_{\sigma(\mathbf{f}_i)}, \mathbf{f}(t_1, u_1, \dots, u_n), \dots, \mathbf{f}(t_{\sigma(\mathbf{f}_i)}, u_1, \dots, u_n), u_1, \dots, u_n) \end{array}$$

is tiered recursive with tiers  $(j, i_1, \dots, i_n) \rightarrow i$

- For every  $i, j, i_1, \dots, i_n \in \mathbb{N}$ , if for every  $\mathbf{f}_i$  there is a function  $\mathbf{g}_i : \mathcal{T}_\Sigma^{\sigma(\mathbf{f}_i)+n} \rightarrow \mathcal{T}_\Sigma$  which is tiered recursive with tiers

$$\begin{array}{c} \sigma(\mathbf{f}_i) \text{ times} \\ ( \overbrace{j, \dots, j}^{\sigma(\mathbf{f}_i) \text{ times}}, i_1, \dots, i_n ) \rightarrow i \end{array}$$

then the function  $\mathbf{f} = \text{cond}(\mathbf{g}_1, \dots, \mathbf{g}_t) : \mathcal{T}_\Sigma^{n+1} \rightarrow \mathcal{T}_\Sigma$  defined as

$$\mathbf{f}(\mathbf{f}_i(t_1, \dots, t_{\sigma(\mathbf{f}_i)}), u_1, \dots, u_n) = \mathbf{g}_i(t_1, \dots, t_{\sigma(\mathbf{f}_i)}, u_1, \dots, u_n)$$

is tiered recursive with tiers  $(j, i_1, \dots, i_n) \rightarrow i$ .

In the following, metavariables like  $\mathbf{l}$  and  $\mathbf{J}$  will be used for expressions like  $(i_1, \dots, i_n) \rightarrow i$ .

Roughly speaking, the rôle of tiers is to single out “a copy” of the signature by a level: this level permits to control the nesting of the recursion. Note that function composition preserves tiers, i.e. we can substitute terms only for variables of equal tier. Moreover, recursion is allowed only on a variable of tier higher than the tier of the function (in the definition,  $i < j$  is required). This construction comes from a *predicative* notion of recurrence. Examples of terms and functions follow.

**Example 1** • Let  $\Sigma_1$  be the signature  $(\{\mathbf{f}_1, \mathbf{f}_2\}, \sigma)$ . If  $\sigma(\mathbf{f}_1) = 0$  and  $\sigma(\mathbf{f}_2) = 1$ , then  $\mathcal{T}_{\Sigma_1}$  is in bijective correspondence with  $\mathbb{N}$ .

- Let  $\Sigma_2$  be the signature  $(\{\mathbf{f}_1, \mathbf{f}_2\}, \sigma)$ . If we take  $\sigma(\mathbf{f}_1) = 0$  and  $\sigma(\mathbf{f}_2) = 2$  then  $\mathcal{T}_{\Sigma_2}$  can be thought as the set of (unlabeled) binary trees.
- Let  $\Sigma_3$  be the signature  $(\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}, \sigma)$  where  $\sigma(\mathbf{f}_1) = \sigma(\mathbf{f}_3) = 0$  and  $\sigma(\mathbf{f}_2) = \sigma(\mathbf{f}_4) = 2$ . Then  $\mathcal{T}_{\Sigma_3}$  is the set of binary trees with binary labels.
- The function  $\text{sum} : \mathcal{T}_{\Sigma_1}^2 \rightarrow \mathcal{T}_{\Sigma_1}$  computing addition on natural numbers can be defined as

$$\begin{aligned} \text{sum}(\mathbf{f}_1, x) &= x; \\ \text{sum}(\mathbf{f}_2(y), x) &= \mathbf{f}_2(\text{sum}(y, x)). \end{aligned}$$

It can be easily proved to be tiered recursive with tiers  $(j, i) \rightarrow i$  whenever  $j > i$ .

- We can define the function  $\text{mirror}$  on  $\mathcal{T}_{\Sigma_2}$  which mirrors a tree (i.e. each left subtree becomes right subtree and viceversa):

$$\begin{aligned} \text{mirror}(\mathbf{f}_1) &= \mathbf{f}_1; \\ \text{mirror}(\mathbf{f}_2(x, y)) &= \mathbf{f}_2(\text{mirror}(y), \text{mirror}(x)); \end{aligned}$$

$\text{mirror}$  can be proved to be tiered recursive with tiers  $(j) \rightarrow i$  whenever  $j > i$ .

### 3 Graph Rewriting

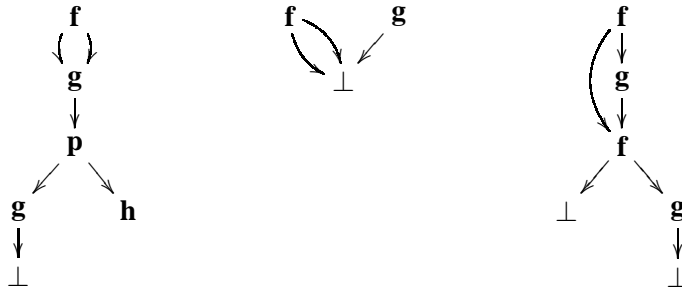
In this section, we introduce term graph rewriting, following [1] but adapting the framework to our specific needs.

**Definition 1 (Labelled Graph)** Given a signature  $\Sigma = (S, \sigma)$ , a labelled graph over  $\Sigma$  consists of a directed acyclic graph together with an ordering on the outgoing edges of each vertex and a (partial) labelling of vertices with symbols from  $\Sigma$  such that the out-degree of each node matches the arity of the corresponding symbols (and is 0 if the labelling is undefined). Formally, a labelled graph is a triple  $G = (V, \alpha, \delta)$  where:

- $V$  is a set of vertices.
- $\alpha : V \rightarrow V^*$  is a (total) ordering function.
- $\delta : V \rightarrow S$  is a (partial) labelling function such that the length of  $\alpha(v)$  is the arity of  $\delta(v)$  if  $\delta(v)$  is defined and is 0 otherwise.

