Labelled Deductive Systems

joint work with

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Road Map

- Introduction: A framework for non-classical logics.
  - Modal, relevance and other non-classical logics: deduction systems (Hilbert, ND, sequent) and Kripke semantics.
  - A labelled deduction framework: why and how?
- Labelled deduction for modal logics.
- Labelled deduction for non-classical logics.
- Encoding non-classical logics in Isabelle.
- Substructural and complexity analysis of labelled non-classical logics.
- Conclusions and outlook.

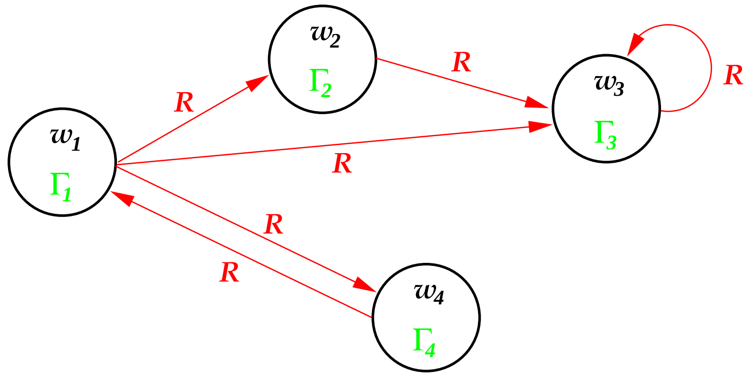
Motivation

- Problem: find uniform deduction systems for non-classical logics.
- Our solution: a framework based on labelling (labelled deduction).
  - Non-classical logics: why?
  - A framework: why and how?
- Modal logics.
- Other non-classical logics: extensions and restrictions (but there are limits).

Why non-classical logics?

Modal, temporal, relevance, linear, substructural, non-monotonic, ...

- Reason about:
  - State and action.
  - Resources.



- Applications in: computer science, artificial intelligence, knowledge representation, mathematics, philosophy, engineering...
  - Programs and circuits.
  - Distributed and concurrent systems.
  - Security.
  - Knowledge and belief.
  - Computational linguistics.
  - ...

The problems

- Specialized approach vs. general methodology.

- 'Explosion' of logics.
  - Each logic demands, at a minimum, a semantics ('truth', $\models A$), a deduction system ($\vdash A$), and metatheorems relating them together ($\models A$ iff $\vdash A$).
  - Specialized or uniform deduction systems?

- Efficient proof search.
  - Specialized or generic provers?
  - Interactive or automated provers?

The problems: a solution (why a framework?)

- Specialized approach vs. general methodology.
  General methodology: how general? $\Rightarrow$ Analysis of the limits.
- 'Explosion' of logics.
  - Each logic demands, at a minimum, a semantics ('truth', $\models A$), a deduction system ($\vdash A$), and metatheorems relating them together ($\models A$ iff $\vdash A$).
  - Specialized or uniform deduction systems?
  - Uniform deduction systems: good' properties?
    $\Rightarrow$ Analysis of structure of deductions and proofs.
- Efficient proof search.
  - Specialized or generic provers?
  - Interactive or automated provers?
  - Interactive generic provers.
    $\Rightarrow$ Uniform implementations (add automation).

A framework: how?

1. Hilbert-style
   − difficult to use in practice
2. Natural deduction systems
   + structured reasoning (normal deductions)
   − lack uniformity
3. Full semantic translation into predicate logic
   + general and uniform
   − lacks structure

A framework: how? A labelled deduction framework

1. Hilbert-style

   – difficult to use in practice

2. Natural deduction systems

   + structured reasoning (normal deductions)
   – lack uniformity

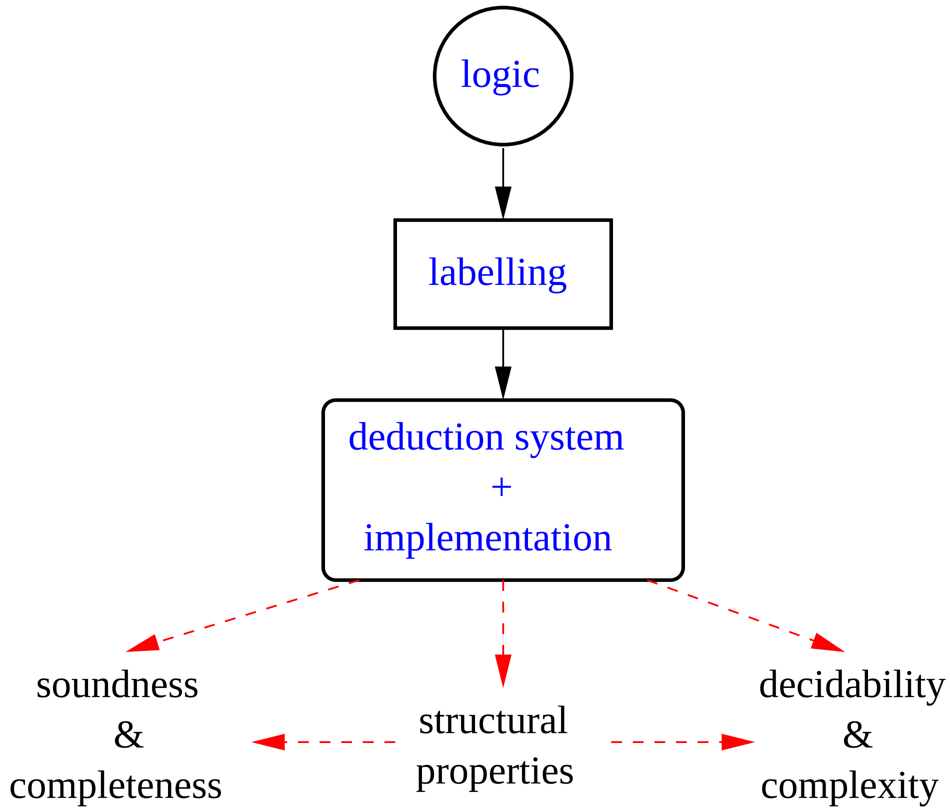
3. Full semantic translation into predicate logic

   + general and uniform
   – lacks structure

4. Combine 2 and 3: partial (controlled) translation

   + uniform & modular, ‘natural’ deduction systems
   + structured reasoning
   – there are limits

The big picture



- Labelling: partial translation:
  - ▶ Lift minimal information from semantics (or "from somewhere else") into syntax.
  - ▶ Investigate the structure of the deduction systems.

Main results

- Methodology:
  - **Presentation:** (modal, relevance, ... logics).
    - Labelled natural deduction (sequent) systems.
    - Uniform & modular: fixed base system + separate theories.
  - **Implementation:** in Isabelle (generic theorem prover).
- Technical contributions:
  - **Soundness and completeness:** parameterized proofs.
  - **Proof theory:**
    - Normalization and subformula property.
    - Structural properties vs. generality.
  - **Substructural analysis:**
    - Decidability and complexity analysis.
    - Bounded space requirements (K, T, $\mathrm{K4}$, $\mathrm{S4}$, ...; B^+, ...).
    - Justification (& refinement) of ‘standard sequent systems’.

What is a deduction system?

- **Hilbert system.**
  - Finitary inductive definitions.
- **Natural deduction system.**
  - Proof under assumption — useful in practice.
- **Sequent calculus system.**
  - Generalized sequent notation — useful for theory.

Propositional arrow logic: Hilbert system $\mathrm{H}(\supset)$

- Want to capture:

$$A \supset B \quad \equiv \quad \text{if } A \text{ then } B$$

- Axioms and modus ponens rule.
  - $\mathrm{K}_{\supset}$: $A \supset B \supset A$
  - $\mathrm{S}_{\supset}$: $(A \supset B) \supset (A \supset B \supset C) \supset (A \supset C)$
  - $\dfrac{A \supset B \quad A}{B}\,\mathrm{MP}$

Propositional arrow logic: ND system $\mathrm{N}(\supset)$

- Want to capture: proof under assumption.

  The 'meaning' of $A \supset B$ is: If A *were* to be true, then B would be true.

- So if, for the sake of argument, I assume that A is true, and show, from that, that B is true, that means that $A \supset B$ is true *irrespective* of whether or not A is true.

  Formally: if A implies B then $A \supset B$.

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B} \supset \mathrm{I}$$

- Similarly, if I know that $A \supset B$ *is* true, and I know that A is true, then I know that B is true.

  Formally: if $A \supset B$ and A, then B.

$$\frac{\begin{array}{c}\vdots \\ A \supset B\end{array} \quad \begin{array}{c}\vdots \\ A\end{array}}{B} \supset \mathrm{E}$$

Sufficiency of Hilbert system $\mathrm{H}(\supset)$

- By induction (using MP):

  if $A \supset B$, then A implies B

- The deduction theorem (again by induction):

  if assuming A then B (if A implies B), then $A \supset B$

  $\cong$ proof under assumption

Equivalence of $\mathrm{H}(\supset)$ and $\mathrm{N}(\supset)$

The natural deduction and Hilbert presentations are equivalent

$$\supset\mathrm{I} + \supset\mathrm{E} \;\equiv\; \mathrm{K}_{\supset} + \mathrm{S}_{\supset} + \mathrm{MP}$$

Proof: easy, given deduction theorem.

Proving $A \supset A$

- In $\mathrm{H}(\supset)$:

$$
\begin{array}{lll}
1. & (A \supset (A \supset A)) \supset (A \supset (A \supset A) \supset A) \supset (A \supset A) & \mathrm{S}_{\supset} \\
2. & A \supset A \supset A & \mathrm{K}_{\supset} \\
3. & A \supset (A \supset A) \supset A & \mathrm{K}_{\supset} \\
4. & (A \supset (A \supset A) \supset A) \supset (A \supset A) & \mathrm{MP}\,1,2 \\
5. & A \supset A & \mathrm{MP}\,3,4
\end{array}
$$

  Using the deduction theorem: A follows from A, so $A \supset A$.

- In $\mathrm{N}(\supset)$: A implies A, thus $A \supset A$.

Infeasibility of Hilbert Systems

- Try to prove: $A \supset B \supset C \supset (A \supset B \supset C \supset D) \supset D$
  - Natural Deduction proof in $\mathrm{N}(\supset)$: trivial (8 steps).
  - Hilbert proof in $\mathrm{H}(\supset)$: definitely not trivial ($\sim 4^4$ steps).
- The situation is even worse for non-classical logics such as modal logics!
- But before let us look at:
  - Extension to propositional classical logic.
  - Sequent systems.

Proof under assumption

$\Gamma \vdash_{\mathrm{N}(\supset)} A$, where Γ is a *set* of formulas, means that in $\mathrm{N}(\supset)$ there is a derivation Π of the formula A from the assumptions Γ, i.e.

$$\begin{array}{c}\Gamma\\ \Pi\\ A\end{array}$$

Example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \supset B \supset C \supset D]^4 \quad [A]^1}{B \supset C \supset D}\supset\mathrm{E} \quad [B]^2}{C \supset D}\supset\mathrm{E} \quad [C]^3}{D}\supset\mathrm{E}}{(A \supset B \supset C \supset D) \supset D}\supset\mathrm{I}^4}{C \supset (A \supset B \supset C \supset D) \supset D}\supset\mathrm{I}^3}{B \supset C \supset (A \supset B \supset C \supset D) \supset D}\supset\mathrm{I}^2}{A \supset B \supset C \supset (A \supset B \supset C \supset D) \supset D}\supset\mathrm{I}^1$$

This is a proof.

A derivation would be: $\{A, B, C, A \supset B \supset C \supset D\} \vdash D$.

Propositional classical logic: Hilbert & ND systems

- $\mathrm{H(PCL)} = \mathrm{H}(\supset) +$

$$A \leftrightarrow \sim\sim A$$

$$\bot \supset A\,, \quad A \wedge B \supset A\,, \quad A \wedge B \supset B\,, \quad \frac{A \quad B}{A \wedge B}\,\mathrm{adjunction}$$

- $\mathrm{N(PCL)} = \mathrm{N}(\supset) +$:

$$\dfrac{\begin{matrix}[A \supset \bot] \\ \vdots \\ \bot\end{matrix}}{A}\,\bot\mathrm{E}$$

where $\sim$, $\wedge$, $\vee$ and other operators (and the corresponding rules) are defined (derived) using $\supset$ and $\bot$ (and the corresponding rules), e.g. $\sim A =_{def} A \supset \bot$, $A \vee B =_{def} (A \supset \bot) \supset B$, $A \wedge B =_{def} (A \supset B \supset \bot) \supset \bot$

$$\dfrac{\begin{matrix}\vdots \\ A \wedge B\end{matrix}}{A}\,\wedge\mathrm{E}\,, \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix} \quad \begin{matrix}\vdots \\ B\end{matrix}}{A \wedge B}\,\wedge\mathrm{I}\,, \qquad \dfrac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}\,\vee\mathrm{E}\,, \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B}\,\vee\mathrm{I} =_{def} \dfrac{\dfrac{\dfrac{[A \supset \bot]^1 \quad \begin{matrix}\vdots \\ A\end{matrix}}{\bot}\,\supset\mathrm{E}}{B}\,\bot\mathrm{E}}{(A \supset \bot) \supset B}\,\supset\mathrm{I}^1$$

Propositional classical logic: Sequent system $\mathrm{S(PCL)}$

Axioms:

$$\frac{}{A \vdash A}\,\mathrm{AX} \qquad \frac{}{\bot \vdash A}\,\bot\mathrm{L}$$

Structural rules:

$$\frac{\Gamma \vdash \Gamma'}{A, \Gamma \vdash \Gamma'}\,\mathrm{WL} \qquad \frac{\Gamma \vdash \Gamma'}{\Gamma \vdash \Gamma', A}\,\mathrm{WR}$$

$$\frac{A, A, \Gamma \vdash \Gamma'}{A, \Gamma \vdash \Gamma'}\,\mathrm{CL} \qquad \frac{\Gamma \vdash \Gamma', A, A}{\Gamma \vdash \Gamma', A}\,\mathrm{CR}$$

Logical rules:

$$\frac{\Gamma \vdash \Gamma', A \quad B, \Gamma \vdash \Gamma'}{A \supset B, \Gamma \vdash \Gamma'}\,{\supset}\mathrm{L} \qquad \frac{A, \Gamma \vdash \Gamma', B}{\Gamma \vdash \Gamma', A \supset B}\,{\supset}\mathrm{R}$$

where Γ and Γ' are *multisets* of formulas
and we can derive

$$\frac{\Gamma \vdash \Gamma', A}{\sim A, \Gamma \vdash \Gamma'}\,{\sim}\mathrm{L} \qquad \frac{A, \Gamma \vdash \Gamma'}{\Gamma \vdash \Gamma', \sim A}\,{\sim}\mathrm{R}$$

Deduction systems for non-classical logics: Problems

- We have 'assumed' that $\supset$ and $\vdash$ have the same properties.
- We have essentially that

$$\text{'follows from' } (\vdash) \equiv \text{'implies' } (\supset)$$

- There are many logics where this may not hold.
  - 'Substructural' (e.g. relevance, linear) logics: $\rightarrow$ has different properties

    $$\nvdash A \rightarrow B \rightarrow A$$

    So $\vdash$ should have different properties if the two are to be the same, e.g.

    $$A, B \nvdash A$$

  - Modal logics: relationship between $\supset$ and $\vdash$ becomes more complex.

Propositional modal logics: Hilbert systems

- We extend our language with $\Box$ (and $\Diamond A =_{def} \sim\Box\sim A$).
- $\mathrm{H(K)}$, a Hilbert system for the basic modal logic K:
  - all axioms schemas of PCL and the rule MP
  - the new axiom schema

$$\mathrm{K}\colon\ \Box(A \supset B) \supset (\Box A \supset \Box B)$$

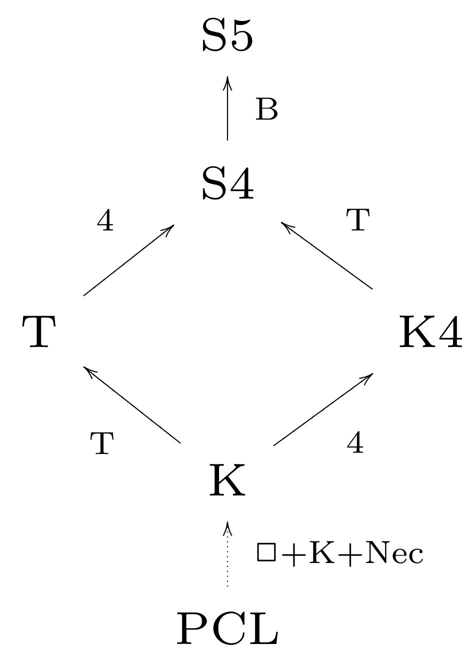
  - and the new rule

$$\frac{A}{\Box A}\,\mathrm{Nec}$$

Propositional modal logics: Hilbert systems (cont.)

- Systems for other logics: we add axioms characterizing $\Box$

D:	$\Box A \supset \Diamond A$
T:	$\Box A \supset A$
4:	$\Box A \supset \Box\Box A$
B:	$A \supset \Box\Diamond A$
5:	$\Diamond A \supset \Box\Diamond A$
2:	$\Diamond\Box A \supset \Box\Diamond A$
M:	$\Box\Diamond A \supset \Diamond\Box A$
Grz:	$\Box(\Box(A \supset \Box A) \supset A) \supset A$

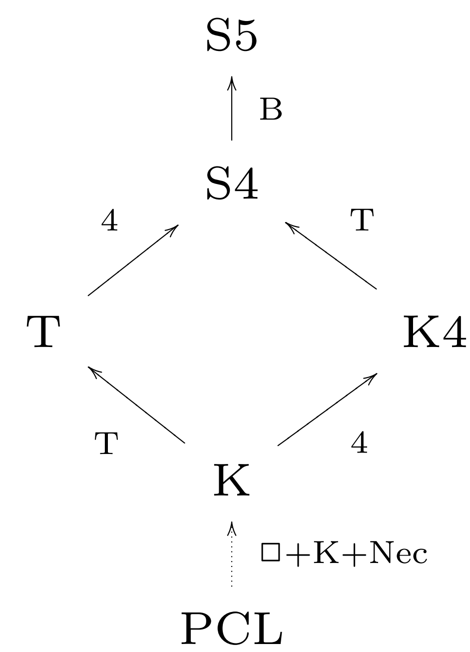


but ...

Propositional modal logics: Hilbert systems (cont.)

- Systems for other logics: we add axioms characterizing $\Box$

$$
\begin{array}{ll}
\text{D:} & \Box A \supset \Diamond A \\
\text{T:} & \Box A \supset A \\
\text{4:} & \Box A \supset \Box\Box A \\
\text{B:} & A \supset \Box\Diamond A \\
\text{5:} & \Diamond A \supset \Box\Diamond A \\
\text{2:} & \Diamond\Box A \supset \Box\Diamond A \\
\text{M:} & \Box\Diamond A \supset \Diamond\Box A \\
\text{Grz:} & \Box(\Box(A \supset \Box A) \supset A) \supset A
\end{array}
$$



but ... the deduction theorem fails!

Not thm: If assuming A then $\Box A$, then $A \supset \Box A$.

A problem with proof under assumption in $\mathrm{S4}$

Imagine we have the deduction theorem in $\mathrm{S4}$.

Then

$$\begin{array}{lll} 1. & \text{from } A \text{ infer } \Box A & \text{Nec}, 1 \\ 2. & A \supset \Box A & \supset \mathrm{I} \end{array}$$

Thus we have

$$A \supset \Box A$$

but we also have (as an axiom)

$$\Box A \supset A$$

and thus that

$$\Box A \leftrightarrow A$$

i.e. $\Box$ is meaningless!

What is going wrong?

An attempted proof of the deduction theorem

We have a proof of B given A, and we want a proof of $A \supset B$.

By induction on the length of the derivation:

Base: B is immediate. Two subcases:

1. B is an axiom. Then B follows without A.
   We also have, as an axiom, $B \supset A \supset B$, so by MP, we have $A \supset B$.
2. B is A. We can prove $A \supset A$ since we have the axioms of PCL and MP.

Step: B is the result of a rule application. Two subcases:

1. B is a result of MP from C and $C \supset B$.
   By the induction hypothesis we have $A \supset C$ and $A \supset C \supset B$, and as an axiom we have $(A \supset C) \supset (A \supset C \supset B) \supset (A \supset B)$, so by two applications of MP we have $A \supset B$.
2. $B = \Box B'$ is the result of Nec from B'.
   By the induction hypothesis we have $A \supset B'$, and we want to get $A \supset \Box B'$.
   How should we do this?

   We can't!

The problem, and solutions

The problem seems to be with the relationship between $\vdash$ and $\supset$.

We have $A \vdash B'$ and can get $A \vdash \Box B'$, but we can't get $A \supset B'$ to $A \supset \Box B'$.

One way (there are others) to proceed:

> assume $\vdash \; \equiv \; \supset$ and try to arrange things so that this makes sense

How do we get $\vdash \equiv \supset$ to work?

- We have

$$A \supset B$$

and we want

$$A \supset \Box B$$

- We can argue

$$\begin{array}{lll} 1. & A \supset B & \\ 2. & \Box(A \supset B) & \text{Nec}, 1 \\ 3. & \Box(A \supset B) \supset \Box A \supset \Box B & \text{K} \\ 4. & \Box A \supset \Box B & \text{MP}\, 2, 3 \end{array}$$

- But remember that we also have, as an axiom

$$\Box A \supset \Box\Box A$$

How do we get $\vdash \; \equiv \; \supset$ to work?