A labelled graph  $(V, \alpha, \delta)$  is closed iff  $\delta$  is a total function.

Consider the signature  $\Sigma = (\{\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{p}\}, \sigma)$ , where arities assigned to  $\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{p}$  by  $\sigma$  are 2, 1, 0, 2. Examples of labelled graphs over the signature  $\Sigma$  are the following ones:

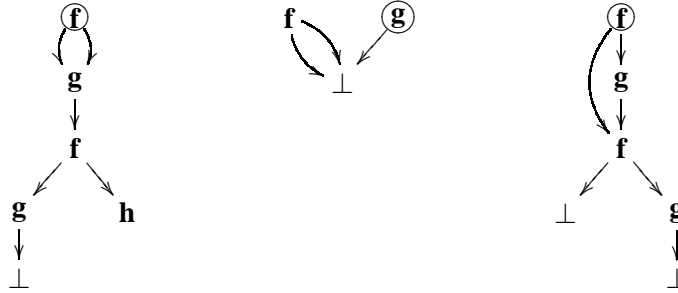


The symbol  $\perp$  denotes vertices where the underlying labelling function is undefined (and, as a consequence, no edge departs from such vertices). Their rôle is similar to the one of variables in terms.

If one of the vertices of a labelled graph is selected as the *root*, we obtain a term graph:

**Definition 2 (Term Graphs)** A term graph, is a quadruple  $G = (V, \alpha, \delta, r)$ , where  $(V, \alpha, \delta)$  is a labelled graph and  $r \in V$  is the root of the term graph.

The following are graphic representations of some term graphs. The root is the only vertex drawn inside a circle.



Given a (closed) term graph  $G$  on  $\Sigma$ ,  $\langle G \rangle$  is simply the term in  $\mathcal{T}_\Sigma$  obtained by unfolding  $G$  starting from its root.

The notion of an homomorphism between labelled graphs is not only interesting mathematically, but will be crucial in defining rewriting:

**Definition 3 (Homomorphisms)** An injective homomorphism between two labelled graphs  $G = (V_G, \alpha_G, \delta_G)$  and  $H = (V_H, \alpha_H, \delta_H)$  over the same signature  $\Sigma$  is a function  $\varphi$  from  $V_G$  to  $V_H$  preserving the labelled graph structure. In particular

$$\begin{aligned} \delta_H(\varphi(v)) &= \delta_G(v); \\ \alpha_H(\varphi(v)) &= \varphi^*(\alpha_G(v)); \end{aligned}$$

for any  $v \in \text{dom}(\delta_G)$ , where  $\varphi^*$  is the obvious generalization of  $\varphi$  to sequences of vertices. Moreover,  $\varphi$  is injective on  $\text{dom}(\delta_G)$ , i.e.,  $\varphi(v) = \varphi(w)$  implies  $v = w$  for every  $v, w \in \text{dom}(\delta_G)$ . An injective homomorphism between two term graphs  $G = (V_G, \alpha_G, \delta_G, r_G)$  and  $H = (V_H, \alpha_H, \delta_H, r_H)$  is an injective homomorphism between  $(V_G, \alpha_G, \delta_G)$  and  $(V_H, \alpha_H, \delta_H)$  such that  $\varphi(r_G) = r_H$ . Two labelled graphs  $G$  and  $H$  are isomorphic iff there is a bijective homomorphism from  $G$  to  $H$ ; in this case, we write  $G \cong H$ . Similarly for term graphs.

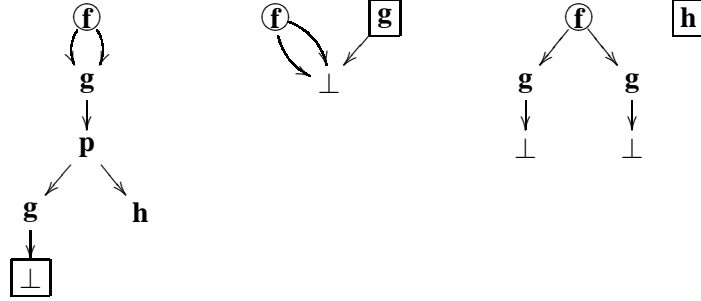
In the sequel, we will always use *homomorphism*, to mean *injective homomorphism*. Injectivity of  $\varphi$  on labelled vertices is not part of the usual definition of an homomorphism between labelled graphs (see [1]). We insist on injectivity because we want a rewriting rule “to match without sharing” (see the notion of redex, Definition 6), which will be crucial for the rôle of unfolding graph rewriting rules (Section 3.1) in the implementation of tiered recursion. Injectivity makes our notion of redex less general than in the usual setting of graph rewriting. This, however, suffices for our purposes. In the following, we will consider term graphs modulo isomorphism, i.e.,  $G = H$  iff  $G \cong H$ . Observe that two isomorphic term graphs have the same graphical representation.

**Definition 4 (Graph rewriting Rules)** A graph rewriting rule over a signature  $\Sigma$  is a triple  $\rho = (G, r, s)$  such that:

- $G$  is a labelled graph;

- $r, s$  are vertices of  $G$ , called the left root and the right root of  $\rho$ , respectively.

The following are three examples of graph rewriting rules, assuming  $\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{p}$  to be function symbols in the underlying signature  $\Sigma$ :



Graphically, the left root is the (unique) node inside a circle, while the right root is the (unique) node inside a square.

**Definition 5 (Subgraphs)** Given a labelled graph  $G = (V_G, \alpha_G, \delta_G)$  and any vertex  $v \in V_G$ , the subgraph of  $G$  rooted at  $v$ , denoted  $G \downarrow v$ , is the term graph  $(V_{G \downarrow v}, \alpha_{G \downarrow v}, \delta_{G \downarrow v}, r_{G \downarrow v})$  where

- $V_{G \downarrow v}$  is the subset of  $V_G$  whose elements are vertices which are reachable from  $v$  in  $G$ .
- $\alpha_{G \downarrow v}$  and  $\delta_{G \downarrow v}$  are the appropriate restrictions of  $\alpha_G$  and  $\delta_G$  to  $V_{G \downarrow v}$ .
- $r_{G \downarrow v}$  is  $v$ .