- Thus, if A is boxed, i.e. A is $\Box A'$

1.	$\Box A' \supset B$	
2.	$\Box(\Box A' \supset B)$	Nec , 1
3.	$\Box(\Box A' \supset B) \supset \Box\Box A' \supset \Box B$	K
4.	$\Box\Box A' \supset \Box B$	MP 2, 3
5.	$(\Box A' \supset \Box\Box A') \supset (\Box\Box A' \supset \Box B) \supset (\Box A' \supset \Box B)$	prop taut
6.	$\Box A' \supset \Box\Box A'$	axiom 4
7.	$(\Box\Box A' \supset \Box B) \supset (\Box A' \supset \Box B)$	MP 5, 6
8.	$\Box A' \supset \Box B$	MP 7, 4

So, we have $A \supset \Box B$ from $A \supset B$ as desired.

How do we get $\vdash \equiv \supset$ to work?

- That is, box-introduction works if all the hypotheses (assumptions) are boxed

$$\dfrac{\begin{array}{c}[\text{only `boxed' assumptions}]\\ \vdots \\ A\end{array}}{\Box A}\,\Box I \qquad \text{i.e.} \quad \dfrac{\begin{array}{c}[\Box\Gamma]\\ \vdots \\ A\end{array}}{\Box A}\,\Box I \qquad \text{i.e.} \quad \left(\dfrac{\Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A, \Sigma'}\right)$$

- For box-elimination we can use the rule

$$\dfrac{\Box A}{A}\,\Box E$$

since

$$\begin{array}{lll} 1. & \Box A & \\ 2. & \Box A \supset A & \mathrm{T} \\ 3. & A & \mathrm{MP}\,1,2 \end{array}$$

A complete natural deduction system for $\mathrm{S4}$

- $\mathrm{N(S4)} = \mathrm{N(PCL)}+$

$$\dfrac{\begin{array}{c}[\Box\Gamma]\\ \vdots\\ A\end{array}}{\Box A}\,\Box\mathrm{I} \quad \text{and} \quad \dfrac{\Box A}{A}\,\Box\mathrm{E}$$

- But what about other logics?

- OK for some logics $(\mathrm{K}, \mathrm{T}, \mathrm{K4}, \mathrm{S5}, \ldots)$,

  but in general there is no ‘easy’ way of coming up with ‘good’ (uniform and modular) systems!

$\Rightarrow$ Look for ‘better’ systems!

Standard deduction systems for non-classical logics: Summary

- Hilbert systems:
  - Simple inductive definitions.
  - Can be hard to use.
  - Very general (a framework).
- Natural deduction systems:
  - Proof under assumption (*consequence*).
  - Easy to use but lack generality (no ‘real’ framework).
- Sequent systems:
  - Special (multiple conclusioned) form of natural deduction with good proof-theoretical properties.

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- Sequent systems:
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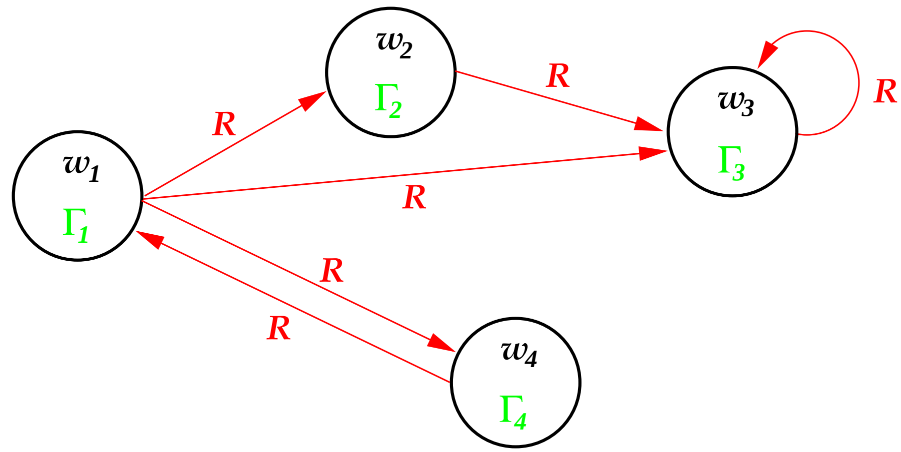
$\Rightarrow$ Looking for a usable framework? Try labelled deduction systems.

Road Map

- Introduction: A framework for non-classical logics.
- Labelled deduction for modal logics.
  - Labelled deduction systems: uniform and modular.
  - Properties: soundness, completeness, normalization, ...
  - A topography of labelled modal logics.
- Labelled deduction for non-classical logics.
- Encoding non-classical logics in Isabelle.
- Substructural and complexity analysis of labelled non-classical logics.
- Conclusions and outlook.

Evolution of state

- Possible worlds (states) $x, y, z, w \in W$.
  - Set of formulas $\Gamma, \Delta, \Theta, \Sigma$.
- Accessibility relation R:
  - Binary transition relation.

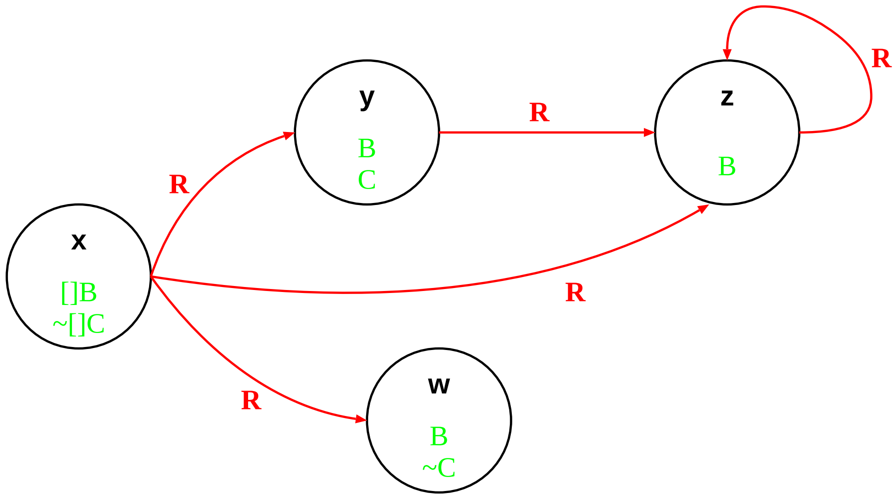


- Kripke semantics:
  - Model $M = (W, R, V)$.
  - Formulas evaluated locally: $M \models x{:}A$ (truth).

$\Rightarrow$ Logics characterized by properties of models.

Modal logics

- Possible worlds (states) $x, y, z, w \in W$.
  - ▶ Set of formulas.
- Accessibility relation R:
  - ▶ Binary transition relation.



- Kripke semantics:
  - ▶ Model $M = (W, R, V)$.
  - ▶ Formulas evaluated locally (truth $\models$):

$$\models^M x{:}\Box A \;\Leftrightarrow\; \text{for all } y.\ x R y \;\Rightarrow \models^M y{:}A$$

$\Rightarrow$ Logics characterized by properties of R.

Modal logics

$$
\begin{tikzcd}
 & \mathrm{S5} & \\
 & \mathrm{S4} \arrow[u, "symm"'] & \\
\mathrm{T} \arrow[ur, "trans"] & & \mathrm{K4} \arrow[ul, "refl"'] \\
 & \mathrm{K} \arrow[ul, "refl"] \arrow[ur, "trans"'] & \\
 & \mathrm{PCL} \arrow[u, dotted, "\Box"] &
\end{tikzcd}
$$

Modal logics: partial translation

- W: a set of labels $(x, y, \ldots)$ representing possible worlds.
- $R \subseteq W \times W$.

labelled formula (lwff)	$x{:}A$	A is provable	iff	$\forall x \in W (\vdash x{:}A)$
relational formula (rwff)	$x\,R\,y$	"x accesses y"		

$\Rightarrow$ Uniform & modular (& natural) deduction systems.

$\Rightarrow$ 'Good' properties (completeness, structure).

$\Rightarrow$ Generalization to relevance and other non-classical logics (but there are limits).

Modal logics: partial translation (cont.)

$$\begin{array}{rcccc} \mathrm{N}(\mathcal{L}) & = & \text{fixed base system} & + & \text{varying relational theory} \\ & = & \mathrm{N}(\mathrm{K}) & + & \mathrm{N}(\mathcal{T}) \end{array}$$

- Base system $\mathrm{N}(\mathrm{K})$:
  - ▶ Natural deduction system formalizing K.
  - ▶ Reason about $x{:}A$.
- Relational theory $\mathrm{N}(\mathcal{T})$:
  - ▶ Describes the behavior of R.
  - ▶ Reason about $x\,R\,y$.
- Separation ⇒ structure ⇒ properties.

Labelled modal logics: definitions

- The language of propositional modal logics consists of a denumerable infinite set of *propositional variables*, the brackets '(' and ')', and the primitive *logical operators*:
  - the classical connectives $\bot$ (falsum) and $\supset$, and
  - the modal operator $\Box$.

- The set of propositional modal formulas is the smallest set that contains the atomic formulas (propositional variables and $\bot$) and is closed under the rules:
  1. if A and B are formulas, then so is $A \supset B$;
  2. if A is a formula, then so is $\Box A$; and
  3. all formulas are given by the above clauses.

  Other operators can be defined in the usual manner, e.g. $\sim A =_{def} A \supset \bot$ and $\Diamond A =_{def} \sim \Box \sim A$.

- Let W be a set of labels and R a binary relation over W. If x and y are labels and A is a propositional modal formula, then $x\,R\,y$ is a relational formula (or rwff) and $x{:}A$ is a labelled formula (or lwff).

Labelled modal logics: the deduction theorem, again

- The deduction theorem

  if assuming A true we can show B true, then $A \supset B$ is true

  fails for implications weaker or substantially different from intuitionistic $\supset$.

- Kripke completeness tells us: A is provable if and only if A is true at every world in every suitable Kripke model $M = (W, R, V)$

$$\vdash A \quad \text{iff} \quad \models^M w{:}A \text{ for all } w \in W\,.$$

- Hence, the deduction theorem corresponds to

$$(\forall w \in W\ (\models^M w{:}A) \Rightarrow \forall w \in W\ (\models^M w{:}B)) \Rightarrow \forall w \in W\ (\models^M w{:}A \supset B)\,.$$

  but this is false. The semantics of $\supset$ in a Kripke model is just the weaker:

$$\forall w \in W\ ((\models^M w{:}A \Rightarrow \models^M w{:}B) \Rightarrow \models^M w{:}A \supset B)\,.$$

- Labelling provides a language in which we can formulate a 'proper' deduction theorem:

  if assuming $w{:}A$ true we can show $w{:}B$ true, then $w{:}A \supset B$ is true.

N(PCL) for propositional classical logic

$$\dfrac{\begin{array}{c}[A \supset \bot] \\ \vdots \\ \bot\end{array}}{A}\,\bot\mathrm{E}$$

$$\dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \supset B}\,\supset\mathrm{I} \qquad \dfrac{A \supset B \qquad A}{B}\,\supset\mathrm{E}$$

The base modal system $\mathrm{N(K)}$

$$\dfrac{\begin{array}{c}[x{:}A \supset \bot]\\ \vdots \\ y{:}\bot\end{array}}{x{:}A}\ \bot\mathrm{E}$$

$$\dfrac{\begin{array}{c}[x{:}A]\\ \vdots \\ x{:}B\end{array}}{x{:}A \supset B}\ \supset\mathrm{I} \qquad \dfrac{x{:}A \supset B \quad x{:}A}{x{:}B}\ \supset\mathrm{E}$$

$$\dfrac{\begin{array}{c}[x\,R\,y]\\ \vdots \\ y{:}A\end{array}}{x{:}\Box A}\ \Box\mathrm{I}\ \ [y \text{ fresh}] \qquad \dfrac{x{:}\Box A \quad x\,R\,y}{y{:}A}\ \Box\mathrm{E}$$

$$M \models x{:}\Box A \;\Leftrightarrow\; \text{for all } y.\ x\,R\,y \;\Rightarrow\; M \models y{:}A$$

Extensions of K

Hilbert systems for other (normal) modal logics are obtained by extending $\mathrm{H(K)}$ with axiom schemas formalizing the behavior of $\Box$.

Name	Axiom schema	Name	Axiom schema
K	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$
D	$\Box A \supset \Diamond A$	R	$\Diamond\Box A \supset (A \supset \Box A)$
T	$\Box A \supset A$	MV	$\Diamond\Box A \vee \Box A$
B	$A \supset \Box\Diamond A$	Löb	$\Box(\Box A \supset A) \supset \Box A$
4	$\Box A \supset \Box\Box A$	Grz	$\Box(\Box(A \supset \Box A) \supset A) \supset A$
5	$\Diamond A \supset \Box\Diamond A$	Go	$\Box(\Box(A \supset \Box A) \supset A) \supset \Box A$
2	$\Diamond\Box A \supset \Box\Diamond A$	M	$\Box\Diamond A \supset \Diamond\Box A$
Cxt	$\Diamond\Box A \supset \Box\Box A$	Z	$\Box(\Box A \supset A) \supset (\Diamond\Box A \supset \Box A)$
X	$\Box\Box A \supset \Box A$	Zem	$\Box\Diamond\Box A \supset (A \supset \Box A)$

Extensions of $\mathrm{N}(\mathrm{K})$

- We extend $\mathrm{N}(\mathrm{K})$ with relational theories (labelling algebras), which axiomatize properties of R formalizing the accessibility relation $\mathfrak{R}$ in Kripke frames.
- Correspondence theory tells us which modal axiom schemas correspond to which axioms for R.
- Should relational theories be axiomatized in higher-order logic ($\Rightarrow$ all normal propositional modal logics), first-order logic, or some subset thereof?
- This is an important decision!
  - Different choices of interface between $\mathrm{N}(\mathrm{K})$ and the relational theory result in essentially different systems.
  - We choose the Horn-fragment: cannot capture all axioms, e.g. 3, M, $\mathrm{L\ddot{o}b}$, but
    - it captures a large family of logics (including most common ones),
    - good normalization properties.

Extensions of $\mathrm{N(K)}$ (cont.)

- Horn relational formula: closed formula of the form

$$\forall x_1 \ldots \forall x_n ((s_1 R t_1 \wedge \ldots \wedge s_m R t_m) \supset s_0 R t_0)$$

where $m \geq 0$, and the s_i and t_i are terms built from the labels $x_1, \ldots, x_n$ and constant function symbols, i.e. Skolem function constants.

- Corresponding Horn relational rule:

$$\frac{s_1 R t_1 \quad \ldots \quad s_m R t_m}{s_0 R t_0}$$

Extensions of $\mathrm{N}(\mathrm{K})$ (cont.)

- Generalized Geach axiom schema $\Diamond^i \Box^m A \supset \Box^j \Diamond^n A$ corresponds to (i, j, m, n) convergency

$$\forall x \forall y \forall z (x\, R^i\, y \wedge x\, R^j\, z \supset \exists u (y\, R^m\, u \wedge z\, R^n\, u))$$

$$\begin{tikzcd}
 & y \arrow[dr, "m"] & \\
x \arrow[ur, "i"] \arrow[dr, "j"'] & & u \\
 & z \arrow[ur, "n"'] &
\end{tikzcd}$$

  where $x\, R^0\, y$ means $x = y$ and $x\, R^{i+1}\, y$ means $\exists v (x\, R\, v \wedge v\, R^i\, y)$.

  Example: transitivity is given by $(0, 2, 1, 0)$.

- Restricted (i, j, m, n) convergency axioms: class of properties of $\mathfrak{R}$ that can be expressed as Horn rules in the theory of one binary predicate R (without $=$)

$$m = n = 0 \text{ implies } i = j = 0$$

- Proposition: If $\mathcal{T}_G$ is a theory corresponding to a collection of restricted (i, j, m, n) convergency axioms, then there is a Horn relational theory $\mathrm{N}(\mathcal{T})$ conservatively extending it.

Some correspondences

Property	(i, j, m, n)	Axiom schema	Horn relational rule
Seriality	$(0,0,1,1)$	D: $\Box A \supset \Diamond A$	$\dfrac{}{x\,R\,f(x)}\ ser$
Reflexivity	$(0,0,1,0)$	T: $\Box A \supset A$	$\dfrac{}{x\,R\,x}\ refl$
Symmetry	$(0,1,0,1)$	B: $A \supset \Box\Diamond A$	$\dfrac{x\,R\,y}{y\,R\,x}\ symm$
Transitivity	$(0,2,1,0)$	4: $\Box A \supset \Box\Box A$	$\dfrac{x\,R\,y \quad y\,R\,z}{x\,R\,z}\ trans$
Euclideaness	$(1,1,0,1)$	5: $\Diamond A \supset \Box\Diamond A$	$\dfrac{x\,R\,y \quad x\,R\,z}{z\,R\,y}\ eucl$
Convergency	$(1,1,1,1)$	2: $\Diamond\Box A \supset \Box\Diamond A$	$\dfrac{x\,R\,y \quad x\,R\,z}{y\,R\,g(x,y,z)}\ conv1$ $\quad \dfrac{x\,R\,y \quad x\,R\,z}{z\,R\,g(x,y,z)}\ conv2$
Contextuality	$(1,2,1,0)$	Cxt: $\Diamond\Box A \supset \Box\Box A$	$\dfrac{x\,R\,y \quad x\,R\,z \quad z\,R\,w}{y\,R\,w}\ cxt$
Density	$(0,1,2,0)$	X: $\Box\Box A \supset \Box A$	$\dfrac{x\,R\,y}{x\,R\,h(x,y)}\ dens1$ $\quad \dfrac{x\,R\,y}{h(x,y)\,R\,y}\ dens2$

f, g and h are (Skolem) function constants.

Some correspondences (cont.)

Property	*Axiom schema*	*Horn relational rule*
Weak reflexivity	$\Box(\Box A \supset A)$	$\dfrac{w\,R\,x}{x\,R\,x}\ \mathit{wrefl}$
Weak symmetry	$\Box(A \supset \Box\Diamond A)$	$\dfrac{w\,R\,x \quad x\,R\,y}{y\,R\,x}\ \mathit{wsymm}$
Weak transitivity	$\Box(\Box A \supset \Box\Box A)$	$\dfrac{w\,R\,x \quad x\,R\,y \quad y\,R\,z}{x\,R\,z}\ \mathit{wtrans}$
Weak euclideaness	$\Box(\Diamond A \supset \Box\Diamond A)$	$\dfrac{w\,R\,x \quad x\,R\,y \quad x\,R\,z}{z\,R\,y}\ \mathit{weucl}$

Relational theory $\mathrm{N}(\mathcal{T})$ (extensions of $\mathrm{N}(\mathrm{K})$)

- Various combinations of Horn relational rules define labelled ND systems for common propositional modal logics.
- The labelled ND system $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$ for the propositional modal logic $\mathcal{L}$ is obtained by extending $\mathrm{N}(\mathrm{K})$ with a Horn relational theory $\mathrm{N}(\mathcal{T})$.

  $\mathrm{N}(\mathcal{T})$ is a collection of relational rules:

$$\frac{x_1 \, R \, y_1 \quad \cdots \quad x_m \, R \, y_m}{x_0 \, R \, y_0}$$

$$\begin{tikzcd}
 & \mathrm{S5} & \\
 & \mathrm{S4} \arrow[u, "symm"'] & \\
\mathrm{T} \arrow[ur, "trans"] & \mathrm{D} & \mathrm{K4} \arrow[ul, "refl"'] \\
 & \mathrm{K} \arrow[ul, "refl"] \arrow[u, "ser"'] \arrow[ur, "trans"'] &
\end{tikzcd}$$

  Examples:

  - $\mathrm{N}(\mathrm{S4}) = \mathrm{N}(\mathrm{K}) \; + \; \dfrac{}{x \, R \, x} \, \mathit{refl} \; + \; \dfrac{x \, R \, y \quad y \, R \, z}{x \, R \, z} \, \mathit{trans}$
  - $\mathrm{N}(\mathrm{D}) = \mathrm{N}(\mathrm{K}) \; + \; \dfrac{}{x \, R \, f(x)} \, \mathit{ser}$

Derivations

- A derivation of an lwff or rwff φ from a set of lwffs Γ and a set of rwffs Δ in a ND system $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$ is a tree formed using the rules in $\mathrm{N}(\mathcal{L})$, ending with φ and depending only on $\Gamma \cup \Delta$.

- We write $\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} \varphi$.

- A derivation of φ in $\mathrm{N}(\mathcal{L})$ depending on the empty set, $\vdash_{\mathrm{N}(\mathcal{L})} \varphi$, is a proof of φ in $\mathrm{N}(\mathcal{L})$ (φ is a $\mathrm{N}(\mathcal{L})$-theorem).

  Fact: When φ is an rwff $x\,R\,y$ we have:

  1. $\Gamma, \Delta \vdash_{\mathrm{N}(\mathrm{K})} x\,R\,y$ iff $x\,R\,y \in \Delta$.
  2. $\Gamma, \Delta \vdash_{\mathrm{N}(\mathrm{K})+\mathrm{N}(\mathcal{T})} x\,R\,y$ iff $\Delta \vdash_{\mathrm{N}(\mathcal{T})} x\,R\,y$.