A term graph  $G = (V, \alpha, \delta, r)$  is said to be a *proper term graph* if  $(V, \alpha, \delta) \downarrow r = G$ .

We are finally able to give the notion of a redex, that represents the occurrence of the lhs of a rewriting rule in a graph:

**Definition 6 (Redexes)** Given a labelled graph  $G$ , a redex for  $G$  is a pair  $(\rho, \varphi)$ , where  $\rho$  is a rewriting rule  $(H, r, s)$  and  $\varphi$  is an homomorphism between  $H \downarrow r$  and  $G$ .

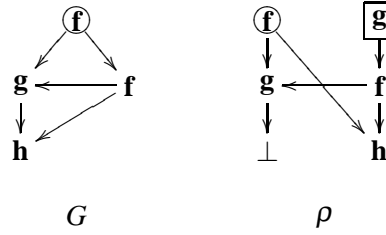
If  $((H, r, s), \varphi)$  is a redex in  $G$ , we say, with a slight abuse of notation, that  $\varphi(r)$  is itself a redex. In most cases, this does not introduce any ambiguity. Given a term graph  $G$  and a redex  $((H, r, s), \varphi)$ , the result of firing the redex is another term graph obtained by successively applying the following three steps to  $G$ :

1. The *build phase*: create an isomorphic copy of the portion of  $H \downarrow s$  not contained in  $H \downarrow r$  (which may contain arcs originating in  $H \downarrow s$  and entering  $H \downarrow r$ ), and add it to  $G$ , obtaining  $J$ . The underlying ordering and labelling functions are defined in the natural way.
2. The *redirection phase*: all edges in  $J$  pointing to  $\varphi(r)$  are replaced by edges pointing to the copy of  $s$ . If  $\varphi(r)$  is the root of  $G$ , then the root of the newly created graph will be the newly created copy of  $s$ . The graph  $K$  is obtained.
3. The *garbage collection phase*: all vertices which are not accessible from the root of  $K$  are removed. The graph  $I$  is obtained.

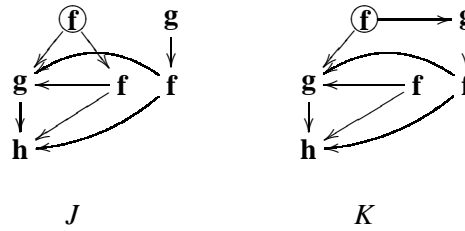
We will write  $G \xrightarrow{(H, r, s)} I$  (or simply  $G \rightarrow I$ , if this does not cause ambiguity) in this case.

**Example 2** As an example, assuming again  $\mathbf{f}, \mathbf{g}, \mathbf{h}$  to be function symbols, consider the term graph  $G$

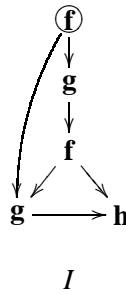
and the rewriting rule  $\rho = (H, r, s)$ :



There is an homomorphism  $\phi$  from  $H \downarrow r$  to  $G$ . In particular,  $\phi$  maps  $r$  to the rightmost vertex in  $G$ . Applying the build phase and the redirection phase we get  $J$  and  $K$  as follows:



Finally, applying the garbage collection phase, we get the result of firing the redex  $(\rho, \phi)$ :



Given a proper graph  $G$  and a redex  $(\rho, \phi)$  for  $G$ , it is easy to prove that the result of firing the redex  $(\rho, \phi)$  is a proper term graph: this is an immediate consequence of how the garbage collection phase is defined.

The notion of innermost and outermost graph rewriting can be defined in a natural way. If  $G \xrightarrow{(H,r,s)} I$  by way of innermost graph rewriting, we'll write  $G \xrightarrow{(H,r,s)}_i I$  (or simply  $G \rightarrow_i I$ ). Similarly, for outermost reduction:  $G \xrightarrow{(H,r,s)}_o I$  or  $G \rightarrow_o I$ .

Given two graph rewriting rules  $\rho = (H, r, s)$  and  $\sigma = (J, p, q)$ ,  $\rho$  and  $\sigma$  are said to be *overlapping* iff there is a term graph  $G$  and two homomorphisms  $\phi$  and  $\psi$  such that  $(\rho, \phi)$  and  $(\sigma, \psi)$  are both redexes in  $G$  with  $\phi(r) = \psi(p)$ .

**Definition 7** A graph rewriting system (GRS) over a signature  $\Sigma$  is a set  $\mathcal{G}$  of non-overlapping graph rewriting rules on  $\Sigma$ .

If  $G \xrightarrow{\rho}_o H$  and  $\rho \in \mathcal{G}$ , we write  $G \xrightarrow{\mathcal{G}}_o H$  or  $G \rightarrow_o H$ , if this doesn't cause any ambiguity. Similarly when  $G \xrightarrow{\rho}_i H$ .

The notion of a term graph can be generalized into the notion of a *multi-rooted* term graph, i.e., a graph with  $n \geq 1$  (not necessarily distinct) roots. Formally, it is a tuple  $G = (V, \alpha, \delta, r_1, \dots, r_n)$ , where  $(V, \alpha, \delta)$  is a labelled graph and  $r_1, \dots, r_n \in V$ . Likewise, we can easily define the subgraph of  $G$  rooted at  $v_1, \dots, v_n$ , denoted  $G \downarrow v_1, \dots, v_n$ , as a multi-rooted term graph. Similarly for homomorphisms.

### 3.1 Unfolding Graph Rewriting Rules

When computing a recursively defined function  $f$  by graph rewriting, we need to take advantage of sharing. In particular, if the recurrence argument is a graph  $G$ , the number of recursive calls generated by calling  $f$  on the graph  $G$  should be equal to the number of vertices of  $G$ , which can in turn be exponentially smaller than the size of  $\langle G \rangle$ . Unfortunately, this cannot be achieved by a finite set of graph rewriting rules: distinct rules, called *unfolding graph rewriting rules* are needed for each possible argument to  $f$ .

Let  $\Sigma = (S, \sigma)$  and  $\Theta = (T, \tau)$  be signatures such that  $S$  and  $T$  are disjoint but in bijective correspondence. Let  $\varphi : S \rightarrow T$  be a bijection and suppose  $\tau(\varphi(v)) = 2\sigma(v) + n$  whenever  $v$  is in  $S$ , for a fixed  $n \in \mathbb{N}$ . Finally, let  $\mathbf{f}$  be a symbol not in  $S \cup T$ .

Under these hypothesis, an *unfolding graph rewriting rule* for  $\Sigma$ ,  $\Theta$  and  $\mathbf{f}$  is a graph rewriting rule  $\rho = (G, r, s)$  where  $G = (V, \alpha, \delta)$  is a labelled graph on a signature  $\Xi \sqsupseteq \Sigma + \Theta$  assigning arity  $n + 1$  to  $\mathbf{f}$ , and satisfying the following constraints:

- The elements of  $V$  are the (pairwise distinct) vertices

$$v_1, \dots, v_m, w_1, \dots, w_m, x_1, \dots, x_n, y$$

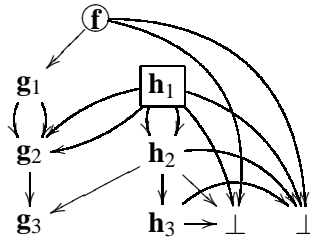
Let  $\chi$  be a function mapping any  $v_i$  to  $w_i$ .

- For every  $1 \leq i \leq m$ ,  $\alpha(v_i)$  is a sequence of vertices from  $\{v_1, \dots, v_m\}$ . Moreover, the set of vertices of  $G \downarrow v_1$  coincides with  $\{v_1, \dots, v_m\}$ .
- For every  $1 \leq i \leq m$ ,  $\delta(v_i) \in S$  and  $\delta(w_i) = \varphi(\delta(v_i))$ ; moreover

$$\alpha(w_i) = \alpha(v_i)\chi^*(\alpha(v_i))x_1 \dots x_n$$

- $\delta(y) = \mathbf{f}$  and  $\alpha(y) = v_1 x_1 \dots x_n$ .
- $\delta(x_i)$  is undefined for every  $i$ .
- $r = y$  and  $s = w_1$ .

**Example 3** Let  $\Sigma = (\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}, \sigma)$ , where arities assigned to  $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$  are 2, 1, 0, respectively. Let  $\Theta = (\{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\}, \sigma)$ , where arities assigned to  $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$  are 6, 4, 2, respectively. If  $\Xi \sqsupseteq \Sigma + \Theta$  is a signature attributing arity 3 to  $\mathbf{f}$ , we are in a position to give examples of unfolding graph rewriting rules for  $\Sigma$ ,  $\Theta$  and  $\mathbf{f}$ . Here is one:



Informally, thus, in an unfolding graph rewriting rule for  $\Sigma$ ,  $\Theta$  and  $\mathbf{f}$ , we may single out four parts. First, the root, labelled with  $\mathbf{f}$ , which is also the left root of the rule; second, a subgraph labelled only on  $\Sigma$  (and this is  $G \downarrow v_1$ ), which represents the recurrence argument to  $\mathbf{f}$ ; third, a subgraph labelled only on  $\Theta$  (it is  $G \downarrow w_1$ ) which is isomorphic to the second part, but for the addition of certain outgoing edges (the root of this part is the right root of the rule); a last part consisting of  $n$  unlabelled vertices, which have only incoming edges, coming from the root and  $G \downarrow w_1$ .



### 3.2 Infinite Graph Rewriting Systems

In [6], two of the authors proved that *finite* graph rewriting systems are polynomially invariant when seen as a computational model. In other words, Turing machines and finite GRSs can simulate each other with a polynomial overhead in both directions, where both computational models are taken with their natural cost models.

In next section, however, we will use *infinite* GRSs (that is, GRSs with an infinite set of rules) to implement recurrence on arbitrary free algebras. We thus need some suitable notion of computability on such infinite systems (which could be even uncomputable). We say that a specific graph rewriting system  $\mathcal{G}$  on the signature  $\Sigma$  is *polytime presentable* if there is a deterministic polytime algorithm  $\mathcal{A}_{\mathcal{G}}$  which, given a term graph  $G$  on  $\Sigma$  returns:

- A term graph  $H$  such that  $G \xrightarrow{\mathcal{G}}_i H$ ;
- The value  $\perp$  if such a graph  $H$  does not exist.

In other words, a (infinite) GRS is polytime presentable iff there is a polytime algorithm which is able to compute any reduct of any given graph, returning an error value if such a reduct does not exist.

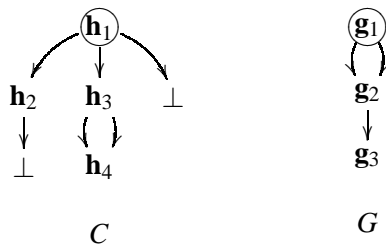
### 3.3 Graph Rewriting in Context

A *context*  $C$  is simply a term graph  $G$ . Given a context  $C$ , let us denote with  $Var(C)$  the set of those vertices of  $G$  which are not labelled.

Given a context  $C$ , another term graph  $G$  and a function  $\xi$  mapping every element of  $V \subseteq Var(C)$  into a vertex of  $G$ , the term graph  $C_{\xi}(G)$  is the one obtained from  $C$  and  $G$  by removing all the vertices in  $V$  and by redirecting to  $\xi(v)$  every edge pointing to  $v \in V$ . More formally, given context  $C = (V, \alpha, \delta, r)$ , term graph  $G = (W, \beta, \varepsilon, s)$  and function  $\xi : X \rightarrow W$  such that  $X \subseteq Var(C)$  and  $s$  is in the range of  $\xi$ ,  $C_{\xi}(G)$  is the term graph  $(Y, \gamma, \zeta, p)$  such that:

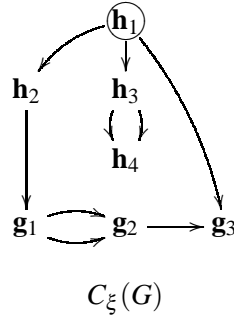
1. The set of vertices  $Y$  is the disjoint union of  $(V - X)$  and  $W$ ;
2. If  $v \in V - X$ , then  $\gamma(v)$  is  $\alpha(v)$  where every occurrence of any  $w \in X$  is replaced by  $\xi(w)$ ;
3. If  $v \in W$ , then  $\gamma(v) = \beta(v)$ ;
4. For every  $v \in V - X$ , it holds that  $\zeta(v) = \delta(v)$ , while for every  $v \in W$ , it holds that  $\zeta(v) = \varepsilon(v)$ ;
5. If  $r \in X$ , then  $p = s$ , otherwise  $p = r$ .