Examples of derivations

- $\mathrm{N(S5)} = \mathrm{N(KT5)} = \mathrm{N(KTB4)} = \mathrm{N(KT45)}$

$$\dfrac{\begin{array}{c}\Pi\\ x\,R\,y\end{array}}{y\,R\,x}\,\mathit{symm} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\Pi\\ x\,R\,y\end{array} \quad \dfrac{}{x\,R\,x}\,\mathit{refl}}{y\,R\,x}\,\mathit{eucl}$$

$$\dfrac{\begin{array}{c}\Pi_1\\ x\,R\,y\end{array} \quad \begin{array}{c}\Pi_2\\ y\,R\,z\end{array}}{x\,R\,z}\,\mathit{trans} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x\,R\,y\end{array} \quad \dfrac{}{x\,R\,x}\,\mathit{refl}}{y\,R\,x}\,\mathit{eucl} \quad \begin{array}{c}\Pi_2\\ y\,R\,z\end{array}}{x\,R\,z}\,\mathit{eucl}$$

Examples of derivations (cont.)

- Derived rules

$$\dfrac{y{:}A \quad x\,R\,y}{x{:}\Diamond A}\,\Diamond\mathrm{I} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{[x{:}\Box \sim A]^1 \quad x\,R\,y}{y{:}\sim A}\,\Box\mathrm{E} \quad y{:}A}{y{:}\bot}\,{\sim}\mathrm{E}}{x{:}\bot}\,\bot\mathrm{E}}{x{:}\sim\Box\sim A}\,{\sim}\mathrm{I}^1$$

$$\dfrac{x{:}\Diamond A \quad \begin{matrix}[y{:}A]\ [x\,R\,y]\\ \Pi\\ z{:}B\end{matrix}}{z{:}B}\,\Diamond\mathrm{E} \quad\rightsquigarrow\quad \dfrac{\dfrac{x{:}\sim\Box\sim A \quad \dfrac{\dfrac{\dfrac{\dfrac{[z{:}B\supset\bot]^3 \quad \begin{matrix}[y{:}A]^1\ [x\,R\,y]^2\\ \Pi\\ z{:}B\end{matrix}}{z{:}\bot}\,{\supset}\mathrm{E}}{y{:}\bot}\,\bot\mathrm{E}}{y{:}\sim A}\,{\sim}\mathrm{I}^1}{x{:}\Box\sim A}\,\Box\mathrm{I}^2}{x{:}\bot}\,{\sim}\mathrm{E}}{z{:}B}\,\bot\mathrm{E}^3$$

Properties of $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$

- Γ a set of labelled formulas, Δ a set of relational formulas.
- Parameterized proofs of
  - Soundness and completeness with respect to Kripke semantics

$$\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} \varphi \quad \Leftrightarrow \quad \Gamma, \Delta \models \varphi$$

  - Faithfulness and adequacy of the implementation

$$\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} \varphi \quad \Leftrightarrow \quad \Gamma, \Delta \vdash \varphi \text{ in } \mathtt{Isabelle}_{\mathrm{N}(\mathcal{L})}$$

- Proof search: normalization and subformula property $\begin{array}{c}\Gamma\ \Delta \\ \vdots\,? \\ \varphi\end{array}$

  Proof is 'normal' (well-defined structure) and contains only subformulas.
  $\Rightarrow$ Restricted proof search.
  $\Rightarrow$ Decidability, complexity? (new proof-theoretical method based on substructural analysis).

Kripke semantics

- A (Kripke) frame for $\mathrm{N}(\mathcal{L})$ is a pair $(\mathfrak{W}, \mathfrak{R})$, where $\mathfrak{W}$ is a non-empty set of worlds and $\mathfrak{R} \subseteq \mathfrak{W} \times \mathfrak{W}$.
- A (Kripke) model for $\mathrm{N}(\mathcal{L})$ is a triple $\mathfrak{M} = (\mathfrak{W}, \mathfrak{R}, \mathfrak{V})$, where
  - $(\mathfrak{W}, \mathfrak{R})$ is a frame for $\mathrm{N}(\mathcal{L})$.
  - The valuation $\mathfrak{V}$ maps an element of $\mathfrak{W}$ and a propositional variable to a truth value (0 or 1).
- Truth for an rwff or lwff φ in a model $\mathfrak{M}$, $\models^{\mathfrak{M}} \varphi$, is the smallest relation $\models^{\mathfrak{M}}$ satisfying:

$$
\begin{array}{lll}
\models^{\mathfrak{M}} x\,R\,y & \text{iff} & (x, y) \in \mathfrak{R} \\
\models^{\mathfrak{M}} x{:}p & \text{iff} & \mathfrak{V}(x, p) = 1 \\
\models^{\mathfrak{M}} x{:}A \supset B & \text{iff} & \models^{\mathfrak{M}} x{:}A \text{ implies } \models^{\mathfrak{M}} x{:}B \\
\models^{\mathfrak{M}} x{:}\Box A & \text{iff} & \text{for all } y,\ \models^{\mathfrak{M}} x\,R\,y \text{ implies } \models^{\mathfrak{M}} y{:}A
\end{array}
$$

Soundness and completeness of $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$

Theorem: $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$ is sound and complete.

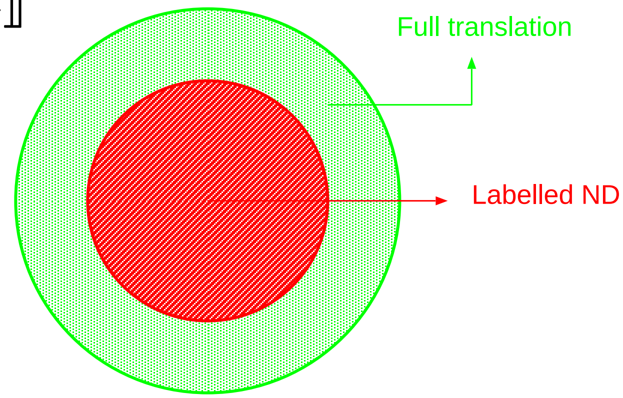
- For Γ a set of labelled formulas, Δ a set of relational formulas, we have

  1. $\Delta \vdash_{\mathrm{N}(\mathcal{L})} x\,R\,y$ iff $\Delta \models x\,R\,y$
  2. $\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} x{:}A$ iff $\Gamma, \Delta \models x{:}A$.

- Proof is parameterized over $\mathrm{N}(\mathcal{T})$.

  - Soundness: By induction on the structure of the derivations.
  - Completeness: By a modified canonical model construction that accounts for the explicit formalization of labels and of the relations between them.

Translations (full vs. partial)



- Full translation: $[\![x{:}\Box A]\!] \rightsquigarrow \forall y.\ x\,R\,y \supset [\![y{:}A]\!]$
  Transitivity: $\forall x.y.z.\ x\,R\,y \wedge y\,R\,z \supset x\,R\,z$

  $+$ generality
  $-$ structure: relations mingled with formulas

- Labelled natural deduction: partial translation

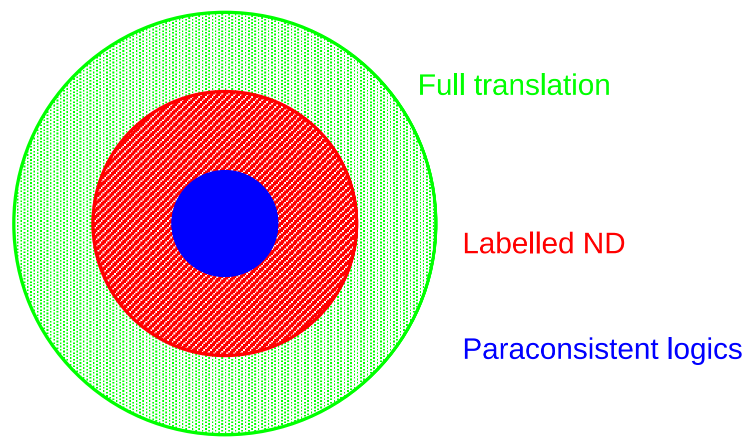
$$\dfrac{\begin{array}{c}[x\,R\,y]\\ \vdots \\ y{:}A\end{array}}{x{:}\Box A}\,\Box\mathrm{I}\quad [y \text{ fresh}] \qquad\qquad \dfrac{x\,R\,y \quad y\,R\,z}{x\,R\,z}\,\mathit{trans}$$

  $-$ less general (but large and extensible)
  $+$ structure (separation)
    rwffs derived from rwffs alone
    lwffs derived from lwffs and rwffs

Extensions and restrictions

Reason about propagation of inconsistency

$\Rightarrow$ vary interface between $\mathrm{N}(\mathrm{K})$ and $\mathrm{N}(\mathcal{T})$.



$$\dfrac{\begin{array}{c}[x{:}A \supset \bot]\\ \vdots \\ y{:}\bot\end{array}}{x{:}A}\,\bot\mathrm{E} \qquad \dfrac{y{:}\bot}{x\,R\,z} \quad \text{and} \quad \dfrac{\neg(x\,R\,z) \quad x\,R\,z}{y{:}\bot} \qquad \dfrac{\begin{array}{c}[x{:}A \supset \bot]\\ \vdots \\ x{:}\bot\end{array}}{x{:}A}$$

$\Rightarrow$ give up some of the properties, e.g. structure, completeness.

Proof search: Normalization and subformula property

- Structure: $\begin{array}{cc} \Gamma & \Delta \\ \vdots & ? \\ \alpha & \end{array}$

- Theorem: Every derivation of $x{:}A$ from Γ, Δ in $\mathrm{N(K)} + \mathrm{N}(\mathcal{T})$ reduces to a derivation in normal form.

  "no detours or irrelevancies"

  example:
$$\dfrac{\dfrac{\begin{array}{c}[x\,R\,y] \\ \Pi \\ y{:}A\end{array}}{x{:}\Box A}\,\Box\mathrm{I} \qquad x\,R\,z}{z{:}A}\,\Box\mathrm{E} \quad \text{reduces to} \quad \begin{array}{c} x\,R\,z \\ \Pi[z/y] \\ z{:}A \end{array}$$

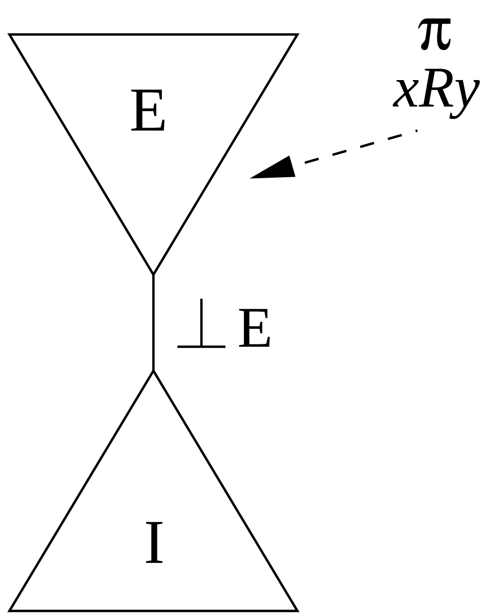
- Corollary: Normal derivations in $\mathrm{N(K)} + \mathrm{N}(\mathcal{T})$ satisfy a subformula property.

$\Rightarrow$ Restricted proof search.

$\Rightarrow$ Decidability, complexity?

Proof search: Tracks

- Thread in a derivation Π in $\mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$: a sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that (i) φ_1 is an assumption of Π, (ii) φ_i stands immediately above φ_{i+1}, for $1 \leq i < n$, and (iii) φ_n is the conclusion of Π.
- Lwff-thread: a thread where $\varphi_1, \ldots, \varphi_n$ are all lwffs.
- Track: initial part of an lwff-thread in Π which stops either at the first minor premise of an elimination rule in the lwff-thread or at the conclusion of the lwff-thread.
- Corollary: The form of tracks in a normal derivation of an lwff in $\mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$ is



A topography of labelled modal logics

3 approaches to falsum:

LOCAL	GLOBAL	UNIVERSAL
"paraconsistent" (modal?) logics	large and well-known class of modal logics (K,T,D,B,S4,S5, ...)	first-order axiomatizable modal logics
inadequate	separation	equivalent to semantic embedding

- Up to now we have used global falsum:

$$\dfrac{\begin{array}{c}[x{:}A \supset \bot] \\ \vdots \\ y{:}\bot\end{array}}{x{:}A}\ \bot E \qquad \equiv \qquad \dfrac{x{:}\bot}{y{:}\bot}\ gf$$

- Falsum propagates between worlds.

$\Rightarrow$ unidirectional interface between $\mathrm{N}(\mathrm{K})$ and $\mathrm{N}(\mathcal{T})$:

  * Rwffs derived from rwffs alone.
  * Lwffs derived from lwffs and rwffs ($\Box E$).

Classes of labelled modal logics

By changing:

- Labelling algebra
  - Different Horn relational theories. $\surd$
  - First-order relational theories, e.g. $\forall x(\sim (R\,x\,x\,))$.
  - Higher-order relational theories.
- Interface
  - Unidirectional. $\surd$
  - Bidirectional.
- Base system
  - Extension: $\mathrm{N}(\mathrm{K}^{uf}) = \mathrm{N}(\mathrm{K})$ with universal falsum.
  - Narrowing: $\mathrm{N}(\mathrm{K}^{lf}) = \mathrm{N}(\mathrm{K})$ with local falsum.

First-order relational theories $\mathrm{N}(\mathcal{T}_{\mathrm{F}}) = \mathrm{N}_R + \mathcal{C}_R$

- N_R: first-order ND system of R

$$\dfrac{\begin{array}{c}[\rho \sqsupset \emptyset] \\ \vdots \\ \emptyset\end{array}}{\rho}\,\emptyset\mathrm{E} \qquad \dfrac{\begin{array}{c}[\rho_1] \\ \vdots \\ \rho_2\end{array}}{\rho_1 \sqsupset \rho_2}\sqsupset\mathrm{I} \qquad \dfrac{\rho_1 \sqsupset \rho_2 \quad \rho_1}{\rho_2}\sqsupset\mathrm{E} \qquad \dfrac{\rho}{\prod x(\rho)}\prod\mathrm{I} \qquad \dfrac{\prod x(\rho)}{\rho[t/x]}\prod\nolimits_{\mathrm{E}}$$

In $\prod\mathrm{I}$, x must not occur free in any open assumption on which ρ depends.

- $\mathcal{C}_R$: collection of rules for relational properties

$$\dfrac{}{\prod x(-\,(x\,R\,x))}\,\mathit{irrefl}$$

$$\dfrac{}{\prod x \prod y \prod z((x\,R^i\,y \sqcap x\,R^j\,z) \sqsupset \coprod u(y\,R^m\,u \sqcap z\,R^n\,u))}\,\mathit{rconv}$$

A problem, the cause, and a solution

- **A problem:**
  - Theorem There are systems $\mathrm{N(K)} + \mathrm{N}(\mathcal{T}_\mathrm{F})$ with $\mathrm{N}(\mathcal{T}_\mathrm{F}) = \mathrm{N}_R + \mathcal{C}_R$ that are incomplete with respect to the corresponding Kripke models with accessibility relation defined by a collection $\mathcal{C}_R$ of first-order axioms.
  - Example: $\mathrm{N}(\mathcal{T}_\mathrm{F}) = \mathrm{N}_R +$

$$\{ \; \prod x \prod y \prod z((x\,R\,y \sqcap x\,R\,z) \sqsupset (y\,R\,z \sqcup z\,R\,y)) \; \}$$

Normalization $\Rightarrow \nvdash_{\mathrm{N(K)}+\mathrm{N}(\mathcal{T}_\mathrm{F})} \mathbf{3}$, i.e.

$$\nvdash_{\mathrm{N(K)}+\mathrm{N}(\mathcal{T}_\mathrm{F})} \sim\Box(\Box A \supset B) \supset \Box(\Box B \supset A)$$

A problem, the cause, and a solution (cont.)

Normalization $\Rightarrow \nvdash_{\mathrm{N(K)+N}(\mathcal{T}_{\mathrm{F}})} 3$, i.e.

$$\nvdash_{\mathrm{N(K)+N}(\mathcal{T}_{\mathrm{F}})} \sim\Box(\Box A \supset B) \supset \Box(\Box B \supset A)$$

since

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x{:}\sim\Box(\Box A \supset B)]^1 \qquad \dfrac{\dfrac{\dfrac{[y{:}\Box B]^3 \qquad \begin{array}{c}[x\,R\,y]^2\ [y{:}A \supset \bot]^4\ [x\,R\,z]^5\ [z{:}\Box A]^6 \\ \Pi_1 \\ y\,R\,z\end{array}}{z{:}B}\,\Box\mathrm{E}}{z{:}\Box A \supset B}\,{\supset}\mathrm{I}^6}{x{:}\Box(\Box A \supset B)}\,\Box\mathrm{I}^5}{x{:}\bot}\,{\sim}\mathrm{E}}{y{:}A}\,\bot\mathrm{E}^4}{y{:}\Box B \supset A}\,{\supset}\mathrm{I}^3}{x{:}\Box(\Box B \supset A)}\,\Box\mathrm{I}^2}{x{:}\sim\Box(\Box A \supset B) \supset \Box(\Box B \supset A)}\,{\supset}\mathrm{I}^1
$$

but

$$x\,R\,y, x\,R\,z \nvdash y\,R\,z \text{ in } \mathrm{N}_R + \mathcal{C}_R$$

A problem, the cause, and a solution (cont.)

- **A problem:**
  - Theorem There are systems $\mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T}_{\mathrm{F}})$ with $\mathrm{N}(\mathcal{T}_{\mathrm{F}}) = \mathrm{N}_R + \mathcal{C}_R$ that are incomplete with respect to the corresponding Kripke models with accessibility relation defined by a collection $\mathcal{C}_R$ of first-order axioms.
  - Example: $\mathrm{N}(\mathcal{T}_{\mathrm{F}}) = \mathrm{N}_R +$

$$\{ \ \prod x \prod y \prod z ((x\,R\,y \sqcap x\,R\,z) \sqsupset (y\,R\,z \sqcup z\,R\,y)) \ \}$$

Normalization $\Rightarrow \nvdash_{\mathrm{N}(\mathrm{K})+\mathrm{N}(\mathcal{T}_{\mathrm{F}})} \mathbf{3}$, i.e.

$$\nvdash_{\mathrm{N}(\mathrm{K})+\mathrm{N}(\mathcal{T}_{\mathrm{F}})} \sim \Box(\Box A \supset B) \supset \Box(\Box B \supset A)$$

  - But: property corresponds to axiom 3!

A problem, the cause, and a solution (cont.)

- **The cause:** global falsum is not enough!
  - Falsum must propagate between base system and labelling algebra. $\Rightarrow$ Bidirectional interface:

$$\dfrac{\dfrac{\dfrac{[y{:}A\supset\bot]^4 \qquad \dfrac{[z{:}\Box A]^6 \qquad \begin{array}{c}[x\,R\,y]^2\ [x\,R\,z]^5\ [y\,R\,z \sqsupset \emptyset]^7\\ \vdots\\ z\,R\,y\end{array}}{y{:}A}\,\Box\mathrm{E}}{y{:}\bot}\,\supset\mathrm{E}}{\emptyset}\,(r)}{y\,R\,z}\,\emptyset\,\mathrm{E}^7$$

since $x\,R\,y, x\,R\,z, y\,R\,z \sqsupset \emptyset \vdash z\,R\,y$ in $\mathrm{N}_R + \mathcal{C}_R$.

A problem, the cause, and a solution (cont.)

- **The cause:** global falsum is not enough!
  - Falsum must propagate between base system and labelling algebra. $\Rightarrow$ Bidirectional interface:

$$\dfrac{\dfrac{\dfrac{[y{:}A \supset \bot]^4 \qquad \dfrac{[z{:}\Box A]^6 \qquad \begin{array}{c}[x\,R\,y]^2\ \ [x\,R\,z]^5\ \ [y\,R\,z \sqsupset \emptyset]^7\\ \vdots\\ z\,R\,y\end{array}}{y{:}A}\,\Box\mathrm{E}}{y{:}\bot}\,\supset\mathrm{E}}{\emptyset}\,(r)}{y\,R\,z}\,\emptyset\,\mathrm{E}^7$$

since $x\,R\,y, x\,R\,z, y\,R\,z \sqsupset \emptyset \vdash z\,R\,y$ in $\mathrm{N}_R + \mathcal{C}_R$.

- **A solution:** collapse $\bot$ and $\emptyset$ (universal falsum)

$$\mathrm{N}(\mathrm{K}^{uf}) \;=\; \mathrm{N}(\mathrm{K}) \;+\; \frac{x{:}\bot}{\emptyset}\,uf_1 \;+\; \frac{\emptyset}{x{:}\bot}\,uf_2$$

But: we lose the separation between the 2 parts of the deduction system.

Universal falsum $\cong$ semantic embedding

Theorem: In $\mathrm{N}(\mathrm{K}^{uf}) + \mathrm{N}(\mathcal{T}_{\mathrm{F}})$ the two parts of the deduction system are not separated: derivations of lwffs can depend on derivations of rwffs, and vice versa.

Universal falsum $\cong$ semantic embedding (cont.)