**Example 4** Let  $C$  and  $G$  be the following graphs



If we take the function  $\xi$  such that  $\xi$  maps the unlabelled node pointed by  $h_2$  to  $g_1$  and the unlabelled

node pointed by  $\mathbf{h}_1$  to  $\mathbf{g}_3$ , then the graph  $C_\xi(G)$  is



When we write  $C_\xi(G) \rightarrow_i C_\theta(H)$ , we are tacitly assuming that rewriting have taken place inside  $G$ . Notice that  $G \rightarrow_i H$  does not imply that  $C_\xi(G) \rightarrow_i C_\theta(H)$  for some  $\theta$ . Moreover, by the very definition of graph rewriting:

**Lemma 1** *If  $C$  and  $G$  are proper and  $C_\xi(G) \rightarrow_i C_\theta(H)$ , then for every  $v_1, \dots, v_s$  such that  $G \downarrow \xi(v_1), \dots, \xi(v_s)$  does not contain any redex, it holds that  $G \downarrow \xi(v_1), \dots, \xi(v_s) \cong H \downarrow \theta(v_1), \dots, \theta(v_s)$ .*

In other words, those portions of  $G$  which do not contain any redex are preserved while performing reduction in  $G$ .

Contexts will be useful when proving that certain GRSs correctly computes tiered recursive functions. In particular, they will allow us to prove those statements by induction on the proof that the functions under consideration are tiered recursive.

## 4 Implementing Tiered Recursion by Term Graph Rewriting

Given a signature  $\Sigma = (S, \sigma)$ ,  $\mathbb{N}\Sigma$  stands for the (infinite) signature

$$(\{\mathbf{f}^i \mid \mathbf{f} \in S \text{ and } i \in \mathbb{N}\}, \tau)$$

where  $\tau(\mathbf{f}^i) = \sigma(\mathbf{f})$  for every  $i \in \mathbb{N}$ . Given a term  $t$  in  $\mathcal{T}_\Sigma$  and  $i \in \mathbb{N}$ ,  $t^i$  denotes the term in  $\mathcal{T}_{\mathbb{N}\Sigma}$  obtained by labelling any function symbol in  $t$  with the specific natural number  $i$ ;  $i\Sigma$  is the subsignature of  $\mathbb{N}\Sigma$  of those function symbols labelled with the particular natural number  $i$ . With  $|G|_i$  we denote the number of vertices of  $G$  labelled with functions in  $i\Sigma$ , whenever  $G$  is a term graph on  $\Theta \sqsupseteq \mathbb{N}\Sigma$ .

Suppose  $f : \mathcal{T}_\Sigma^n \rightarrow \mathcal{T}_\Sigma$  and suppose the term graph rewriting system  $\mathcal{G}$  over a signature  $\Theta \sqsupseteq \mathbb{N}\Sigma$  (including a symbol  $f$  of arity  $n$ ) is such that whenever  $f(t_1, \dots, t_n) = u$ , and  $\langle G \rangle = \mathbf{f}(t_1^i, \dots, t_n^i)$ , it holds that  $G \rightarrow_i^* H$  where  $\langle H \rangle = u^i$ . Then we say  $\mathcal{G}$  represents  $f$  with respect to  $(i_1, \dots, i_n) \rightarrow i$ .

Now, let  $\mathcal{G}$  be a term graph rewriting system representing  $f$  with respect to  $(i_1, \dots, i_n) \rightarrow i$  and let  $p : \mathbb{N} \rightarrow \mathbb{N}$  be a polynomial. We say that  $\mathcal{G}$  is bounded by  $p$  iff whenever  $\langle G \rangle = \mathbf{f}(t_1^i, \dots, t_n^i)$  and  $G \rightarrow_i^m H$ , it holds that  $m, |H| \leq p(|G|)$ .

The main result of this paper is the following:

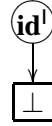
**Theorem 1** *For every signature  $\Sigma$  and for every tiered recursive function  $f : \mathcal{T}_\Sigma^n \rightarrow \mathcal{T}_\Sigma$  with tiers  $l = (i_1, \dots, i_n) \rightarrow i$  there are a term graph rewriting system  $\mathcal{G}_f^l$  on  $\Theta \sqsupseteq \mathbb{N}\Sigma$  and a polynomial  $p : \mathbb{N}^n \rightarrow \mathbb{N}$  such that  $\mathcal{G}_f^l$  represents  $f$  with respect to  $l$ , being bounded by  $p$ . Moreover,  $\mathcal{G}_f^l$  is polytime presentable.*

In other words, every tiered recursive function is represented by a GRS which is potentially infinite, but which is polytime presentable. Moreover, appropriate polynomial bounds hold for the number of

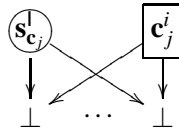
innermost rewriting steps necessary to compute the normal form of term graphs and for the size of any intermediate results produced during computation.

In the rest of this section, we will give a proof of Theorem 1. This will be a constructive proof, i.e. we define  $\mathcal{G}_f^l$  by induction on the structure of  $f$  as a tiered function (i.e. on the structure of the proof that  $f$  is a tiered function). Let  $\Sigma = (\{c_1, \dots, c_t\}, \sigma)$ .  $\mathcal{G}_f^l$  is defined as follows:

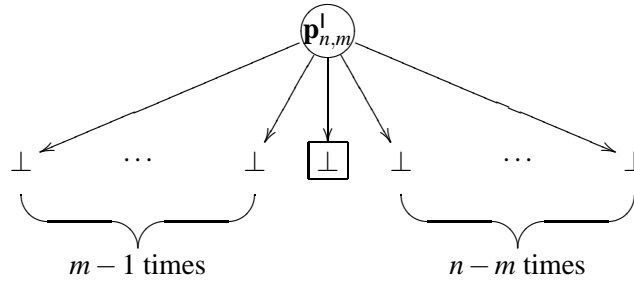
- For every  $l = (i) \rightarrow i$ ,  $\mathcal{G}_{id}^l$  is a GRS whose only rule is



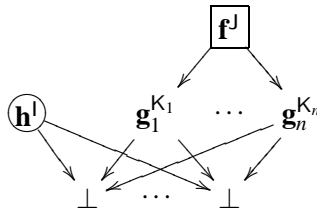
- For every  $l = (i, \dots, i) \rightarrow i$ ,  $\mathcal{G}_{s_{c_j}}^l$  is a GRS whose only rule is



- For every  $l$ ,  $\mathcal{G}_{p_{n,m}}^l$  is a GRS whose only rule is



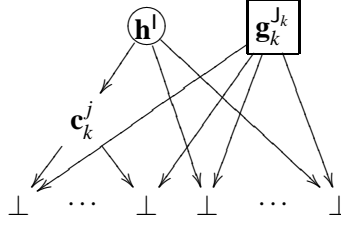
- Let  $h$  be  $\text{comp}(f, g_1, \dots, g_n)$  and suppose  $h$  is tiered recursive with tiers  $l$ . Then  $\mathcal{G}_h^l$  is the GRS  $\mathcal{G}_f^J \cup \mathcal{G}_{g_1}^{K_1} \cup \dots \cup \mathcal{G}_{g_n}^{K_n} \cup \{\rho\}$ , where  $\rho$  is the rule



and  $J, K_1, \dots, K_n$  are the tiers of  $f, g_1, \dots, g_n$ , respectively.

- Let  $h : \mathcal{F}_\Sigma^{n+1} \rightarrow \mathcal{F}_\Sigma$  be  $\text{rec}(g_1, \dots, g_t)$  and suppose  $h$  is tiered recursive with tiers  $l = (j, i_1, \dots, i_n) \rightarrow i$ . Let  $J_1, \dots, J_t$  be the tiers of  $g_1, \dots, g_t$ , respectively. Moreover, let  $\Theta = (\{g_1^{J_1}, \dots, g_t^{J_t}\}, \tau)$ , where  $\tau(g_i^{J_i}) = 2\sigma(\mathbf{f}_i) + n$ . Then,  $\mathcal{G}_h^l$  is the GRS  $\mathcal{G}_{g_1}^{J_1} \cup \dots \cup \mathcal{G}_{g_t}^{J_t} \cup \mathcal{H}$  where  $\mathcal{H}$  is the set of all unfolding graph rewriting rules for  $j\Sigma, \Theta$  and  $h^l$ .
- Let  $h$  be  $\text{cond}(g_1, \dots, g_t)$  and suppose  $h$  is tiered recursive with tiers  $l = (j, i_1, \dots, i_n) \rightarrow i$ . Then  $\mathcal{G}_h^l$  is

the GRS  $\mathcal{G}_{g_1}^{J_1} \cup \dots \cup \mathcal{G}_{g_t}^{J_t} \cup \{\rho_1, \dots, \rho_t\}$ , where  $\rho_k$  is the rule



and  $J_1, \dots, J_t$  are the tiers of  $g_1, \dots, g_t$ , respectively.

The *extensional* soundness of the above encoding can be verified relatively easily. More interesting, and difficult, is the study of its complexity properties.

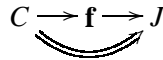
Theorem 1 is a direct consequence of the following:

**Proposition 1** *Suppose  $f : \mathcal{T}_\Sigma^n \rightarrow \mathcal{T}_\Sigma$  is tiered recursive with tiers  $l = (i_1, \dots, i_n) \rightarrow i$  and let  $\mathcal{G}_f$  be the GRS on  $\Theta \sqsupseteq \mathbb{N}\Sigma$  defined as above. Then there is a polynomial with natural coefficients  $p : \mathbb{N} \rightarrow \mathbb{N}$  such that for every proper context  $C$ , for every  $\xi$  and for every proper term graph  $G$  such that  $\langle G \rangle = \mathbf{f}^l(t_1^{i_1}, \dots, t_n^{i_n})$ , it holds that  $C_\xi(G) \rightarrow_i C_{\theta_1}(H_1) \rightarrow_i \dots \rightarrow_i C_{\theta_m}(H_m)$  where:*

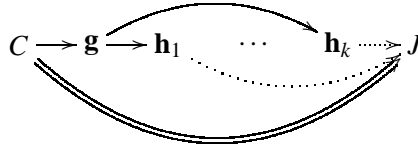
1.  $\langle H_m \rangle = (f(t_1, \dots, t_n))^i$ ;
2.  $m \leq p(|G|)$ ;
3.  $|H_j| \leq p(|G|)$  for every  $j$ ;
4.  $|H_m|_i \leq |G|_i + p(\sum_{k=i+1}^\infty |G|_k)$ .

**Proof.** By induction on the structure of  $f$  as a tiered function. In this proof, we use notations like  $J \rightarrow K$ , meaning there is *at least* an arc from  $J$  to  $K$ ,  $J \Rightarrow K$ , meaning there is *some* (but possibly zero) arcs from  $J$  to  $K$  and  $J \dashrightarrow K$ , meaning all the vertices in  $K$  are reachable from  $J$ . Tiering information is omitted whenever possible, e.g.,  $\mathbf{f}$  often takes the place of  $\mathbf{f}^l$ . We only give the most interesting inductive cases.