- In fact, $\mathrm{N}(\mathrm{K}^{uf}) + \mathrm{N}(\mathcal{T}_\mathrm{F})$, unlike $\mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$, is essentially equivalent to the usual semantic embedding of propositional modal logics in first-order logic.
  - Translation $[\![\cdot]\!]$ of formulas of $\mathrm{N}(\mathrm{K}^{uf}) + \mathrm{N}(\mathcal{T}_\mathrm{F})$ into formulas of first-order logic:

$$\begin{array}{rcl@{\qquad}rcl}
[\![\emptyset]\!] & \leadsto & \bot\,; & [\![x{:}\bot]\!] & \leadsto & \bot\,; \\
[\![x\,R\,y]\!] & \leadsto & R(x,y)\,; & [\![x{:}p]\!] & \leadsto & P(x)\,; \\
[\![\rho_1 \sqsupset \rho_2]\!] & \leadsto & [\![\rho_1]\!] \supset [\![\rho_2]\!]\,; & [\![x{:}A \supset B]\!] & \leadsto & [\![x{:}A]\!] \supset [\![x{:}B]\!]\,; \\
[\![\textstyle\prod x(\rho)]\!] & \leadsto & \forall x([\![\rho]\!])\,; & [\![x{:}\Box A]\!] & \leadsto & \forall y(R(x,y) \supset [\![y{:}A]\!])\,; \\
[\![\Delta]\!] & \leadsto & \{[\![\rho]\!] \mid \rho \in \Delta\}\,; & [\![\Gamma]\!] & \leadsto & \{[\![x{:}A]\!] \mid x{:}A \in \Gamma\}\,.
\end{array}$$

  - The following are then equivalent:
    1. $\Gamma, \Delta \vdash \varphi$ in $\mathrm{N}(\mathrm{K}^{uf}) + \mathrm{N}_R + \mathcal{C}_R$.
    2. $\mathcal{C}_R, [\![\Gamma]\!], [\![\Delta]\!] \vdash [\![\varphi]\!]$ in (the ND system for) first-order logic.

Local falsum: Paraconsistent modal logics

$\mathrm{N}(\mathrm{K}^{lf})$ is $\mathrm{N}(\mathrm{K})$ with $\bot\mathrm{E}$ restricted so that falsum is local and cannot move arbitrarily between worlds:

$$\dfrac{\begin{array}{c}[x{:}A \supset \bot] \\ \vdots \\ x{:}\bot\end{array}}{x{:}A}\ lf$$

Local falsum: Paraconsistent modal logics

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$$\dfrac{\begin{array}{c}[x{:}A \supset \bot] \\ \vdots \\ x{:}\bot\end{array}}{x{:}A}\,lf$$

lf propagates $\bot$ forward indirectly (and backward, when R symmetric):

$$\dfrac{\dfrac{x{:}\bot}{x{:}\Box\bot}\,lf \qquad x\,R\,y}{y{:}\bot}\,\Box\mathrm{E}$$

but not to an arbitrary world: $x{:}\bot \nvdash_{\mathrm{N}(\mathrm{K}^{lf})} y{:}\bot$

$\Rightarrow$ $\Box$ and $\Diamond$ are not interdefinable in $\mathrm{N}(\mathrm{K}^{lf})$!

They are not even ‘intuitionistically’ related (e.g. $\Box\sim A$ does not imply $\sim\Diamond A$).

Local falsum: Paraconsistent modal logics (cont.)

$\Rightarrow$ $\mathrm{N}(\mathrm{K}^{lf})$ in general not suitable for formalizing modal logics.

- Only certain logics (e.g. if R *universal* as for $\mathrm{S5}$ — where $x\,R\,y$ for all x, y).

- But: resulting formalization is unsatisfactory, since it lacks important metatheoretical properties that we get in $\mathrm{N}(\mathrm{K})$.

- Proposition: Derivations in $\mathrm{N}(\mathrm{K}^{lf})$ do not have normal forms satisfying the subformula property.

  For example:

$$\dfrac{\dfrac{x{:}\bot}{x{:}\Box\bot}\,lf \qquad x\,R\,y}{y{:}\bot}\,\Box\mathrm{E}$$

- Can be fixed, but ...

Summary

- A labelled deduction framework for (propositional) modal logics.
  - Labelled (natural) deduction systems: uniform and modular.
  - Structural properties vs. generality.
  - Structure $\Rightarrow$ implementation, decidability, complexity.

Falsum	Base system	Interface	Labelling algebra	Presentation
local	$\mathrm{N}(\mathrm{K}^{lf})$	unidirectional (only $\Box\mathrm{E}$)	separate $\mathrm{N}(\mathcal{T})$	inadequate
global	$\mathrm{N}(\mathrm{K})$	unidirectional ($\Box\mathrm{E} + \bot\mathrm{E}$)	separate $\mathrm{N}(\mathcal{T})$ separate $\mathrm{N}(\mathcal{T}_{\mathrm{F}})$	complete incomplete
universal	$\mathrm{N}(\mathrm{K}^{uf})$	bidirectional	$\mathrm{N}(\mathcal{T}_{\mathrm{F}})$, NOT separate	complete BUT equivalent to semantic embedding

- Other non-classical logics: extensions and restrictions (but there are limits).

Extension to quantified modal logics

- Two degrees of freedom:
  - Properties of the accessibility relation (as in propositional case).
  - How the domains of individuals change between worlds: varying, increasing, decreasing, or constant domains.
- Hence: extend fixed base ND system $\mathrm{N(QK)}$ with relational theory (as before) and with domain theory formalizing the behavior of the domains of quantification.
  - Introduce labelled terms $w{:}t$ expressing the existence of the term t at world w.
  - Adopt quantifier rules similar to those of *free logic*

$$\dfrac{\begin{array}{c}[w{:}t]\\ \vdots\\ w{:}A[t/x]\end{array}}{w{:}\forall x(A)}\,\forall\mathrm{I} \quad \text{and} \quad \dfrac{w{:}\forall x(A) \quad w{:}t}{w{:}A[t/x]}\,\forall\mathrm{E}$$

where, in $\forall\mathrm{I}$, t does not occur in any assumption on which $w{:}A[t/x]$ depends other than $w{:}t$.

Generalization to non-classical logics

- Modal logics $\rightsquigarrow$ non-classical logics.
  Unary $\Box$ with binary R $\rightsquigarrow$ n-ary modality $\mathcal{M}$ with $n+1$-ary relation R

$$\dfrac{\begin{array}{c}[x\,R\,y]\\ \vdots\\ y{:}A\end{array}}{x{:}\Box A}\,\Box\mathrm{I}\ [y\text{ fresh}] \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}[x_1{:}A_1]\ldots[x_{n-1}{:}A_{n-1}]\,[R\,x\,x_1\ldots x_n]\\ \vdots\\ x_n{:}A_n\end{array}}{x{:}\mathcal{M}A_1\ldots A_n}\,\mathcal{M}\mathrm{I}\ [x_1,\ldots,x_n\text{ fresh}]$$

$$\dfrac{x{:}\Box A \quad x\,R\,y}{y{:}A}\,\Box\mathrm{E} \quad\rightsquigarrow\quad \dfrac{x{:}\mathcal{M}A_1\ldots A_n \quad x_1{:}A_1 \cdots x_{n-1}{:}A_{n-1} \quad R\,x\,x_1\ldots x_n}{x_n{:}A_n}\,\mathcal{M}\mathrm{E}$$

- Example: relevance logics, binary $\rightarrow$ with ternary R

$$\dfrac{\begin{array}{c}[y{:}A]\,[R\,x\,y\,z]\\ \vdots\\ z{:}B\end{array}}{x{:}A\rightarrow B}\rightarrow\mathrm{I}\ [y,z\text{ fresh}] \qquad \dfrac{x{:}A\rightarrow B \quad y{:}A \quad R\,x\,y\,z}{z{:}B}\rightarrow\mathrm{E}$$

Road Map

- Introduction: A framework for non-classical logics.
- Labelled deduction for modal logics.
- Labelled deduction for non-classical logics.
  - ▶ Propositional relevance logics and quantified modal logics.
    - ∗ Labelled deduction systems: uniform and modular.
    - ∗ Properties: soundness, completeness, normalization, ...
    - ∗ A first step towards the combination of non-classical logics.
- Encoding non-classical logics in Isabelle.
- Substructural and complexity analysis of labelled non-classical logics.
- Conclusions and outlook.

A labelling recipe for non-classical logics

- We have seen labelled presentations of propositional modal logics:
  - The deduction machinery is minimal (a minimal fragment of first-order logic).
  - Derivations are strictly separated.
  - Derivations normalize and satisfy a subformula property.

- We will now see a recipe to present non-classical logics in an analogous way:
  - Introduce labelling.
  - Give ND rules for the operators, distinguishing 'local' and 'non-local' ones.
  - Introduce quantifiers.

$\Rightarrow$ labelled ND presentations with 'good' properties.

Labelled deduction for propositional non-classical logics

We distinguish local and non-local logical operators.

- The truth of $a{:}\mathcal{O}A_1 \ldots A_n$, where $\mathcal{O}$ is a local operator, depends only on the local truth of $a{:}A_1, \ldots, a{:}A_n$.
- Examples: $\supset$, $\wedge$, $\vee$, $\sim$, ...

$$
\begin{array}{lll}
\models^{\mathfrak{M}} a{:}A \wedge B & \text{iff} & \models^{\mathfrak{M}} a{:}A \text{ and } \models^{\mathfrak{M}} a{:}B; \\
\models^{\mathfrak{M}} a{:}A \vee B & \text{iff} & \models^{\mathfrak{M}} a{:}A \text{ or } \models^{\mathfrak{M}} a{:}B; \\
\models^{\mathfrak{M}} a{:}A \supset B & \text{iff} & \models^{\mathfrak{M}} a{:}A \text{ implies } \models^{\mathfrak{M}} a{:}B; \\
\models^{\mathfrak{M}} a{:} \sim A & \text{iff} & \not\models^{\mathfrak{M}} a{:}A.
\end{array}
$$

Non-local operators

- A non-local operator $\mathcal{M}$ is associated with an $n+1$–ary relation R on worlds

$\Rightarrow$ truth of $a{:}\mathcal{M}A_1 \ldots A_n$ is evaluated non-locally at worlds R-accessible from a

i.e. in terms of the truth of $a_1{:}A_1, \ldots, a_n{:}A_n$ where $R\, a\, a_1 \ldots a_n$.

- Examples:
  - unary $\Box$ (and $\Diamond$) and binary R,
  - binary relevant $\rightarrow$ and ternary compossibility relation R,
  - (binary intuitionistic $\rightarrow$ and binary partial order $R = \sqsubseteq$),
  - ...

- We extend $\models^{\mathfrak{M}}$ so that: $\models^{\mathfrak{M}} R\, a\, a_1 \ldots a_n$ iff $(a, a_1, \ldots, a_n) \in \mathfrak{R}$

and distinguish 'universal' and 'existential' non-local operators.

Non-local operators (cont.)

- $\mathcal{M}$ is a universal non-local operator when the metalevel quantification in the evaluation clause is universal (and the body is an implication):

$$\models^{\mathfrak{M}} a{:}\mathcal{M}A_1 \ldots A_n \text{ iff } \begin{array}{l} \text{for all } a_1, \ldots, a_n \\ ((\models^{\mathfrak{M}} R\, a\, a_1 \ldots a_n \text{ and } \models^{\mathfrak{M}} a_1{:}A_1 \text{ and} \ldots \text{and } \models^{\mathfrak{M}} a_{n-1}{:}A_{n-1}) \\ \text{ imply } \models^{\mathfrak{M}} a_n{:}A_n) \end{array}$$

  Examples:

$$\models^{\mathfrak{M}} a{:}\Box A_1 \text{ iff for all } a_1\, (\models^{\mathfrak{M}} R\, a\, a_1 \text{ implies } \models^{\mathfrak{M}} a_1{:}A_1)$$
$$\models^{\mathfrak{M}} a{:}A_1 \rightarrow A_2 \text{ iff for all } a_1, a_2\, ((\models^{\mathfrak{M}} R\, a\, a_1\, a_2 \text{ and } \models^{\mathfrak{M}} a_1{:}A_1) \text{ imply } \models^{\mathfrak{M}} a_2{:}A_2)$$

- $\mathcal{M}$ is an existential non-local operator when the metalevel quantification in the evaluation clause is existential (and the body is a conjunction):

$$\models^{\mathfrak{M}} a{:}\mathcal{M}A_1 \ldots A_n \text{ iff } \begin{array}{l} \text{there exist } a_1, \ldots, a_n \\ (\models^{\mathfrak{M}} R\, a\, a_1 \ldots a_n \text{ and } \models^{\mathfrak{M}} a_1{:}A_1 \text{ and} \ldots \text{and } \models^{\mathfrak{M}} a_{n-1}{:}A_{n-1} \\ \text{ and } \models^{\mathfrak{M}} a_n{:}A_n) \end{array}$$

  Example: $\models^{\mathfrak{M}} a{:}\Diamond A_1$ iff there exists $a_1\, (\models^{\mathfrak{M}} R\, a\, a_1$ and $\models^{\mathfrak{M}} a_1{:}A_1)$.

Non-local negation

- In relevance (and other) logics, both a formula and its 'negation' may be true at a world.
- This cannot be the case with $\sim$.
  $\Rightarrow$ Introduce a new operator: non-local negation $\neg$ is formalized by a unary function $*$ on worlds

$$\models^{\mathfrak{M}} a{:}\neg A \text{ iff } \not\models^{\mathfrak{M}} a^*{:}A$$

  Informally: a^* is the world that does not deny what a asserts, i.e. a and a^* are compatible worlds.
- We generalize this to

$$\models^{\mathfrak{M}} a{:}\neg A \text{ iff for all } b\,(\models^{\mathfrak{M}} a^*{:}A \text{ implies } \models^{\mathfrak{M}} b{:}\perp\!\!\!\perp)$$

  where the constant $\perp\!\!\!\perp$ expresses incoherence of compatible worlds, i.e. $\not\models^{\mathfrak{M}} b{:}\perp\!\!\!\perp$ for every world b.

On negation and incoherence

- Equivalent to approaches based on incompatibility relation N between worlds:

$$\models^{\mathfrak{M}} a{:}\neg A \ \text{ iff } \ \text{for all } b(\models^{\mathfrak{M}} b{:}A \text{ implies } b\ N\ a)$$

$\Rightarrow$ a^* is the 'strongest' world b for which $b\ N\ a$ does *not* hold

- Given relevant implication, we can define

$$a{:}\neg A \ \text{ as } \ a{:}A \rightarrow \perp\!\!\!\perp$$

and postulate that for every b

$$R\, a\, a^*\, b$$

That a and a^* are 'compossible' according to every b is justified by the meaning of $*$.

- When $a = a^*$:

$$\begin{array}{ccc} \perp\!\!\!\perp & \text{reduces to} & \perp \\ \neg & \text{reduces to} & \sim \end{array}$$

Language of a non-classical logic $\mathcal{L}$ (and of $\mathrm{N}(\mathcal{L})$)

$(W, *, S, O, F)$

- W is a set of labels closed under $*$ of type $W \supset W$.
  (We assume that $0 \in W$ is a label denoting the actual world $\mathfrak{o}$.)
- S a denumerably infinite set of propositional variables.
- O is the set whose members are
  - the constant $\perp\!\!\!\perp$ (and/or $\perp$);
  - local and/or non-local negation (or neither for positive logics);
  - a set of local operators $\mathcal{C}_1, \mathcal{C}_2, \ldots$; and
  - a set of non-local operators $\mathcal{M}_1, \mathcal{M}_2, \ldots$ with associated relations $R_1, R_2, \ldots$ of the appropriate arities.
- F is the set of rwffs $R_i\, a\, a_1 \ldots a_n$ and lwffs $a{:}A$.

Remark: NO assumptions on interrelationships between R_i and R_j!

Characterization of a non-classical logic L

- By language.
- By models $\mathfrak{M} = (\mathfrak{W}, \mathfrak{o}, \mathfrak{R}_1, \mathfrak{R}_2, \ldots, *, \mathfrak{V})$.
  - ▶ Independent conditions on $*$ and each $\mathfrak{R}_i$.
  - ▶ Moreover: truth is monotonic in some logics.

    $\Rightarrow$ Define a partial order $\sqsubseteq$ on worlds

$$\begin{aligned} \text{relevance logics:} &\quad x \sqsubseteq y \text{ iff } R\,0\,x\,y \\ \text{intuitionistic logic:} &\quad x \sqsubseteq y \text{ iff } x\,R\,y \\ \text{(modal logics:} &\quad x \sqsubseteq y \text{ iff } x \equiv y) \end{aligned}$$

and add conditions

* if $\models^{\mathfrak{M}} a_i{:}A$ and $\models^{\mathfrak{M}} a_i \sqsubseteq a_j$, then $\models^{\mathfrak{M}} a_j{:}A$,
* for all $j < n$, if $\models^{\mathfrak{M}} R_i\,a_0 \ldots a_{j-1}\,a_j\,a_{j+1} \ldots a_n$ and $\models^{\mathfrak{M}} a \sqsubseteq a_j$, then $\models^{\mathfrak{M}} R_i\,a_0 \ldots a_{j-1}\,a\,a_{j+1} \ldots a_n$
* if $\models^{\mathfrak{M}} R_i\,a_0 \ldots a_{n-1}\,a_n$ and $\models^{\mathfrak{M}} a_n \sqsubseteq a$, then $\models^{\mathfrak{M}} R_i\,a_0 \ldots a_{n-1}\,a$.

Monotony and persistency

- There are logics for which

$$\text{if } \models^{\mathfrak{M}} a_i{:}A \text{ and } \models^{\mathfrak{M}} a_i \sqsubseteq a_j\,, \text{ then } \models^{\mathfrak{M}} a_j{:}A$$

  does not hold.

- Example: intuitionistic logic (with $\rightarrow$) plus classical implication $\supset$.
  - Monotony holds for $A \rightarrow B$, but not for $A \supset B$.
  - Solution: restrict rule $monl$ to persistent formulas, e.g. A is persistent if
    - it is atomic,
    - it is $B \rightarrow C$ or $\neg B$, where $\neg$ is intuitionistic (and thus non-local) negation,
    - it is $B \wedge C$ or $B \vee C$, and B, C are persistent.

The base system $\mathrm{N}(\mathcal{B})$

As for modal logics: Kripke semantics 'suggests' ND rules.

- Rules for local operators are trivial, e.g.

$$\dfrac{\begin{array}{c}[a{:}A]\\ \vdots\\ a{:}B\end{array}}{a{:}A \supset B}\supset\mathrm{I} \qquad \dfrac{a{:}A \supset B \quad a{:}A}{a{:}B}\supset\mathrm{E} \qquad \cdots$$

The base system $\mathrm{N}(\mathcal{B})$ (cont.)

- For the non-local operators $\mathcal{M}^u$ and $\mathcal{M}^e$ we give the rules

$$\dfrac{\begin{array}{c}[a_1{:}A_1] \cdots [a_{u-1}{:}A_{u-1}]\,[R^u\, a\, a_1 \ldots a_u] \\ \vdots \\ a_u{:}A_u\end{array}}{a{:}\mathcal{M}^u A_1 \ldots A_u}\ \mathcal{M}^u\mathrm{I}$$

$$\dfrac{a{:}\mathcal{M}^u A_1 \ldots A_u \quad a_1{:}A_1 \ \cdots \ a_{u-1}{:}A_{u-1} \quad R^u\, a\, a_1 \ldots a_u}{a_u{:}A_u}\ \mathcal{M}^u\mathrm{E}$$

$$\dfrac{a_1{:}A_1 \cdots a_e{:}A_e \quad R^e\, a\, a_1 \ldots a_e}{a{:}\mathcal{M}^e A_1 \ldots A_e}\ \mathcal{M}^e\mathrm{I} \qquad \dfrac{a{:}\mathcal{M}^e A_1 \ldots A_e \quad \begin{array}{c}[a_1{:}A_1] \cdots [a_e{:}A_e]\,[R^e\, a\, a_1 \ldots a_e] \\ \vdots \\ b{:}B\end{array}}{b{:}B}\ \mathcal{M}^e\mathrm{E}$$

In $\mathcal{M}^u\mathrm{I}$ and $\mathcal{M}^e\mathrm{E}$, each a_k and each a_l, for $1 \leq k \leq u$ and $1 \leq l \leq e$, is *fresh*.

Note that the rules are independent of the properties of R^u and R^e!

The base system $\mathrm{N}(\mathcal{B})$ (cont.)

- Negation rules:

$$\dfrac{\begin{array}{c}[a^*{:}A]\\ \vdots\\ b{:}\perp\!\!\!\perp\end{array}}{a{:}\neg A}\ \neg\mathrm{I} \qquad\qquad \dfrac{a{:}\neg A \quad a^*{:}A}{b{:}\perp\!\!\!\perp}\ \neg\mathrm{E}$$

reflect the semantics and capture only a minimal non-local negation.

For intuitionistic or classical non-local negation we must also add

$$\dfrac{b{:}\perp\!\!\!\perp}{a{:}A}\ \perp\!\!\!\perp\mathrm{Ei} \qquad\qquad \dfrac{\begin{array}{c}[a{:}\neg A]\\ \vdots\\ b{:}\perp\!\!\!\perp\end{array}}{a^*{:}A}\ \perp\!\!\!\perp\mathrm{Ec}$$

- Monotony at the level of lwffs:

$$\dfrac{a_i{:}A \quad a_i \sqsubseteq a_j}{a_j{:}A}\ monl$$

where A is a persistent.

Relational theories (Labelling algebras)

- Relational theories axiomatize the properties of $*$ and of the relations $\mathfrak{R}_i$.

  (We can again exploit correspondence theory.)

- We restrict again our attention to Horn relational rules

$$\frac{R_i\, t_0^1 \ldots t_n^1 \qquad \cdots \qquad R_i\, t_0^m \ldots t_n^m}{R_i\, t_0^0 \ldots t_n^0}$$

where the t_k^j are terms built from labels and (Skolem) function symbols, e.g.

$$\frac{}{R\,0\,a\,a}\,iden \qquad \frac{R\,a\,b\,x \quad R\,x\,c\,d}{R\,b\,c\,f(a,b,c,d,x)}\,assoc1 \qquad \frac{R\,a\,b\,x \quad R\,x\,c\,d}{R\,a\,f(a,b,c,d,x)\,d}\,assoc2$$

Relational theories (Labelling algebras; cont.)