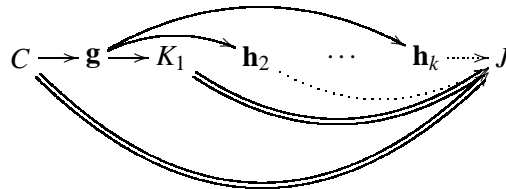
- Suppose  $f = \text{comp}(g, h_1, \dots, h_k)$  where  $g$  is tiered recursive with tiers  $(j_1, \dots, j_k) \rightarrow i$  and  $h_h$  is tiered recursive with tiers  $(i_1, \dots, i_n) \rightarrow j_h$ . Let  $p_g, p_{h_1}, \dots, p_{h_k}$  be some polynomials satisfying the properties above, whose existence follows from the inductive hypothesis. We can write  $G$  as  $\mathbf{f} \rightarrow J$  and so we start from



In one rewriting step the graph becomes



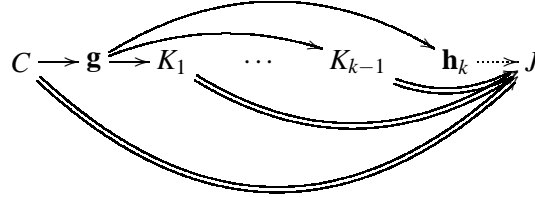
After some  $m_1$  rewriting steps, we get to



By the induction hypothesis, the pointers coming from  $C$  remain unaltered. Notice that  $m_1 \leq p_{h_1}(|J| + 1) \leq p_{h_1}(|G|)$ . Moreover,  $K_1$  can only contain vertices labelled with  $\mathbf{c}_k^{j_1}$ , because the whole graph is still proper. Then, by the inductive hypothesis,

$$|K_1|_i \leq p_{h_1} \left( \sum_{s=j_1+1}^{\infty} |G|_s \right)$$

The size of any intermediate graph produced in these  $m_1$  steps (not considering  $|C|$ ) is  $k + p_{h_1}(|J| + 1) \leq k + p_{h_1}(|G|)$ . Likewise, after  $m_2 + \dots + m_{k-1}$  rewriting steps, again by induction hypothesis, we get to



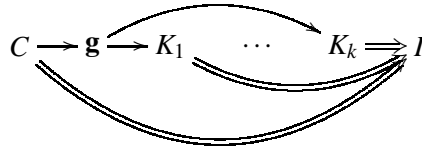
where for every  $s$ ,  $m_s \leq p_{h_s}(|J| + 1) \leq p_{h_s}(|G|)$ ,  $|K_s|_t = 0$  whenever  $t \neq j_s$  and

$$|K_s|_{j_s} \leq p_{h_s} \left( \sum_{s=j_s+1}^{\infty} |G|_s \right)$$

Moreover, the size of any intermediate graph produced in the  $m_s$  steps (not considering  $|C|$ ) is at most

$$k + \sum_{r=1}^s p_{h_r}(|G|).$$

After  $m_k$  steps, we reach



Again,  $m_k \leq p_{h_k}(|J| + 1) \leq p_{h_k}(|G|)$ ,  $|K_k|_t = 0$  whenever  $t \neq j_k$  and

$$|K_k|_{j_k} \leq p_{h_k} \left( \sum_{s=j_k+1}^{\infty} |G|_s \right)$$

This time, however, we cannot claim that  $J$  remains unchanged. Indeed, it's replaced by  $I$ , which anyway only contains a subset of the vertices of  $J$ . As usual, the size of any intermediate result is at most

$$k + \sum_{r=1}^k p_{h_r}(|G|).$$

The graph above can be written as follows:



where  $L$  is a graph such that

$$|L|_t \leq r \left( \sum_{s=t+1}^{\infty} |G|_s \right) + |G|_t$$

for every  $t$  and  $r$  is a fixed polynomial not depending on  $G$ . Finally, after  $l$  steps, we get to  $C \Rightarrow M$ , where  $\langle M \rangle = g(u_1, \dots, u_k)^i$  and the pointers coming from  $C$  remain unaltered. Moreover:

$$\begin{aligned} l \leq p_g(|L|) &\leq p_g \left( 1 + \sum_{t=1}^{\infty} |L|_t \right) \leq p_g \left( 1 + \sum_{t=1}^{\infty} \left( r \left( \sum_{s=t+1}^{\infty} |G|_s \right) + |G|_t \right) \right) \\ &\leq p_g \left( r \left( 1 + \sum_{t=1}^{\infty} \sum_{s=t}^{\infty} |G|_s \right) \right) \leq p_g \left( r(\max\{i_1, \dots, i_n\} |G|) \right) = q(|G|) \end{aligned}$$

where  $q$  is a polynomial. By induction hypothesis,

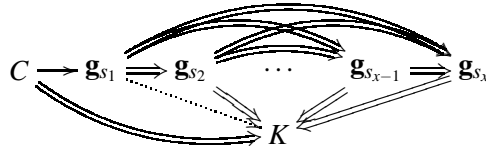
$$\begin{aligned} |M|_i &\leq p_g \left( \sum_{s=i+1}^{\infty} |L|_s \right) + |L|_i \leq p_g \left( \sum_{s=i+1}^{\infty} \left( r \left( \sum_{t=s+1}^{\infty} |G|_t \right) + |G|_s \right) \right) + |L|_i \leq p_g \left( \sum_{s=i+1}^{\infty} \left( r \left( \sum_{t=s}^{\infty} |G|_t \right) \right) \right) + |L|_i \\ &\leq p_g \left( r \left( \sum_{s=i+1}^{\infty} \sum_{t=s}^{\infty} |G|_t \right) \right) + |L|_i \leq p_g \left( r(\max\{i_1, \dots, i_n\} \left( \sum_{s=i+1}^{\infty} |G|_s \right)) \right) + |L|_i \\ &\leq p_g \left( r(\max\{i_1, \dots, i_n\} \left( \sum_{s=i+1}^{\infty} |G|_s \right)) \right) + r \left( \sum_{s=i+1}^{\infty} |G|_s \right) + |G|_i = z \left( \sum_{s=i+1}^{\infty} |G|_s \right) + |G|_i \end{aligned}$$

where  $z$  is a polynomial. The size of intermediate results is itself bound by  $q(|G|)$ . We can choose  $p_h$  to be just  $q + z$ .