- For negation, we give Horn rules that impose different behaviors on $*$, e.g.

$$\frac{}{a \sqsubseteq a^{**}}\ **\mathrm{i} \qquad \frac{}{a^{**} \sqsubseteq a}\ **\mathrm{c} \qquad \frac{}{a \sqsubseteq a^{*}}\ \mathit{ortho1} \qquad \frac{}{a^{*} \sqsubseteq a}\ \mathit{ortho2}$$

encode intuitionistic ($**\mathrm{i}$), classical ($**\mathrm{i}$ and $**\mathrm{c}$), or ortho ($ortho\, i$) negation.

- For monotony at the level of rwffs ($0 \leq j < n$):

$$\frac{R_i\, a_0 \ldots a_{j-1}\, a_j\, a_{j+1} \ldots a_n \quad a \sqsubseteq a_j}{R_i\, a_0 \ldots a_{j-1}\, a\, a_{j+1} \ldots a_n}\ monR_i(j)$$

$$\frac{R_i\, a_0 \ldots a_{n-1}\, a_n \quad a_n \sqsubseteq a}{R_i\, a_0 \ldots a_{n-1}\, a}\ monR_i(n)$$

Labelled ND systems for prop. non-classical logics

- Our framework presents large families of (fragments of and full) non-classical logics.

- The labelled ND system $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathcal{B}) + \mathrm{N}(\mathcal{T})$ for the propositional non-classical logic $\mathcal{L}$ is the extension of an appropriate base system $\mathrm{N}(\mathcal{B})$ with a given Horn relational theory $\mathrm{N}(\mathcal{T})$.

  By considering the rules for $\perp\!\!\!\perp$, we distinguish 3 families of systems according to their treatment of non-local negation: minimal, intuitionistic, or classical.

$\mathrm{N}(\mathcal{L})$	$\mathrm{N}(\mathcal{B})$	$\mathrm{N}(\mathcal{T})$ *(includes at least)*
$\mathrm{N}(\mathcal{ML})$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e$, $\neg$ $monI$	$monR_i$ rules (for R^u and R^e)
$\mathrm{N}(\mathcal{JL})$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e$, $\neg$ $monI$ $\perp\!\!\!\perp\mathrm{Ei}$	$monR_i$ rules (for R^u and R^e) $**\mathrm{i}$
$\mathrm{N}(\mathcal{CL})$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e$, $\neg$ $monI$ $\perp\!\!\!\perp\mathrm{Ec}$	$monR_i$ rules (for R^u and R^e) $**\mathrm{i}$, $**\mathrm{c}$

Examples of propositional non-classical logics

- Not all non-classical logics expressible in our framework.

  (Not all relational theories expressible as Horn theories.)

- But: large and well-known families of non-classical logics:
  - Modal logics in the Geach hierarchy: K, D, T, B, $\mathrm{S4}$, $\mathrm{S4.2}$, $\mathrm{KD45}$, $\mathrm{S5}$, ...

    and their (simple) multimodal versions.
  - Many relevance logics: B, N, T, R, ...
  - 'Independent' combinations of the above.
  - Fragments and full logics.

$\mathrm{H(B^+)}$, a Hilbert system for $\mathrm{B^+}$

- Axiom schemas:

A1: $A \rightarrow A$.

A2: $A \wedge B \rightarrow A$.

A3: $A \wedge B \rightarrow B$.

A4: $A \rightarrow A \vee B$.

A5: $B \rightarrow A \vee B$.

A6: $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$.

A7: $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$.

A8: $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$.

- Inference rules:

R1: $\dfrac{A \rightarrow B \quad A}{B}\, \mathit{modus\ ponens}$,

R2: $\dfrac{A \quad B}{A \wedge B}\, \mathit{adjunction}$,

R3: $\dfrac{A \rightarrow B \quad C \rightarrow D}{(B \rightarrow C) \rightarrow (A \rightarrow D)}\, \mathit{affixing}$,

along with their disjunctive forms, where if $\dfrac{A_1 \ \cdots \ A_n}{B}$ is a rule, then its *disjunctive form* is the rule $\dfrac{C \vee A_1 \ \cdots \ C \vee A_n}{C \vee B}$

$\mathrm{N}(\mathrm{B}^+)$, a labelled ND system for B^+

$$\frac{a{:}A \quad a{:}B}{a{:}A \wedge B}\wedge\mathrm{I} \qquad \frac{a{:}A \wedge B}{a{:}A}\wedge\mathrm{E1} \qquad \frac{a{:}A \wedge B}{a{:}B}\wedge\mathrm{E2}$$

$$\frac{a{:}A}{a{:}A \vee B}\vee\mathrm{I1} \qquad \frac{a{:}B}{a{:}A \vee B}\vee\mathrm{I2} \qquad \frac{a{:}A \vee B \quad \begin{array}{c}[a{:}A]\\ \vdots \\ c{:}C\end{array} \quad \begin{array}{c}[a{:}B]\\ \vdots \\ c{:}C\end{array}}{c{:}C}\vee\mathrm{E}$$

$$\frac{\begin{array}{c}[b{:}A]\ [R\,a\,b\,c]\\ \vdots \\ c{:}B\end{array}}{a{:}A \to B}\to\mathrm{I} \quad (b, c \text{ fresh}) \qquad \frac{a{:}A \to B \quad b{:}A \quad R\,a\,b\,c}{c{:}B}\to\mathrm{E}$$

$$\frac{R\,a\,b\,c \quad R\,0\,x\,a}{R\,x\,b\,c}\,monR(1) \qquad \frac{R\,a\,b\,c \quad R\,0\,x\,b}{R\,a\,x\,c}\,monR(2) \qquad \frac{R\,a\,b\,c \quad R\,0\,c\,x}{R\,a\,b\,x}\,monR(3)$$

$$\frac{a{:}A \quad R\,0\,a\,b}{b{:}A}\,monl \qquad \frac{}{R\,0\,a\,a}\,iden$$

Some correspondences

Name	Axiom schema/Inference rule	Property	Horn relational rules
A9	$A \wedge (A \to B) \to B$	$R\,a\,a\,a$ or $R\,0\,a\,b \supset R\,a\,a\,b$ (*idempotence*)	$\dfrac{}{R\,a\,a\,a}\ idem$ or $\dfrac{R\,0\,a\,b}{R\,a\,a\,b}\ idem$
A11	$(A \to B) \to ((B \to C) \to (A \to C))$	$R^2\,a\,b\,c\,d \supset R^2\,b\,(ac)\,d$ (*suffixing*)	$\dfrac{R\,a\,b\,x \quad R\,x\,c\,d}{R\,a\,c\,f_2(a,b,c,d,x)}\ suff1$ $\dfrac{R\,a\,b\,x \quad R\,x\,c\,d}{R\,b\,f_2(a,b,c,d,x)\,d}\ suff2$
A12	$(A \to B) \to ((C \to A) \to (C \to B))$	$R^2\,a\,b\,c\,d \supset R^2\,a\,(bc)\,d$ (*associativity* or *prefixing*)	$\dfrac{R\,a\,b\,x \quad R\,x\,c\,d}{R\,b\,c\,f_3(a,b,c,d,x)}\ assoc1$ $\dfrac{R\,a\,b\,x \quad R\,x\,c\,d}{R\,a\,f_3(a,b,c,d,x)\,d}\ assoc2$
A13	$(A \to (A \to B)) \to (A \to B)$	$R\,a\,b\,c \supset R^2\,a\,b\,b\,c$ (*contraction*)	$\dfrac{R\,a\,b\,c}{R\,a\,b\,f_4(a,b,c)}\ cont1$ $\dfrac{R\,a\,b\,c}{R\,f_4(a,b,c)\,b\,c}\ cont2$
A14	$((A \to A) \to B) \to B$	$R\,a\,0\,a$ (*specialized assertion*)	$\dfrac{}{R\,a\,0\,a}\ specassert$
A15	$A \to ((A \to B) \to B)$	$R\,a\,b\,c \supset R\,b\,a\,c$ (*commutativity* or *assertion*)	$\dfrac{R\,a\,b\,c}{R\,b\,a\,c}\ comm$

- $R^2\,a\,b\,c\,d =_{def} \exists x(R\,a\,b\,x \wedge R\,x\,c\,d)$ and $R^2\,a\,(bc)\,d =_{def} \exists x(R\,b\,c\,x \wedge R\,a\,x\,d)$.
- All the properties of R are outermost universally quantified.
- Using the definition of the partial order we could write $a \sqsubseteq b$ for $R\,0\,a\,b$.

Some correspondences (cont.)

Name	Axiom schema/Inference rule	Property	Horn relational rules
A16	$A \to (A \to A)$	$R\,a\,b\,c \supset (R\,0\,a\,c \vee R\,0\,b\,c)$ or $R\,0\,0\,a \vee R\,0\,0\,a^*$ (*mingle*)	no Horn relational rules! (requires universal $\perp\!\!\!\perp$)
A17	$A \to (B \to B)$	$R\,0\,0\,a$ or $R\,a\,b\,c \supset R\,0\,b\,c$ (*thinning*)	$\dfrac{}{R\,0\,0\,a}\,thin$ or $\dfrac{R\,a\,b\,c}{R\,0\,b\,c}\,thin$
A18	$A \to (B \to A)$	$R\,a\,b\,c \supset R\,0\,a\,c$ (*positive paradox*)	$\dfrac{R\,a\,b\,c}{R\,0\,a\,c}\,pospar$
R4	$\dfrac{A \to \neg B}{B \to \neg A}\,contraposition$	$R\,0\,a\,b \supset R\,0\,b^*\,a^*$ (*antitonicity*)	$\dfrac{R\,0\,a\,b}{R\,0\,b^*\,a^*}\,anti$
A19	$(A \to \neg B) \to (B \to \neg A)$	$R\,a\,b\,c \supset R\,a\,c^*\,b^*$ (*inversion*)	$\dfrac{R\,a\,b\,c}{R\,a\,c^*\,b^*}\,inv$
A20	$\neg\neg A \to A$	$a^{**} = a$ (*period two*)	$\dfrac{}{R\,0\,a\,a^{**}}\,{**}\mathrm{i}$ $\dfrac{}{R\,0\,a^{**}\,a}\,{**}\mathrm{c}$
A21	$A \vee \neg A$	$R\,0\,0^*0$ (*excluded middle*)	$\dfrac{}{R\,0\,0^*0}\,exmid$

Extensions of B^+: Hilbert and labelled ND systems

Logic $\mathcal{L}$	Hilbert system $\mathrm{H}(\mathcal{L})$	Labelled ND system $\mathrm{N}(\mathcal{L})$
N^+	$\mathrm{H}(\mathrm{B}^+) + \{\mathrm{A11}, \mathrm{A12}\}$	$\mathrm{N}(\mathrm{B}^+) + \{\textit{suff1}, \textit{suff2}, \textit{assoc1}, \textit{assoc2}\}$
T^+	$\mathrm{H}(\mathrm{N}^+) + \{\mathrm{A13}\}$	$\mathrm{N}(\mathrm{N}^+) + \{\textit{cont1}, \textit{cont2}\}$
E^+	$\mathrm{H}(\mathrm{T}^+) + \{\mathrm{A14}\}$	$\mathrm{N}(\mathrm{T}^+) + \{\textit{specassert}\}$
R^+	$\mathrm{H}(\mathrm{E}^+) + \{\mathrm{A15}\}$	$\mathrm{N}(\mathrm{E}^+) + \{\textit{comm}\}$
$\mathrm{S4}^+$	$\mathrm{H}(\mathrm{E}^+) + \{\mathrm{A17}\}$	$\mathrm{N}(\mathrm{E}^+) + \{\textit{thin}\}$
J^+	$\mathrm{H}(\mathrm{R}^+) + \{\mathrm{A17}\} = \mathrm{H}(\mathrm{S4}^+) + \{\mathrm{A15}\}$	$\mathrm{N}(\mathrm{R}^+) + \{\textit{thin}\} = \mathrm{N}(\mathrm{S4}^+) + \{\textit{comm}\}$
B	$\mathrm{H}(\mathrm{B}^+) + \{\mathrm{A20}, \mathrm{R4}\}$	$\mathrm{N}(\mathrm{B}^+) + \{\neg\mathrm{I}, \neg\mathrm{E}, \perp\!\!\perp\mathrm{Ec}, {*}{*}\mathrm{i}, {*}{*}\mathrm{c}, \textit{anti}\}$
R	$\mathrm{H}(\mathrm{B}) + \{\mathrm{A11}, \mathrm{A13}, \mathrm{A15}, \mathrm{A19}\}$ $= \mathrm{H}(\mathrm{B}^+) + \{\mathrm{A11}, \mathrm{A13},$ $\mathrm{A15}, \mathrm{A19}, \mathrm{A20}\}$	$\mathrm{N}(\mathrm{B}) + \{\textit{suff1}, \textit{suff2}, \textit{cont1}, \textit{cont2}, \textit{comm}, \textit{inv}\}$ $= \mathrm{N}(\mathrm{B}^+) + \{\neg\mathrm{I}, \neg\mathrm{E}, \perp\!\!\perp\mathrm{Ec}, {*}{*}\mathrm{i}, {*}{*}\mathrm{c},$ $\textit{suff1}, \textit{suff2}, \textit{cont1}, \textit{cont2}, \textit{comm}, \textit{inv}\}$
G	$\mathrm{H}(\mathrm{B}) + \{\mathrm{A21}\}$	$\mathrm{N}(\mathrm{B}) + \{\textit{exmid}\}$
C	$\mathrm{H}(\mathrm{R}) + \{\mathrm{A17}\}$	$\mathrm{N}(\mathrm{R}) + \{\textit{thin}\}$

J^+ is positive intuitionistic logic, G is ‘basic’ classical logic and C is ‘full’ classical logic.

Extensions of B^+: Hilbert and labelled ND systems

Note that we have chosen the 'economical' system $\mathrm{H(R)}$, where, e.g., $\mathrm{R4}$ is redundant as it can be derived using $\mathrm{A19}$ and $\mathrm{R1}$; similarly, in $\mathrm{N(R)}$ we can trivially derive the rule $anti$ using inv, and the rule $idem$ using identity and contraction:

$$\dfrac{\dfrac{\dfrac{}{R\,0\,a\,a}\,iden}{R\,f_4(0\,,a\,,a)\,a\,a}\,cont2 \qquad \dfrac{\dfrac{}{R\,0\,a\,a}\,iden}{R\,0\,a\,f_4(0\,,a\,,a)}\,cont1}{R\,a\,a\,a}\,monR(1)$$

Alternative, equivalent, axiomatizations are possible, for R and other logics.

An advantage of our approach

- Routley and Meyer have shown that
  - $\mathrm{H}(\mathrm{R}^+)$ is a subsystem of the system $\mathrm{H}(\mathrm{J}^+)$ for positive intuitionistic logic J^+,
  - but $\mathrm{H}(\mathrm{R})$ is a subsystem only of the system $\mathrm{H}(\mathrm{C})$ for 'full' classical logic C.
  - That is: $\mathrm{H}(\mathrm{J})$ for 'full' intuitionistic logic J cannot be modularly obtained by simply adding new axioms to $\mathrm{H}(\mathrm{R})$.
- This is not the case with our systems!
  - Extending $\mathrm{N}(\mathrm{R})$ with the rule $\dfrac{}{R\,0\,0\,a}\,\mathit{thin}$ yields $\mathrm{N}(\mathrm{C})$,
  - but we have $\mathrm{N}(\mathrm{R}) = \mathrm{N}(\mathcal{C}\mathrm{R})$
  - and we can restore the modularity, we just need to consider the system $\mathrm{N}(\mathcal{J}\mathrm{R})$, i.e. $\mathrm{N}(\mathrm{R})$ with an intuitionistic treatment of negation.
  - Indeed: $\mathrm{N}(\mathrm{R}^+) \subset \mathrm{N}(\mathcal{J}\mathrm{R}) \subset \mathrm{N}(\mathrm{R})$ and $\mathrm{N}(\mathcal{J}\mathrm{R}) + \mathit{thin} = \mathrm{N}(\mathrm{J})$.

Extending $\mathrm{N(R)}$ with *thin* yields $\mathrm{N(C)}$

We show that we are then able to prove $R\,0\,a\,0$, so that, essentially, all the worlds collapse; i.e. $a = a^* = a^{**}$, $\rightarrow$ reduces to $\supset$, and $\neg$ to $\sim$.

$$
\dfrac{\overset{\Pi}{R\,0\,a\,0^*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R\,0^*\,0^*\,0^*}\,\mathit{idem}}{R\,0^*\,0^{**}\,0^{**}}\,\mathit{inv} \qquad \dfrac{}{R\,0\,0^{**}\,0}\,{**}\mathrm{c}}{R\,0^*\,0^{**}\,0}\,\mathit{monR}(3) \qquad \dfrac{}{R\,0\,0\,0^{**}}\,{**}\mathrm{i}}{R\,0^*\,0\,0}\,\mathit{monR}(2)}{R\,0\,0^*\,0}\,\mathit{comm}}{R\,0\,a\,0}\,\mathit{monR}(3)
$$

where Π is

$$
\dfrac{\dfrac{\dfrac{}{R\,0\,0\,a^*}\,\mathit{thin}}{R\,0\,a^{**}\,0^*}\,\mathit{anti} \qquad \dfrac{}{R\,0\,a\,a^{**}}\,{**}\mathrm{i}}{R\,0\,a\,0^*}\,\mathit{monR}(2)
$$

Note that we have:

Fact: $\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{B})+\mathrm{N}(\mathcal{T})} R_i\,a\,a_1 \ldots a_n$ iff $\Delta \vdash_{\mathrm{N}(\mathcal{T})} R_i\,a\,a_1 \ldots a_n$.

Example derivations

$$
\dfrac{
\dfrac{
[a{:}A \rightarrow \neg B]^3 \quad [d^*{:}A]^1 \quad
\dfrac{
\dfrac{[R\,b\,c\,d]^2 \quad [R\,0\,a\,b]^3}{R\,a\,c\,d}\, monR(1)
}{R\,a\,d^*\,c^*}\, inv
}{c^*{:}\neg B}\, {\rightarrow}\mathrm{E}
\qquad
\dfrac{[c{:}B]^2 \quad \dfrac{}{R\,0\,c\,c^{**}}\, {**}\mathrm{i}}{c^{**}{:}B}\, monl
}{
\dfrac{
\dfrac{
\dfrac{e{:}\bot}{d{:}\neg A}\, \neg\mathrm{I}^1
}{b{:}B \rightarrow \neg A}\, {\rightarrow}\mathrm{I}^2
}{0{:}(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)}\, {\rightarrow}\mathrm{I}^3
}\, \neg\mathrm{E}
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{[a{:}\neg\neg A]^2 \quad [R\,0\,a\,b]^2}{b{:}\neg\neg A}\, monl
\qquad [b^*{:}\neg A]^1
}{c{:}\bot}\, \neg\mathrm{E}
}{b^{**}{:}A}\, \bot\mathrm{Ec}^1
\qquad
\dfrac{}{R\,0\,b^{**}\,b}\, {**}\mathrm{c}
}{b{:}A}\, monl
}{0{:}\neg\neg A \rightarrow A}\, {\rightarrow}\mathrm{I}^2
$$

Soundness and completeness of $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathcal{B}) + \mathrm{N}(\mathcal{T})$

- Theorem $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathcal{B}) + \mathrm{N}(\mathcal{T})$ is sound and complete.
- For Γ a set of labelled formulas, Δ a set of relational formulas, we have

  1. $\Delta \vdash_{\mathrm{N}(\mathcal{L})} R_i\, a\, a_1 \ldots a_n$ iff $\Delta \models R_i\, a\, a_1 \ldots a_n$
  2. $\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} a{:}A$ iff $\Gamma, \Delta \models a{:}A$.

- Proof is parameterized over $\mathrm{N}(\mathcal{T})$.
  - ▶ Soundness: By induction on the structure of the derivations.
  - ▶ Completeness: By a modified canonical model construction that accounts for the explicit formalization of labels and of the relations between them.
    - ∗ To account for positive (negation-less) fragments, we build the canonical model by extending disjoint theory – counter-theory.
    - ∗ That is: we do not define maximality in terms of consistency.

Normalization and subformula property

- Theorem: Every derivation of $x{:}A$ from Γ, Δ in $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathcal{B}) + \mathrm{N}(\mathcal{T})$ reduces to a derivation in normal form.
- Normal form of a derivation $\equiv$ "no detours or irrelevancies".
  Two forms of detour
  - proper reductions for $\mathcal{M}^u$, $\mathcal{M}^e$ and $\neg\mathrm{E}$, like for modal logics

$$
\dfrac{\dfrac{\begin{array}{c}[x\,R\,y]\\ \Pi \\ y{:}A\end{array}}{x{:}\Box A}\,\Box\mathrm{I} \qquad x\,R\,z}{z{:}A}\,\Box\mathrm{E} \qquad \text{reduces to} \qquad \begin{array}{c}x\,R\,z\\ \Pi[z/y]\\ z{:}A\end{array}
$$

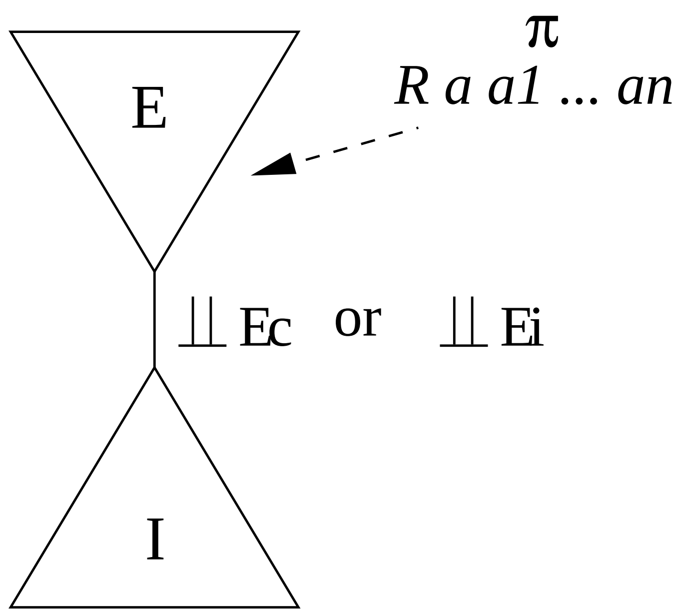
  - and permutative reductions for $\mathcal{M}^e\mathrm{E}$, $\vee\mathrm{E}$, $\perp\!\!\!\perp\mathrm{Ei}$ and $monl$
    (for lwffs that potentially interact in a proper reduction but are too far apart in a derivation).
- Corollary: Normal derivations in $\mathrm{N}(\mathcal{B}) + \mathrm{N}(\mathcal{T})$ satisfy a subformula property.