- Suppose  $f = \text{rec}(g_1, \dots, g_t)$  where  $g_i$  is tiered recursive with tiers  $(i, \dots, i, j, \dots, j, i_1, \dots, i_n) \rightarrow i$ . Let  $p_{g_1}, \dots, p_{g_n}$  be some polynomials satisfying the properties above, whose existence follows from the inductive hypothesis. We can write  $G$  as  $\mathbf{f} \Rightarrow J$  and so we start from

$$C \rightarrow \mathbf{f} \Rightarrow J$$

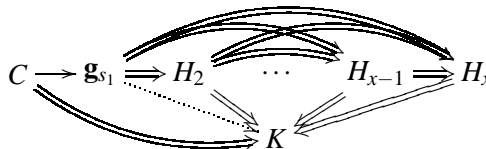
In one (unfolding) rewriting step the graph becomes



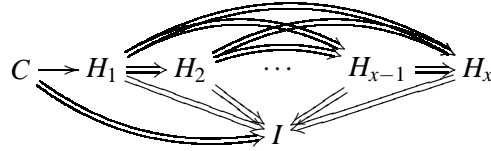
where:

- $\mathbf{c}_{s_1}, \dots, \mathbf{c}_{s_x}$  are the vertices of  $J$  reachable from  $\mathbf{f}$  by following its leftmost outgoing arc, ordered topologically;
- $x \leq |J|_j \leq |G|_j$
- $|K|_t \leq |J|_t \leq |G|_t$  for every  $t$ .

In  $m_x + m_{x-1} + \dots + m_2$  rewriting steps, the graph becomes



Finally, in  $m_1$  rewriting steps, we get to



where  $|I|_t \leq |K|_t$ . The graph above can be written as follows

$$C \xrightarrow{\quad} L$$

Reasoning exactly as in the previous inductive case, we can get the following bounds for every  $1 \leq s \leq x$ :

$$\begin{aligned} |H_s|_i &\leq p_{i_s} \left( \sum_{t=i+1}^{\infty} |K|_t \right) \leq p_{i_s} \left( \sum_{t=i+1}^{\infty} |G|_t \right) \leq p_{i_s} (|G|) \\ |H_s|_t &= 0 \text{ whenever } t \neq i \\ |L|_i &= \sum_{s=1}^x |H_s|_i + |I|_i \leq \sum_{s=1}^x p_{i_s} \left( \sum_{t=i+1}^{\infty} |G|_t \right) + |G|_i \\ m_s &\leq p_{i_s} (|K| + 1 + \sum_{t=s+1}^x |H_t|) \leq p_{i_s} (|G| + 1 + \sum_{t=s+1}^x p_{i_t} (|G|)) \end{aligned}$$

A bound for the size of the intermediate values produced in the any of the  $k$  groups of steps can be obtained analogously. The thesis follows.  $\square$

This concludes the proof.  $\square$

Theorem 1 follows easily from Proposition 1, once we observe that any  $\mathcal{G}_f$  is polytime presentable. Indeed, even if  $\mathcal{G}_f$  is infinite whenever  $f$  is defined by tiered recursion, the rules in  $\mathcal{G}_f$  are very “regular” and an algorithm  $\mathcal{A}_{\mathcal{G}_f}$  with the required properties can be defined naturally.

## 5 Conclusions

We proved that Leivant’s characterization of polynomial time functions holds for any free algebra (the original result was proved only for algebras with unary constructors, i.e. for word algebras). The representation of the terms of the algebras as term graphs permits to avoid uncontrolled duplication of shared subterms, thus preserving polynomial bounds.

The main contribution of the paper is the implementation of tiered recursion via term graph rewriting. The proofs of the related theorems and propositions are non-trivial. We introduce graph unfolding rules and graph contexts, in order to implement recursion efficiently and to prove inductively our main result. Moreover, the result is given on infinite graph rewriting systems – in presence of an infinite set of rewriting rules, some well-known computability results are lost.

## References

- [1] H. P. Barendregt, M. C. J. D. van Eekelen, J. R. W. Glauert, J. R. Kennaway, M. J. Plasmeijer, and M. R. Sleep. Term graph rewriting. In *PARLE. Parallel architectures and languages Europe, Vol. II (Eindhoven, 1987)*, volume 259 of *Lecture Notes in Computer Science*, pages 141–158. Springer, Berlin, 1987.

- [2] Stephen Bellantoni and Stephen Cook. A new recursion-theoretic characterization of the polytime functions. *Computational Complexity*, 2(2):97–110, 1992.
- [3] Vuokko-Helena Caseiro. An equational characterization of the poly-time functions on any constructor data structure. Technical report, Department of Informatics, University of Oslo, 1996.
- [4] Alan Cobham. The intrinsic computational difficulty of functions. In Y. Bar-Hillel, editor, *Logic, Methodology and Philosophy of Science, proceedings of the second International Congress, held in Jerusalem, 1964*, Amsterdam, 1965. North-Holland.
- [5] Martin Hoffman. Programming languages capturing complexity classes. *ACM SIGACT News* 31(1), 11:31–42, 2000.
- [6] Ugo Dal Lago and Simone Martini. Derivational complexity is an invariant cost model. Presented at International Workshop on Foundational and Practical Aspects of Resources Analysis, FOPARA, Eindhoven, 2009.
- [7] Daniel Leivant. Stratified functional programs and computational complexity. In *Proceedings of the 20th ACM SIGPLAN-SIGACT symposium on Principles of programming languages*, pages 325–333. ACM, 1993.
- [8] Daniel Leivant. Ramified recurrence and computational complexity, I: Word recurrence and poly-time. In *Feasible mathematics, II (Ithaca, NY, 1992)*, pages 320–343. Birkhauser, 1994.
- [9] Jean-Yves Marion. Analysing the implicit complexity of programs. *Information and Computation*, 183(1):2–18, 2003.
- [10] Christos H. Papadimitriou. *Computational Complexity*. Addison Wesley, 1993.
- [11] Harold Simmons. The realm of primitive recursion. *Archive for Mathematical Logic*, 27(2):177–188, 1988.