$\Rightarrow$ Restricted proof search.

$\Rightarrow$ Decidability, complexity?

Proof search: Tracks

Corollary: The form of tracks in a normal derivation of an lwff in $\mathrm{N}(\mathcal{B}) + \mathrm{N}(\mathcal{T})$ is



Positive fragments and interrelated relations

- Consider the positive modal logic K with $\Box$ and $R^{\Box}$, $\Diamond$ and $R^{\Diamond}$.
- Theorem: If our restriction is withdrawn, and $R^{\Box}$ and $R^{\Diamond}$ are related, then incompleteness may arise:

$$x{:}\Box(A \vee B) \supset (\Diamond A \vee \Box B)$$

corresponds to but is not provable in systems containing

$$\frac{x\, R^{\Box} y}{x\, R^{\Diamond} y}\,(\mathfrak{R}^{\Box} \subseteq \mathfrak{R}^{\Diamond})$$

- ▶ By exploiting normalization results.
- ▶ Hilbert-style presentations suffer from the same problem.
- ▶ Solution: give up fixed base system and add rule

$$\frac{}{x{:}\Box(A \vee B) \supset (\Diamond A \vee \Box B)}$$

Quantified modal logics

- Two, independent, degrees of freedom (two-dimensional space of possible logics):
  - ▶ properties of the accessibility relation (as in propositional case),
  - ▶ how the domains of individuals change between worlds: varying, increasing, decreasing, or constant domains.

  Other dimensions are possible, e.g. non-rigid designators.

- Standard approaches: piecemeal fashion or lack uniformity.
  Problems:
  - ▶ Hilbert systems: standard quantifier rules automatically require domains to be increasing (because of Converse Barcan formula).
  - ▶ Incompleteness with respect to Kripke semantics is common.
  - ▶ Meta-results (e.g. completeness) are not proved in uniform way.

- Labelled deduction systems: no problems.

Labelled quantified modal logics

$$\begin{array}{lcccccc} \mathrm{N}(\mathrm{Q}\mathcal{L}) & = & \text{base system} & + & \text{relational theory} & + & \text{domain theory} \\ & = & \text{fixed } \mathrm{N}(\mathrm{QK}) & + & \text{varying } \mathrm{N}(\mathcal{T}) & + & \text{varying } \mathrm{N}(\mathcal{D}) \end{array}$$

- Base system $\mathrm{N}(\mathrm{QK})$:
  - Natural deduction system formalizing QK.
  - Reason about $w{:}A$.
- Relational theory $\mathrm{N}(\mathcal{T})$:
  - Describes the behavior of R.
  - Reason about $w_i R w_j$.
- Domain theory $\mathrm{N}(\mathcal{D})$:
  - Describes the behavior of domains of quantification behavior.
  - Reason about labelled terms $w{:}t$ (t exists at w).
- Separation $\Rightarrow$ structure $\Rightarrow$ properties.

The base system $\mathrm{N(QK)}$ for quantified K

$$\dfrac{\begin{matrix}[w_i{:}A \supset \bot] \\ \vdots \\ w_j{:}\bot\end{matrix}}{w_i{:}A}\,\bot\mathrm{E} \qquad \dfrac{\begin{matrix}[w{:}A] \\ \vdots \\ w{:}B\end{matrix}}{w{:}A \supset B}\,\supset\mathrm{I} \qquad \dfrac{w{:}A \supset B \quad w{:}A}{w{:}B}\,\supset\mathrm{E}$$

$$\dfrac{\begin{matrix}[w_i R w_j] \\ \vdots \\ w_j{:}A\end{matrix}}{w_i{:}\Box A}\,\Box\mathrm{I} \qquad \dfrac{w_i{:}\Box A \quad w_i R w_j}{w_j{:}A}\,\Box\mathrm{E} \qquad \dfrac{\begin{matrix}[w{:}t] \\ \vdots \\ w{:}A[t/x]\end{matrix}}{w{:}\forall x(A)}\,\forall\mathrm{I} \qquad \dfrac{w{:}\forall x(A) \quad w{:}t}{w{:}A[t/x]}\,\forall\mathrm{E}$$

In $\Box\mathrm{I}$, w_j is different from w_i and does not occur in any assumption on which $w_j{:}A$ depends other than $w_i R w_j$.
In $\forall\mathrm{I}$, t does not occur in any assumption on which $w{:}A[t/x]$ depends other than $w{:}t$.

Derived rules of $\mathrm{N(QK)}$

$$\frac{w_j{:}A \quad w_i \, R \, w_j}{w_i{:}\Diamond A}\,\Diamond\mathrm{I} \qquad\qquad \frac{w{:}A[t/x] \quad w{:}t}{w{:}\exists x(A)}\,\exists\,\mathrm{I}$$

$$\frac{w_i{:}\Diamond A \qquad \begin{array}{c}[w_j{:}A]\ [w_i \, R \, w_j]\\ \vdots \\ w_k{:}B\end{array}}{w_k{:}B}\,\Diamond\mathrm{E} \qquad \frac{w_i{:}\exists x(A) \qquad \begin{array}{c}[w_i{:}A[t/x]]\ [w_i{:}t]\\ \vdots \\ w_j{:}B\end{array}}{w_j{:}B}\,\exists\,\mathrm{E}$$

In $\Diamond\mathrm{E}$, w_j is different from w_i and w_k, and does not occur in any assumption on which the upper occurrence of $w_k{:}B$ depends other than $w_j{:}A$ and $w_i \, R \, w_j$.

In $\exists\,\mathrm{E}$, t does not occur in any assumption on which the upper occurrence of $w_j{:}B$ depends other than $w_i{:}A[t/x]$ and $w_i{:}t$.

Extensions of $\mathrm{N(QK)}$

- Relational theories axiomatize properties of R (as in the propositional case).
- Domain theories: different combinations of the rules

$$\frac{w_i R w_j \quad w_i{:}t}{w_j{:}t}\, id$$

increasing domains, corresponds to CBF $\Box\forall x(A) \supset \forall x(\Box A)$

$$\frac{w_i R w_j \quad w_j{:}t}{w_i{:}t}\, dd$$

decreasing domains, corresponds to BF $\forall x(\Box A) \supset \Box\forall x(A)$

yield different labelled ND systems for quantified modal logics.

The labelled ND system $\mathrm{N(Q\mathcal{L})} = \mathrm{N(QK)} + \mathrm{N}(\mathcal{T}) + \mathrm{N}(\mathcal{D})$ is obtained by extending $\mathrm{N(QK)} + \mathrm{N}(\mathcal{T})$ with a given domain theory $\mathrm{N}(\mathcal{D})$ generated by a subset of $\{id, dd\}$.

Two-dimensional uniformity

$$
\begin{tikzcd}
\mathrm{N(QK)} + \mathrm{N}(\mathcal{D}) \arrow[r, "\text{accessibility}", "\text{relation}"'] & \mathrm{N(QK)} + \mathrm{N}(\mathcal{T}) + \mathrm{N}(\mathcal{D}) \\
\mathrm{N(QK)} \arrow[u, "\text{domains}"] \arrow[r, "\text{accessibility}", "\text{relation}"'] & \mathrm{N(QK)} + \mathrm{N}(\mathcal{T}) \arrow[u, "\text{domains}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathrm{N(QKT4.c)} & \\
\mathrm{N(QKT4.i)} \arrow[ur, "dd"] & & \mathrm{N(QKT4.d)} \arrow[ul, "id"'] \\
 & \mathrm{N(QKT4)} \arrow[ul, "id"] \arrow[ur, "dd"'] &
\end{tikzcd}
$$

Example derivations

CBF is a theorem of (any extension of) $\mathrm{N(QK.i)}$:

$$
\dfrac{\dfrac{\dfrac{[w{:}\Box\forall x(A)]^1 \quad [w\,R\,w_1]^3}{w_1{:}\forall x(A)}\,\Box\mathrm{E} \qquad \dfrac{[w\,R\,w_1]^3 \quad [w{:}t]^2}{w_1{:}t}\,id}{\dfrac{\dfrac{w_1{:}A[t/x]}{w{:}\Box A[t/x]}\,\Box\mathrm{I}^3}{w{:}\forall x(\Box A)}\,\forall\mathrm{I}^2}\,\forall\mathrm{E}}{w{:}\Box\forall x(A) \supset \forall x(\Box A)}\,\supset\mathrm{I}^1
$$

We can prove similarly that BF is a theorem of (any extension of) $\mathrm{N(QK.d)}$.

Remark: id and dd are interderivable when the accessibility relation is symmetric.

Labelled quantified modal logics: Properties

1. Labelled deduction systems are uniform and modular.
2. Labelled deduction systems are sound and complete.

   For Θ a set of lterms:
   (a) $\Delta \vdash_{\mathrm{N(Q}\mathcal{L})} w_i R w_j$ iff $\Delta \models w_i R w_j$,
   (b) $\Delta, \Theta \vdash_{\mathrm{N(Q}\mathcal{L})} w{:}t$ iff $\Delta, \Theta \models w{:}t$
   (c) $\Gamma, \Delta, \Theta \vdash_{\mathrm{N(Q}\mathcal{L})} w{:}A$ iff $\Gamma, \Delta, \Theta \models w{:}A$.
3. The deduction machinery is minimal.
4. Derivations are strictly separated.
   (a) A derivation of an lwff can depend on a derivation of an rwff (via an application of $\Box\mathrm{E}$), but not vice versa.
   (b) A derivation of an lwff can depend on a derivation of an lterm (via an application of $\forall\mathrm{E}$), but not vice versa.
   (c) A derivation of an lterm can depend on a derivation of an rwff (via an application of id or dd), but not vice versa.
5. Derivations normalize and satisfy a subformula property.

   As in the propositional case.

Falsum

- Also local and universal falsum generalize.
- Moreover, we may also need universal falsum for lterms:

$$\frac{w_i{:}\bot}{w_j{:}\emptyset}\, uft_1 \qquad\qquad \frac{w_j{:}\emptyset}{w_i{:}\bot}\, uft_2$$

  allow us to mingle derivations of lwffs with derivations of lterms.

- Needed, for example, to prove

$$w{:}\forall x(A) \supset \exists x(A)$$

  when we extend a first-order domain theory with

$$\frac{}{w{:}\bigsqcup x(x)}\, \textit{non-empty}$$

Road Map

- Introduction: A framework for non-classical logics.
- Labelled deduction for modal logics.
- Labelled deduction for non-classical logics.
- Encoding non-classical logics in Isabelle.
- Substructural and complexity analysis of labelled non-classical logics.
- Conclusions and outlook.

Encoding non-classical logics in Isabelle

- Isabelle: a generic theorem prover.
- Metalogic *Meta*: a natural deduction presentation of minimal implicational predicate logic with universal quantification over all higher-types.

  (universal quantifier $\bigwedge$ or `!!`, implication $\Rightarrow$ or `==>`)
- Object logics encoded by declaring a theory, composed of a signature and axioms, which are formulas in the language of *Meta*.
  - Theories in Isabelle correspond to instances of an abstract datatype in ML and Isabelle provides means for creating elements of these types, extending them, and combining them.
  - Axioms establish the validity of judgements (assertions about syntactic objects declared in the signature).
  - Derivations are constructed by deduction in the metalogic.

Encoding propositional modal logics

```
K = Pure +          (*  K extends Pure (Isabelle's metalogic)  *)
                    (* with the following signature and axioms *)
types (* Definition of type constructors *)
  label,o 0
arities (* Addition of the arity `logic' to the existing types *)
  label, o :: logic
consts
  (* Logical operators *)
  falsum   :: "o"
  imp      :: "[o, o] => o"              ("_ --> _" [25,26] 25)
  not      :: "o => o"                   ("~ _" [40] 40)
  box      :: "o => o"                   ("[]_" [50] 50)
  dia      :: "o => o"                   ("<>_" [50] 50)
  (* Judgements *)
  LF       :: "[label, o] => prop"       ("(_ : _)" [0,0] 100)
  RF       :: "[label, label] => prop"   ("(_ R _)" [0,0] 100)
rules
  (* Axioms representing the object-level rules *)
  falsumE  "(x:A --> falsum ==> y: falsum) ==> x:A"
  impI     "(x:A ==> x:B) ==> x:A --> B"
  impE     "x:A --> B ==> x:A ==> x:B"
  boxI     "(!!y. (x R y ==> y:A)) ==> x:[]A"
  boxE     "x:[]A ==> x R y ==> y:A"
  (* Definitions *)
  not_def  "x: ~A == x: A --> falsum"
  dia_def  "x: <>A == x: ~([](~A))"
end
```

- Two types: `label` and `o` (unlabelled modal formulas).
- Operators: typed constants over this signature.
- Two judgements: `LF` and `RF`.
- Mixfix annotations: abbreviate `imp` with `-->`, `LF(x,A)` with `x:A`.
- In axioms, free variables are implicitly outermost universally quantified.

Extensions of K

Addition of Horn axioms:

```
KT = K +
rules
  refl       "x R x"
end


K4 = K +
rules
  trans      "x R y ==> y R z ==> x R z"
end


KT4 = K4 +
rules
  refl       "x R x"
end


K2 = K +
consts
  g          :: "[label,label,label] => label"
rules
  conv1      "x R y ==> x R z ==> y R g(x,y,z)"
  conv2      "x R y ==> x R z ==> z R g(x,y,z)"
end
```

Logics inherit theorems and derived rules from their ancestors,

e.g. $x{:}\Box A \leftrightarrow \Box\Box A$ in `KT4`

Faithfulness and adequacy

- $Meta_{\mathrm{N}(\mathcal{L})}$ is faithful (with respect to $\mathrm{N}(\mathcal{L})$) iff

  1. $\mathcal{RF}(\Delta) \vdash_{Meta_{\mathrm{N}(\mathcal{L})}} \mathcal{RF}(x,y)$ implies $\Delta \vdash_{\mathrm{N}(\mathcal{L})} x\,R\,y$, and
  2. $\mathcal{LF}(\Gamma), \mathcal{RF}(\Delta) \vdash_{Meta_{\mathrm{N}(\mathcal{L})}} \mathcal{LF}(x,A)$ implies $\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} x{:}A$.

- $Meta_{\mathrm{N}(\mathcal{L})}$ is adequate (with respect to $\mathrm{N}(\mathcal{L})$) iff the converses hold, i.e. iff

  1. $\Delta \vdash_{\mathrm{N}(\mathcal{L})} x\,R\,y$ implies $\mathcal{RF}(\Delta) \vdash_{Meta_{\mathrm{N}(\mathcal{L})}} \mathcal{RF}(x,y)$, and
  2. $\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} x{:}A$ implies $\mathcal{LF}(\Gamma), \mathcal{RF}(\Delta) \vdash_{Meta_{\mathrm{N}(\mathcal{L})}} \mathcal{LF}(x,A)$.

- Theorem: $Meta_{\mathrm{N}(\mathcal{L})}$ is faithful and adequate.
  By induction on structure of (object/meta) derivations.

Isabelle proof session

- Isabelle manipulates rules. A rule is a formula

```
!! v1 ... vm. A1 ==> ... ==> (An ==> A)
```

  which is also displayed as

```
!! v1 ... vm. [| A1; ...; An|] ==> A
```

- Rules represent proof states where `A` is the goal to be established and the `Ai`'s are the subgoals to be proved.
- Isabelle supports proof construction through higher-order resolution
  - given a proof state with subgoal `B` and a rule,
  - we treat the `vi`'s of the rule as variables for unification,
  - and higher-order unify `A` with `B`.
  - If this succeeds, then unification yields a substitution σ,
  - and the proof state is updated replacing `B` with the subgoals `A1`, . . . , `An` and applying σ to the whole proof state.

Examples

- An interactive proof.

```
> goal K4.thy "x:[]A --> [][]A";
x : []A --> [][]A
 1. x : []A --> [][]A

> by (rtac impI 1);
x : []A --> [][]A
 1. x : []A ==> x : [][]A

> by (rtac boxI 1);
x : []A --> [][]A
 1. !!y. [| x : []A; x R y |] ==> y : []A

> by (rtac boxI 1);
x : []A --> [][]A
 1. !!y ya. [| x : []A; x R y; y R ya |] ==> ya : A

> by (etac boxE 1);
x : []A imp [][]A
 1. !!y ya. [| x R y; y R ya |] ==> x R ya

> by (etac trans 1);
x : []A --> [][]A
 1. !!y ya. y R ya ==> y R ya

> by (atac 1);
x : []A --> [][]A
No subgoals!

> qed "BoxImpliesBoxBox";
val BoxImpliesBoxBox = "?x : []?A --> [][]?A"
```

Examples (cont.)

- We can also derive new rules

```
> val [major,minor] =
   goalw K.thy [dia_def] "[| y:A; x R y |] ==> x: <>A";
x : <>A
 1. x : ~ [](~ A)
val major = "y : A  [y : A]" : thm
val minor = "x R y  [x R y]" : thm

...

> qed "diaE";
val diaE = "[| ?x : <>?A; !!y. [| y : ?A; ?x R y |] ==> ?z : ?B |] ==> ?z : ?B" : thm
```

- We can use Isabelle's built-in tacticals such as `EVERY`, `THEN`, `REPEAT`
- We can increase automation by writing tactics.

Encoding propositional non-classical logics

```
Rplus = Pure +
types   (* Definition of type constructors *)
  label, o 0
arities (* Addition of the arity 'logic' to the existing types *)
  label, o :: logic
consts  (* Labels, Logical operators, Judgements *)
  act  :: "label"
  f2   :: "[label,label,label,label,label] => label"
  f3   :: "[label,label,label,label,label] => label"
  f4   :: "[label,label,label] => label"
  inc  :: "o"
  and  :: "[o, o] => o"       (infixr 35)
  or   :: "[o, o] => o"       (infixr 30)
  imp  :: "[o, o] => o"       (infixr 25)
  LF   :: "[label, o] => prop"           ("(_ : _)" [0,0] 100)
  RF   :: "[label, label, label] => prop" ("(R _ _ _)" [0,0,0] 100)
rules (* Base system and Properties of the compossibility relation R *)
  conjI   "[| a:A; a:B |] ==> a: A and B"
  conjE1  "a: A and B ==> a:A"
  conjE2  "a: A and B ==> a:B"
  disjI1  "a:A ==> a: A or B"
  disjI2  "a:B ==> a: A or B"
  disjE   "[| a: A or B;  a:A ==> c:C;  a:B ==> c:C |] ==> c:C"
  impI    "[| !!b c. [| b:A; R a b c |] ==> c:B |] ==> a: A imp B"
  impE    "[| a: A imp B; b:A; R a b c |] ==> c:B"
  monl    "[| a:A; R act a b |] ==> b:A"
  monR1      "[| R a b c; R act x a |] ==> R x b c"
  monR2      "[| R a b c; R act x b |] ==> R a x c"
  monR3      "[| R a b c; R act c x |] ==> R a b x"
  iden       "R act a a"
  suff1      "[| R a b x; R x c d |] ==> R a c f2(a,b,c,d,x)"
  suff2      "[| R a b x; R x c d |] ==> R b f2(a,b,c,d,x) d"
  assoc1     "[| R a b x; R x c d |] ==> R b c f3(a,b,c,d,x)"
  assoc2     "[| R a b x; R x c d |] ==> R a f3(a,b,c,d,x) d"
  cont1      "R a b c ==> R a b f4(a,b,c)"
  cont2      "R a b c ==> R f4(a,b,c) b c"
  specassert "R a act a"
  comm       "R a b c ==> R b a c"
end
```

Encoding quantified modal logics

```
QK = Pure +

classes
  term < logic
default
  term
types  (* Definition of type constructors *)
  label, o 0
arities (* Addition of the arity ‘logic’ to the existing types *)
  label, o :: logic

consts
  falsum   :: "o"
  imp      :: "[o, o] => o"              ("_ --> _" [25,26] 25)
  not      :: "o => o"                   ("~ _" [40] 40)
  box      :: "o=> o"                    ("[]_" [50] 50)
  dia      :: "o=> o"                    ("<>_" [50] 50)
  All      :: "('a => o) => o"           (binder "ALL " 10)
  Ex       :: "('a => o) => o"           (binder "EX " 10)
  LF       :: "[label, o] => prop"       ("(_ : _)" [0,0] 100)
  RF       :: "[label, label] => prop"   ("(_ R _)" [0,0] 100)
  LT       :: "[label, 'a] => prop"      ("(_ E _)" [0,0] 100)

rules
  falsumE  "(w:A --> falsum ==> v: falsum) ==> w:A"
  impI     "(w:A ==> w:B) ==> w:(A --> B)"
  impE     "w: A --> B ==> w:A ==> w:B"
  boxI     "(!!v. (w R v ==> v:A)) ==> w:([]A)"
  boxE     "w:[]A ==> w R v ==> v:A"
  allI     "(!!t. (w E t ==> w: A(t))) ==> (w: ALL x.A(x))"
  allE     "w: ALL x. A(x) ==> w E t ==> w:A(t)"

  (* Definitions *)
  not_def  "w: ~A == w: A --> falsum"
  dia_def  "w: <>A == w: ~([](~A))"
  ex_def   "w: EX x. A(x) == w: ~(ALL x. ~A(x))"
end
```

Road Map

- Introduction: A framework for non-classical logics.
- Labelled deduction for modal logics.
- Labelled deduction for non-classical logics.
- Encoding non-classical logics in Isabelle.
- Substructural and complexity analysis of labelled non-classical logics.
  - Substructural analysis of labelled sequent systems.
  - A new proof-theoretic method (a *recipe*) for establishing decidability and bounding the complexity of non-classical logics.
  - Justification (and partial refinement) of rules of standard sequent systems.
- Conclusions and outlook.

Properties of $\mathrm{N}(\mathcal{L}) = \mathrm{N}(\mathrm{K}) + \mathrm{N}(\mathcal{T})$

- Γ a set of labelled formulas, Δ a set of relational formulas.
- Parameterized proofs of
  - Soundness and completeness with respect to Kripke semantics

$$\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} \varphi \quad \Leftrightarrow \quad \Gamma, \Delta \models \varphi$$

  - Faithfulness and adequacy of the implementation

$$\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} \varphi \quad \Leftrightarrow \quad \Gamma, \Delta \vdash \varphi \text{ in } \mathtt{Isabelle}_{\mathrm{N}(\mathcal{L})}$$

- Proof search: normalization and subformula property $\begin{array}{c}\Gamma\ \Delta\\ \vdots\,?\\ \varphi\end{array}$

  Proof is 'normal' (well-defined structure) and contains only subformulas.
  $\Rightarrow$ Restricted proof search.
  $\Rightarrow$ Decidability, complexity? (new proof-theoretical method based on substructural analysis).

Proof search: Normalization and subformula property

- Structure: $\begin{array}{c}\Gamma\ \ \Delta\\ \vdots\ ?\\ \alpha\end{array}$

- Theorem: Every derivation of $x{:}A$ from Γ, Δ in $\mathrm{N(K)} + \mathrm{N}(\mathcal{T})$ reduces to a derivation in normal form.

  "no detours or irrelevancies"

  example:
$$\dfrac{\dfrac{\begin{array}{c}[x\,R\,y]\\ \Pi\\ y{:}A\end{array}}{x{:}\Box A}\,\Box\mathrm{I} \qquad x\,R\,z}{z{:}A}\,\Box\mathrm{E} \qquad \text{reduces to} \qquad \begin{array}{c}x\,R\,z\\ \Pi[z/y]\\ z{:}A\end{array}$$

- Corollary: Normal derivations in $\mathrm{N(K)} + \mathrm{N}(\mathcal{T})$ satisfy a subformula property.

$\Rightarrow$ Restricted proof search.

$\Rightarrow$ Decidability, complexity?

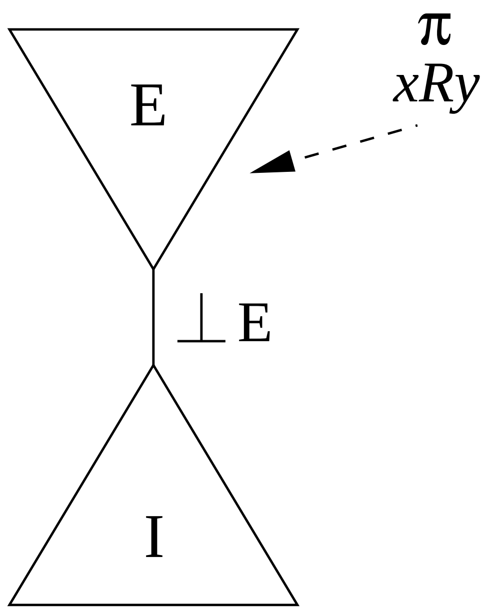
Proof search: Tracks

- Thread in a derivation Π in $\mathrm{N(K)} + \mathrm{N}(\mathcal{T})$: a sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that (i) φ_1 is an assumption of Π, (ii) φ_i stands immediately above φ_{i+1}, for $1 \leq i < n$, and (iii) φ_n is the conclusion of Π.

- Lwff-thread: a thread where $\varphi_1, \ldots, \varphi_n$ are all lwffs.

- Track: initial part of an lwff-thread in Π which stops either at the first minor premise of an elimination rule in the lwff-thread or at the conclusion of the lwff-thread.

- Corollary: The form of tracks in a normal derivation of an lwff in $\mathrm{N(K)} + \mathrm{N}(\mathcal{T})$ is



Proof search: structural analysis $\begin{array}{cc}\Gamma & \Delta \\ \vdots & ? \\ \alpha & \end{array}$

- Normalization & subformula property $\Rightarrow$ restricted proof search.
- Further restriction by exploiting labels.
  Structural analysis of proofs in normal form.
  $\Rightarrow$ bounds on formulas in proofs:

Q: Which formulas?

A: Subformulas!

Q: How many formulas?

$$\begin{array}{c}\beta \\ \vdots \\ \alpha\end{array} \quad \text{or} \quad \begin{array}{c}\beta, \beta \\ \vdots \\ \alpha\end{array} \quad \text{or} \quad \ldots \quad \text{or} \quad \begin{array}{c}\beta, \ldots, \beta \\ \vdots \\ \alpha\end{array} \quad \text{or} \quad \begin{array}{c}\beta, \ldots, \beta, \ldots \\ \vdots \\ \alpha\end{array} \quad ?$$

A: this kind of analysis is more easily performed when logics are presented using sequent systems, which allow for a finer grained control of structural information via their structural rules.

Proof search: details (recipe)

- A new proof-theoretical method for bounding the complexity of the decision problem for propositional non-classical logics.

  1. Logics presented as cut-free labelled sequent systems.
  2. Guidelines to provide bounds on
     - structural reasoning (structural rules: contraction, ...),
     - relational reasoning (accessibility relation).

$\Rightarrow$ Decision procedures with bounded space requirements (PSPACE bounds: new/compare well with best currently known)

- $O(n \log n)$-space for K, $\mathrm{B}[\rightarrow, \wedge]$, B^{+}, ...
- $O(n^2 \log n)$-space for T, ...
- $O(n^4 \log n)$-space $\mathrm{K4}$ and $\mathrm{S4}$, ...

Labelled sequent systems for non-classical logics

Our normalizing labelled natural deduction systems yield equivalent cut-free labelled sequent systems that

1. allow us to present non-classical logics in a uniform and modular way;
2. are decomposed into two separated parts: a base system fixed for related logics, and a labelling algebra, which we extend to generate particular logics;
3. contain left and right rules for each logical operator (except for falsum $\perp$ and incoherence $\perp\!\!\!\perp$), independent of the relation(s) R_i and of the other operators;
4. satisfy a subformula property; and
5. provide the basis of a general proof-theoretical method for bounding the complexity of the decision problem for propositional non-classical logics.

We consider (some) modal logics in detail and discuss extensions for other logics.

Base modal sequent system $\mathrm{S(K)}$

- Language is the same as for modal $\mathrm{N(K)}$, but now
  - Γ is a finite multiset of labelled formulas,
  - Δ is a finite multiset of relational formulas.

Axioms:

$$\frac{}{x{:}A \vdash x{:}A}\,\mathrm{AXl} \qquad \frac{}{y{:}\bot \vdash x{:}A}\,\bot\mathrm{L} \qquad \frac{}{x\,R\,y \vdash x\,R\,y}\,\mathrm{AXr}$$

Structural rules:

$$\frac{\Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{WlL} \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}A}\,\mathrm{WlR}$$

$$\frac{x{:}A, x{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL} \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\,\mathrm{ClR}$$

$$\frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta, x\,R\,y \vdash \Gamma'}\,\mathrm{WrL} \qquad \frac{\Delta, x\,R\,y, x\,R\,y \vdash u\,R\,v}{\Delta, x\,R\,y \vdash u\,R\,v}\,\mathrm{CrL}$$

Logical rules:

$$\frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \supset B, \Gamma, \Delta \vdash \Gamma'}\,\supset\mathrm{L} \qquad \frac{x{:}A, \Gamma, \Delta \vdash \Gamma', x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \supset B}\,\supset\mathrm{R}$$

$$\frac{\Delta \vdash x\,R\,y \quad y{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}\Box A, \Gamma, \Delta \vdash \Gamma'}\,\Box\mathrm{L} \qquad \frac{\Gamma, \Delta, x\,R\,y \vdash y{:}A, \Gamma'}{\Gamma, \Delta \vdash x{:}\Box A, \Gamma'}\,\Box\mathrm{R}\ [y\ \text{fresh}]$$

Relational theory $\mathrm{S}(\mathcal{T})$: Extensions of $\mathrm{S}(\mathrm{K})$

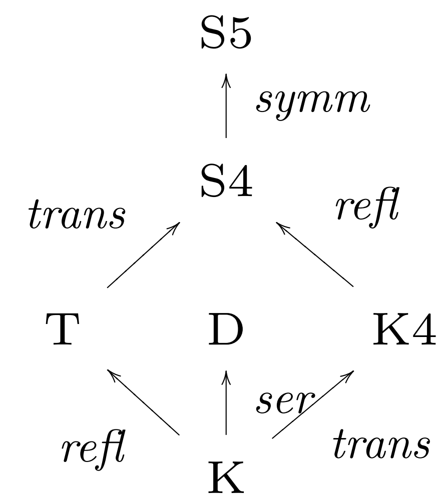
$\mathrm{N}(\mathcal{T})$ is a collection of relational rules ('intuitionistic' sequents)

$$\frac{\Delta \vdash s_1\, R\, t_1 \quad \cdots \quad \Delta \vdash s_m\, R\, t_m}{\Delta \vdash s_0\, R\, t_0}$$

Examples:

$$\mathrm{S}(\mathrm{S4}) = \mathrm{S}(\mathrm{K}) \; + \; \frac{}{\vdash x\, R\, x}\,\mathit{refl} \quad + \quad \frac{\Delta \vdash x\, R\, y \quad \Delta \vdash y\, R\, z}{\Delta \vdash x\, R\, z}\,\mathit{trans}$$

$$\mathrm{S}(\mathrm{D}) = \mathrm{S}(\mathrm{K}) \; + \; \frac{}{\vdash x\, R\, f(x)}\,\mathit{ser}$$



We can again exploit correspondence theory.

Derived rules of $\mathrm{N(K)}$

$$\frac{\Gamma, \Delta \vdash \Gamma', x{:}A}{x{:}\sim A, \Gamma, \Delta \vdash \Gamma'} \sim\mathrm{L} \qquad \frac{x{:}A, \Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}\sim A} \sim\mathrm{R}$$

$$\frac{x{:}A, x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \wedge B, \Gamma, \Delta \vdash \Gamma'} \wedge\mathrm{L} \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad \Gamma, \Delta \vdash \Gamma', x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \wedge B} \wedge\mathrm{R}$$

$$\frac{x{:}A, \Gamma, \Delta \vdash \Gamma' \quad x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \vee B, \Gamma, \Delta \vdash \Gamma'} \vee\mathrm{L} \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \vee B} \vee\mathrm{R}$$

$$\frac{y{:}A, \Gamma, \Delta, x\,R\,y \vdash \Gamma'}{x{:}\Diamond A, \Gamma, \Delta \vdash \Gamma'} \Diamond\mathrm{L} \qquad \frac{\Delta \vdash x\,R\,y \quad \Gamma, \Delta \vdash \Gamma', y{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}\Diamond A} \Diamond\mathrm{R}$$

In $\Diamond\mathrm{L}$, y does not occur in $x{:}\Diamond A, \Gamma, \Delta \vdash \Gamma'$.

Examples of derivations

$$\dfrac{\begin{matrix}\Pi\\ y{:}A,\Gamma,\Delta,x\,R\,y\vdash\Gamma'\end{matrix}}{x{:}\Diamond A,\Gamma,\Delta\vdash\Gamma'}\,\Diamond\mathrm{L} \qquad\rightsquigarrow\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\Pi\\ y{:}A,\Gamma,\Delta,x\,R\,y\vdash\Gamma'\end{matrix}}{\Gamma,\Delta,x\,R\,y\vdash\Gamma',y{:}\sim A}\,{\sim}\mathrm{R}}{\Gamma,\Delta\vdash\Gamma',x{:}\Box\sim A}\,\Box\mathrm{R}}{x{:}\sim\Box\sim A,\Gamma,\Delta\vdash\Gamma'}\,{\sim}\mathrm{L}$$

Side condition is 'inherited' from $\Box\mathrm{R}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x\,R\,y\vdash x\,R\,y}\,\mathrm{AXr}\quad \dfrac{\dfrac{}{x\,R\,y\vdash x\,R\,y}\,\mathrm{AXr}\quad \dfrac{\dfrac{\dfrac{\dfrac{}{y{:}A\vdash y{:}A}\,\mathrm{AXl}}{y{:}A\vdash y{:}B,y{:}A}\,\mathrm{WlL}\quad \dfrac{\dfrac{}{y{:}B\vdash y{:}B}\,\mathrm{AXl}}{y{:}B,y{:}A\vdash y{:}B}\,\mathrm{WlL}}{y{:}A\supset B,y{:}A\vdash y{:}B}\,{\supset}\mathrm{L}}{y{:}A\supset B,y{:}A,x\,R\,y\vdash y{:}B}\,\mathrm{WrL}}{y{:}A\supset B,x{:}\Box A,x\,R\,y\vdash y{:}B}\,\Box\mathrm{L}}{x{:}\Box(A\supset B),x{:}\Box A,x\,R\,y\vdash y{:}B}\,\Box\mathrm{L}}{x{:}\Box(A\supset B),x{:}\Box A\vdash x{:}\Box B}\,\Box\mathrm{R}}{x{:}\Box(A\supset B)\vdash x{:}\Box A\supset\Box B}\,{\supset}\mathrm{R}}{\vdash x{:}\Box(A\supset B)\supset(\Box A\supset\Box B)}\,{\supset}\mathrm{R}$$

Labelled sequent systems for non-classical logics

We proceed like for ND systems.

Quantifier rules:

$$\frac{\Delta, \Theta \vdash w{:}t \quad w{:}A[t/x], \Gamma, \Delta, \Theta \vdash \Gamma'}{w{:}\forall x(A), \Gamma, \Delta, \Theta \vdash \Gamma'} \forall \mathrm{L} \qquad \frac{\Gamma, \Delta, \Theta, w{:}t \vdash \Gamma' w{:}A[t/x]}{\Gamma, \Delta, \Theta \vdash \Gamma', w{:}\forall x(A)} \forall \mathrm{R}$$

where

- Θ is a multiset of labelled terms,
- in $\forall\mathrm{R}$, t does not occur in $\Gamma, \Delta, \Theta \vdash \Gamma', w{:}\forall x(A)$.

Domain rules:

$$\frac{\Delta \vdash w_i R w_j \quad \Delta, \Theta \vdash w_i{:}t}{\Delta, \Theta \vdash w_j{:}t}\, id \qquad \frac{\Delta \vdash w_i R w_j \quad \Delta, \Theta \vdash w_j{:}t}{\Delta, \Theta \vdash w_i{:}t}\, dd$$

Non-local operators

$$\frac{\Delta \vdash R^u\, a\, a_1 \ldots a_u \quad \Gamma, \Delta \vdash \Gamma', a_1{:}A_1 \ \cdots \ \Gamma, \Delta \vdash \Gamma', a_{u-1}{:}A_{u-1} \quad a_u{:}A_u, \Gamma, \Delta \vdash \Gamma'}{a{:}\mathcal{M}^u A_1 \ldots A_u, \Gamma, \Delta \vdash \Gamma'}\ \mathcal{M}^u\mathrm{L}$$

$$\frac{a_1{:}A_1, \ldots, a_{u-1}{:}A_{u-1}, \Gamma, \Delta, R^u\, a\, a_1 \ldots a_u \vdash \Gamma', a_u{:}A_u}{\Gamma, \Delta \vdash \Gamma', a{:}\mathcal{M}^u A_1 \ldots A_u}\ \mathcal{M}^u\mathrm{R}$$

In $\mathcal{M}^u\mathrm{R}$, $a_1, \ldots, a_u$ are all different from a and each other, and do not occur in $\Gamma, \Delta \vdash \Gamma', a{:}\mathcal{M}^u A_1 \ldots A_u$.

Examples (in $\rightarrow\mathrm{R}$, b and c are different from a and each other, and do not occur in $\Gamma, \Delta \vdash \Gamma', a{:}A \rightarrow B$):

$$\frac{\Delta \vdash R\,a\,b\,c \quad \Gamma, \Delta \vdash \Gamma', b{:}A \quad c{:}B, \Gamma, \Delta \vdash \Gamma'}{a{:}A \rightarrow B, \Gamma, \Delta \vdash \Gamma'}\rightarrow\mathrm{L} \qquad \frac{b{:}A, \Gamma, \Delta, R\,a\,b\,c \vdash \Gamma', c{:}B}{\Gamma, \Delta \vdash \Gamma', a{:}A \rightarrow B}\rightarrow\mathrm{R}$$

$$\frac{\Gamma, \Delta \vdash \Gamma', a^*{:}A}{a{:}\neg A, \Gamma, \Delta \vdash \Gamma'}\neg\mathrm{L} \quad \frac{a^*{:}A, \Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', a{:}\neg A}\neg\mathrm{R} \qquad \frac{\Delta \vdash R\,a\,b\,c}{\Delta \vdash R\,a\,c^*\,b^*}\ inv$$

$$\frac{}{\vdash R\,0\,a\,a^{**}}\ {**}\mathrm{i} \quad \frac{}{\vdash R\,0\,a^{**}\,a}\ {**}\mathrm{c}$$

Labelled seq. sys. for the basic relevance logic B^+

$$\frac{}{x{:}A \vdash x{:}A}\,\mathrm{AXl} \qquad \frac{}{R\,x\,y\,z \vdash R\,x\,y\,z}\,\mathrm{AXr}$$

$$\frac{x{:}A, x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \wedge B, \Gamma, \Delta \vdash \Gamma'}\,\wedge\mathrm{L} \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad \Gamma, \Delta \vdash \Gamma', x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \wedge B}\,\wedge\mathrm{R}$$

$$\frac{x{:}A, \Gamma, \Delta \vdash \Gamma' \quad x{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \vee B, \Gamma, \Delta \vdash \Gamma'}\,\vee\mathrm{L} \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \vee B}\,\vee\mathrm{R}$$

$$\frac{\Delta \vdash R\,x\,y\,z \quad \Gamma, \Delta \vdash \Gamma', y{:}A \quad z{:}B, \Gamma, \Delta \vdash \Gamma'}{x{:}A \rightarrow B, \Gamma, \Delta \vdash \Gamma'}\,{\rightarrow}\mathrm{L} \qquad \frac{y{:}A, \Gamma, \Delta, R\,x\,y\,z \vdash \Gamma', z{:}B}{\Gamma, \Delta \vdash \Gamma', x{:}A \rightarrow B}\,{\rightarrow}\mathrm{R}\ [b, c \text{ fresh}]$$

$$\frac{\Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{WlL} \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma', x{:}A}\,\mathrm{WlR} \qquad \frac{\Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta, R\,x\,y\,z \vdash \Gamma'}\,\mathrm{WrL}$$

$$\frac{x{:}A, x{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL} \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\,\mathrm{ClR} \qquad \frac{\Delta, R\,a\,b\,c, R\,a\,b\,c \vdash R\,x\,y\,z}{\Delta, R\,a\,b\,c \vdash R\,x\,y\,z}\,\mathrm{CrL}$$

$$\frac{\Delta \vdash R\,0\,x\,y \quad \Gamma, \Delta \vdash \Gamma', x{:}A}{\Gamma, \Delta \vdash \Gamma', y{:}A}\,monl \qquad \frac{}{\vdash R\,0\,x\,x}\,iden$$

$$\frac{\Delta \vdash R\,0\,a\,x \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,a\,y\,z}\,monr1 \qquad \frac{\Delta \vdash R\,0\,a\,y \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,a\,z}\,monr2 \qquad \frac{\Delta \vdash R\,0\,z\,a \quad \Delta \vdash R\,x\,y\,z}{\Delta \vdash R\,x\,y\,a}\,monr3$$

Properties of $\mathrm{S}(\mathcal{L}) = \mathrm{S}(\mathrm{K}) + \mathrm{S}(\mathcal{T})$

- Cut-free: cut is an admissible rule

$$\frac{\Gamma, \Delta \vdash \Gamma', x{:}A \quad x{:}A, \Gamma, \Delta \vdash \Gamma'}{\Gamma, \Delta \vdash \Gamma'}\, cut$$

- Normalizing ND systems and cut-free sequent systems are 'equivalent'.
  - Theorem:
    - $\Gamma, \Delta \vdash_{\mathrm{N}(\mathcal{L})} x{:}A$ iff $\Gamma, \Delta \vdash x{:}A$ is provable in $\mathrm{S}(\mathcal{L})$.
    - $\Delta \vdash_{\mathrm{N}(\mathcal{L})} x\,R\,y$ iff $\Delta \vdash x\,R\,y$ is provable in $\mathrm{S}(\mathcal{L})$.
  - Theorem: $\mathrm{N}(\mathcal{L})$ is sound and complete.
  - Corollary: $\mathrm{S}(\mathcal{L})$ is sound and complete.

Sequent systems as refutation systems

- The progressive (backwards) construction of a derivation $\begin{array}{c}\Pi \\ S_0 = \Gamma_0, \Delta_0 \vdash \Gamma'_0\end{array}$ is associated to the progressive construction of a (partial) model $\mathfrak{M} = (\mathfrak{W}, \mathfrak{R}, \mathfrak{V})$ such that for each $S_i = \Gamma_i, \Delta_i \vdash \Gamma'_i$ in Π, with $i \geq 0$,
  - the worlds of $\mathfrak{M}$ are connected according to Δ_i, i.e. $(x, y) \in \mathfrak{R}$ iff $\Delta_i \vdash x\,R\,y$,
  - $\mathfrak{M}$ satisfies all lwffs $x{:}A \in \Gamma_i$, i.e. $\models^{\mathfrak{M}} x{:}A$, and
  - $\mathfrak{M}$ falsifies all lwffs $x{:}B \in \Gamma'_i$, i.e. $\not\models^{\mathfrak{M}} x{:}B$.
- Then we have:
  - if S_0 is provable, then $\mathfrak{M}$ is inconsistent (i.e. it contains an inconsistent world),
  - if S_0 is not provable, then $\mathfrak{M}$ is a counter-model for it.

  $\mathfrak{M}$ is partial in the sense that the truth values of some propositional variables might be missing from the model, but we can univocally determine these values from the values of the composite formulas of S_i they appear in (e.g. $\models^{\mathfrak{M}} x{:}\sim p$, for p a propositional variable, implies $\not\models^{\mathfrak{M}} x{:}p$, i.e. $\mathfrak{V}(x, p) = 0$).

Example

We can represent the inconsistent model $\mathfrak{M}$ spawned by

$$
\dfrac{
\dfrac{}{\vdash x_1 \, R \, x_1}\,\mathit{refl}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{x_1 \, R \, x_2 \vdash x_1 \, R \, x_2}\,\mathrm{AXr}
\qquad
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{}{x_2{:}B \vdash x_2{:}B}\,\mathrm{AXl} \\
\vdots\,\mathrm{W} \\
x_1{:}B, x_2{:}B, x_1 \, R \, x_2 \vdash x_2{:}B, x_2{:}\Box B
\end{array}
}{x_1{:}B, x_1 \, R \, x_2 \vdash x_2{:}B, x_2{:}B \supset \Box B}\,\supset\mathrm{R}
}{x_2{:}\sim(B \supset \Box B), x_1{:}B, x_1 \, R \, x_2 \vdash x_2{:}B}\,\sim\mathrm{L}
}{x_1{:}\Box\sim(B \supset \Box B), x_1{:}B, x_1 \, R \, x_2 \vdash x_2{:}B}\,\Box\mathrm{L}
}{x_1{:}\Box\sim(B \supset \Box B), x_1{:}B \vdash x_1{:}\Box B}\,\Box\mathrm{R}
}{
\dfrac{x_1{:}\Box\sim(B \supset \Box B) \vdash x_1{:}B \supset \Box B}{x_1{:}\sim(B \supset \Box B), x_1{:}\Box\sim(B \supset \Box B) \vdash}\,\sim\mathrm{L}
}\,\supset\mathrm{R}
}{
\dfrac{
\dfrac{x_1{:}\Box\sim(B \supset \Box B), x_1{:}\Box\sim(B \supset \Box B) \vdash}{x_1{:}\Box\sim(B \supset \Box B) \vdash}\,\mathrm{ClL}
}{\vdash x_1{:}\sim\Box\sim(B \supset \Box B)}\,\sim\mathrm{R}
}\,\Box\mathrm{L}
$$

Example (cont.)

with the diagram:

$$\begin{tikzcd}
x \arrow[loop above] \arrow[rr] & & y \arrow[loop above] \\
\Box \sim (B \supset \Box B) \arrow[rr, "4", "\Box\mathrm{L}, \sim\mathrm{L}, \supset\mathrm{R}"'] \arrow[d, "1"', "\mathrm{ClL}"] & & B, \sim \Box B \\
\Box \sim (B \supset \Box B) \arrow[d, "2"', "\Box\mathrm{L}, \sim\mathrm{L}, \supset\mathrm{R}"] & & \\
B, \sim \Box B \arrow[rr, "3", "\Box\mathrm{R}"'] & & \sim B
\end{tikzcd}$$

$\mathfrak{M}$ is inconsistent since $\models^{\mathfrak{M}} y{:} \sim B$ and $\models^{\mathfrak{M}} y{:}B$

Further examples: Using relational rules and contraction in $\mathrm{S}(\mathrm{K4})$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{AXr}
\UnaryInfC{$x_1 \, R \, x_2 \vdash x_1 \, R \, x_2$}
\AxiomC{Π}
\noLine
\UnaryInfC{$\Delta \vdash x_1 \, R \, x_3$}
\AxiomC{}
\RightLabel{AXr}
\UnaryInfC{$x_2 \, R \, x_3 \vdash x_2 \, R \, x_3$}
\RightLabel{WrL}
\UnaryInfC{$\Delta \vdash x_2 \, R \, x_3$}
\AxiomC{}
\RightLabel{AXl}
\UnaryInfC{$x_3{:}\Box B \vdash x_3{:}\Box B$}
\RightLabel{W}
\dottedLine
\UnaryInfC{$x_3{:}\Box B, \Delta \vdash x_3{:}B, x_3{:}\Box B$}
\RightLabel{$\Box$L}
\BinaryInfC{$x_2{:}\Box\Box B, \Delta \vdash x_3{:}B, x_3{:}\Box B$}
\RightLabel{$\sim$L}
\UnaryInfC{$x_3{:} \sim\Box B, x_2{:}\Box\Box B, \Delta \vdash x_3{:}B$}
\RightLabel{$\Box$L}
\BinaryInfC{$x_1{:}\Box \sim\Box B, x_2{:}\Box\Box B, \Delta \vdash x_3{:}B$}
\RightLabel{$\Box$R}
\UnaryInfC{$x_1{:}\Box \sim\Box B, x_2{:}\Box\Box B, x_1 \, R \, x_2 \vdash x_2{:}\Box B$}
\RightLabel{$\sim$L}
\UnaryInfC{$x_2{:} \sim\Box B, x_1{:}\Box \sim\Box B, x_2{:}\Box\Box B, x_1 \, R \, x_2 \vdash$}
\RightLabel{$\Box$L}
\BinaryInfC{$x_1{:}\Box \sim\Box B, x_1{:}\Box \sim\Box B, x_2{:}\Box\Box B, x_1 \, R \, x_2 \vdash$}
\RightLabel{ClL}
\UnaryInfC{$x_1{:}\Box \sim\Box B, x_2{:}\Box\Box B, x_1 \, R \, x_2 \vdash$}
\RightLabel{$\sim$R}
\UnaryInfC{$x_1{:}\Box \sim\Box B, x_1 \, R \, x_2 \vdash x_2{:} \sim\Box\Box B$}
\RightLabel{$\Box$R}
\UnaryInfC{$x_1{:}\Box \sim\Box B \vdash x_1{:}\Box \sim\Box\Box B$}
\RightLabel{$\supset$R}
\UnaryInfC{$\vdash x_1{:}\Box \sim\Box B \supset \Box \sim\Box\Box B$}
\end{prooftree}
$$

where $\Delta = \{x_1 \, R \, x_2, x_2 \, R \, x_3\}$ and Π is

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{AXr}
\UnaryInfC{$x_1 \, R \, x_2 \vdash x_1 \, R \, x_2$}
\RightLabel{WrL}
\UnaryInfC{$x_1 \, R \, x_2, x_2 \, R \, x_3 \vdash x_1 \, R \, x_2$}
\AxiomC{}
\RightLabel{AXr}
\UnaryInfC{$x_2 \, R \, x_3 \vdash x_2 \, R \, z$}
\RightLabel{WrL}
\UnaryInfC{$x_1 \, R \, x_2, x_2 \, R \, x_3 \vdash x_2 \, R \, x_3$}
\RightLabel{\textit{trans}}
\BinaryInfC{$x_1 \, R \, x_2, x_2 \, R \, x_3 \vdash x_1 \, R \, x_3$}
\end{prooftree}
\quad .
$$

Proof search: problems

Let A be a formula that is not trivially provable and consider an attempted proof of the non-theorem $x{:}\Box(A \supset \Box A)$ in $\mathrm{S}(\mathrm{K})$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ {???}}{y{:}A,\, x\,R\,y,\, y\,R\,z \vdash z{:}A}}{y{:}A,\, x\,R\,y \vdash y{:}\Box A}\,\Box\mathrm{R}}{x\,R\,y \vdash y{:}A \supset \Box A}\,{\supset}\mathrm{R}}{\vdash x{:}\Box(A \supset \Box A)}\,\Box\mathrm{R}$$

and its associated 'putative' counter-model (model or counter-model?)

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ \sim\Box(A \supset \Box A) & \xrightarrow[\Box\mathrm{R},\,\supset\mathrm{R}]{1} & A, \sim\Box A & \xrightarrow[\Box\mathrm{R}]{2} & \sim A \end{array}$$

Q: Since contraction is always applicable, how can we guarantee that proof search terminates?

A: We have seen that contraction is not (always) eliminable, but in some cases we can bound its application!

Proof search (proof of $\vdash x_1{:}D$)

- Simplifying rules: size is a decreasing measure, e.g.

$$\frac{\Delta \vdash x\,R\,y \quad y{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}\Box A, \Gamma, \Delta \vdash \Gamma'}\,\Box\mathrm{L} \qquad \text{(subformula property)}$$

- Non-simplifying rules: size is not a decreasing measure, e.g.

$$\frac{x{:}A, x{:}A, \Gamma, \Delta \vdash \Gamma'}{x{:}A, \Gamma, \Delta \vdash \Gamma'}\,\mathrm{ClL} \qquad \frac{\Gamma, \Delta \vdash \Gamma', x{:}A, x{:}A}{\Gamma, \Delta \vdash \Gamma', x{:}A}\,\mathrm{ClR}$$

and relational rules: CrL, $\dfrac{\Delta \vdash x\,R\,y \quad \Delta \vdash y\,R\,z}{\Delta \vdash x\,R\,z}\,\mathit{trans}$, ...

- Bounding proof search $\rightsquigarrow$ bounding non-simplifying rules.
  - Substructural and relational analysis of $\mathrm{S}(\mathcal{L})$.
    $\Rightarrow$ decreasing measure $\Rightarrow$ bounds on space complexity of decision procedures:
  - combine bounds on contraction with bounds on number of labels, rwffs and lwffs generated in proofs,
  - apply and extend standard techniques.

Logic-independent bounds (proof of $\vdash x_1{:}D$)

1. Theorem: CrL is eliminable in $\mathrm{S}(\mathcal{L})$.
   Just remove $\mathrm{WrL} - \mathrm{CrL}$ pairs (delete, collapse):

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{AXr}
\UnaryInfC{$x\,R\,y \vdash x\,R\,y$}
\RightLabel{WrL}
\UnaryInfC{$x\,R\,y, y\,R\,z \vdash x\,R\,y$}
\AxiomC{}
\RightLabel{AXr}
\UnaryInfC{$y\,R\,z \vdash y\,R\,z$}
\RightLabel{WrL}
\UnaryInfC{$x\,R\,y, y\,R\,z \vdash y\,R\,z$}
\RightLabel{\textit{trans}}
\BinaryInfC{$x\,R\,y, y\,R\,z \vdash x\,R\,z$}
\RightLabel{WrL}
\UnaryInfC{$x\,R\,y, x\,R\,y, y\,R\,z \vdash x\,R\,z$}
\RightLabel{CrL}
\UnaryInfC{$x\,R\,y, y\,R\,z \vdash x\,R\,z$}
\AxiomC{}
\RightLabel{AXl}
\UnaryInfC{$z{:}A \vdash z{:}A$}
\RightLabel{WrL}
\UnaryInfC{$z{:}A, y\,R\,z \vdash z{:}A$}
\RightLabel{WrL}
\UnaryInfC{$z{:}A, x\,R\,y, y\,R\,z \vdash z{:}A$}
\RightLabel{$\Box$L}
\BinaryInfC{$x{:}\Box A, x\,R\,y, y\,R\,z \vdash z{:}A$}
\RightLabel{$\Box$R}
\UnaryInfC{$x{:}\Box A, x\,R\,y \vdash y{:}\Box A$}
\RightLabel{$\Box$R}
\UnaryInfC{$x{:}\Box A \vdash x{:}\Box\Box A$}
\RightLabel{$\supset$R}
\UnaryInfC{$\vdash x{:}\Box A \supset \Box\Box A$}
\end{prooftree}
$$

2. Theorem: We can always transform a proof of $\vdash x_1{:}D$ so that it does not contain contractions, except for contractions of labelled formulas of the form $x{:}\mathcal{M}A_1 \dots A_n$.

   That is: contractions of $x{:}\Box A$, $x{:}A \rightarrow B$, ...

3. ...

Logic-independent bounds (proof of $\vdash x_1{:}D$; cont.)

- Permutations: invert order of rules.

  Example:

$$\dfrac{\dfrac{u{:}A, \Gamma, \Delta, x\,R\,y \vdash \Gamma', u{:}B, y{:}C}{\Gamma, \Delta, x\,R\,y \vdash \Gamma', u{:}A \supset B, y{:}C}\supset\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', u{:}A \supset B, x{:}\Box C}\Box\mathrm{R} \quad \text{permutes to} \quad \dfrac{\dfrac{u{:}A, \Gamma, \Delta, x\,R\,y \vdash \Gamma', u{:}B, y{:}C}{u{:}A, \Gamma, \Delta \vdash \Gamma', u{:}B, x{:}\Box C}\Box\mathrm{R}}{\Gamma, \Delta \vdash \Gamma', u{:}A \supset B, x{:}\Box C}\supset\mathrm{R}$$

- Fact: Every 'lwff-rule' permutes w.r.t. any other 'lwff-rule', with the exception of $\Box\mathrm{L}$ which does not permute w.r.t. $\Box\mathrm{R}$.

$$\dfrac{\dfrac{\Delta, x\,R\,y \vdash x\,R\,y \quad y{:}A, \Gamma, \Delta, x\,R\,y \vdash \Gamma', y{:}B}{x{:}\Box A, \Gamma, \Delta, x\,R\,y \vdash \Gamma', y{:}B}\Box\mathrm{L}}{x{:}\Box A, \Gamma, \Delta, \vdash \Gamma', x{:}\Box B}\Box\mathrm{R}$$

  Analogous problem for $\mathcal{M}\mathrm{L}$ and $\mathcal{M}\mathrm{R}$.

The recipe for an arbitrary non-classical logic $\mathcal{L}$

1. Give a cut-free labelled sequent system for $\mathrm{S}(\mathcal{L})$.
   (a) Distinguish *simplifying* and *non-simplifying rules*.
   (b) Apply *logic-independent bounds* to restrict non-simplifying rules.

2. Provide (*logic-dependent*) bounds for the remaining non-simplifying rules.
   (a) By following our guidelines and examples.
   (b) Possibly bringing in *relational oracles* to decide $\Delta \vdash R\,x\,x_1 \ldots x_n$.

3. Compute the space requirements of the decision procedure.
   (a) Based on the results of step *(2)* and our guidelines.

Space complexity of proof search (proof of $\vdash x_1{:}D$)

- Combine bounds on non-simplifying rules with bounds on number of labels and relational formulas generated in proofs.

- Adapt and extend standard techniques:
  - Rather than storing entire proofs (branches),
  - store a sequent and a stack that maintains information sufficient to reconstruct branching points
    (stack entry: indices for rules, principal formulas and branching points),
  - each rule application generates a new sequent and extends the stack,
  - if necessary, bring in oracle to decide relational queries.

Space complexity of proof search (proof of $\vdash x_1{:}D$)

- Overall space required is $O((\mathtt{l}\,\mathtt{e}) + \mathtt{s} + \mathtt{r})$:
  - length $\mathtt{l}$ of the stack,
  - size $\mathtt{e}$ of a stack entry,
  - size $\mathtt{s}$ required to store any single sequent that could arise in the proof,
  - space requirement $\mathtt{r}$ of oracle.
- Measure $\mathtt{m}$ bounds:
  - length $\mathtt{l}$ of the stack (proof depth),
  - number of labels, labelled formulas and relational formulas in the proof,
- $\mathtt{e}$ is bounded by $O(\log \mathtt{m})$,
- represent subformulas with indices $\Rightarrow$ $\mathtt{s}$ is $O(\mathtt{m}\,\log \mathtt{m})$.

$\Rightarrow$ Overall space required is $O(\mathtt{m}\,\log \mathtt{m} + \mathtt{r})$.

Logic-dependent bounds (Measure m and oracle r)

Guidelines:

- Contractions: annotate sequents with contraction index, e.g.

$$\frac{x{:}\mathcal{M}A_1 \ldots A_n, x{:}\mathcal{M}A_1 \ldots A_n, \Gamma, \Delta \vdash^{s-1} \Gamma'}{x{:}\mathcal{M}A_1 \ldots A_n, \Gamma, \Delta \vdash^{s} \Gamma'} \mathrm{ClL_s} \ (s > 0)$$

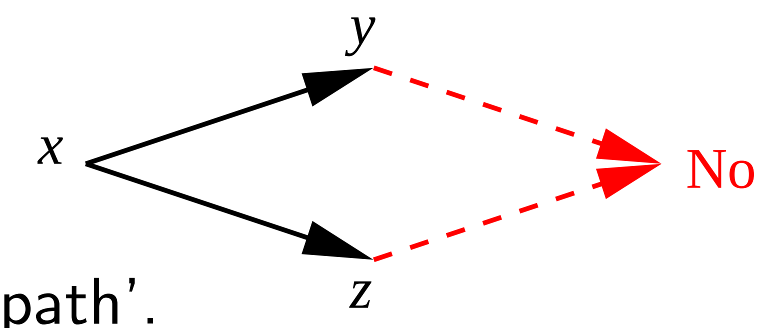
  $\Rightarrow$ (lexicographically ordered) measure (s, Σ), where Σ is size of sequent.

- Relational reasoning: compute space requirement r of oracle.

- ...

Logic-dependent bounds (Measure m and oracle r)

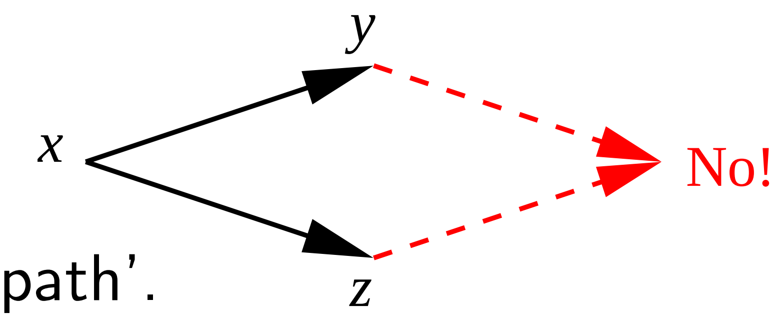
- Theorem ($\Box$-disjunction property): If $\mathrm{S}(\mathcal{L})$ is ‘divergent’, then ClR is eliminable.
  - I.e. every $\vdash x_1{:}D$ provable in $\mathrm{S}(\mathcal{L})$ has a proof with no applications of ClR.
  - Intuition: divergent $=$ ‘follow only one path’.



$$\begin{array}{c} \vdots \\ \dfrac{\dfrac{\dfrac{\ldots, x\,R\,y, x\,R\,z \vdash \ldots, y{:}A, z{:}A}{\ldots, x\,R\,y \vdash \ldots, y{:}A, x{:}\Box A}\;\Box\mathrm{R}}{\ldots \vdash \ldots, x{:}\Box A, x{:}\Box A}\;\Box\mathrm{R}}{\ldots \vdash \ldots, x{:}\Box A}\;\mathrm{ClR} \\ \vdots \\ \vdash x_1{:}D \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \vdots \\ \dfrac{\ldots, x\,R\,\color{red}{y} \vdash \ldots, \color{red}{y}{:}A}{\ldots \vdash \ldots, x{:}\Box A}\;\Box\mathrm{R} \\ \vdots \\ \vdash x_1{:}D \end{array}$$

Logic-dependent bounds (Measure `m` and oracle `r`)

- Theorem ($\Box$-disjunction property): If $\mathrm{S}(\mathcal{L})$ is 'divergent', then ClR is eliminable.
  - I.e. every $\vdash x_1{:}D$ provable in $\mathrm{S}(\mathcal{L})$ has a proof with no applications of ClR.
  - Intuition: divergent = 'follow only one path'.



$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\ldots, x\,R\,y, x\,R\,z \vdash \ldots, y{:}A, z{:}A}}{\ldots, x\,R\,y \vdash \ldots, y{:}A, x{:}\Box A}\,\Box\mathrm{R}}{\ldots \vdash \ldots, x{:}\Box A, x{:}\Box A}\,\Box\mathrm{R}}{\ldots \vdash \ldots, x{:}\Box A}\,\mathrm{ClR}
\qquad \rightsquigarrow \qquad
\dfrac{\dfrac{\vdots}{\ldots, x\,R\,z \vdash \ldots, z{:}A}}{\ldots \vdash \ldots, x{:}\Box A}\,\Box\mathrm{R}
$$

$$
\begin{array}{c}\vdots\\ \vdash x_1{:}D\end{array}
\qquad\qquad
\begin{array}{c}\vdots\\ \vdash x_1{:}D\end{array}
$$

- Divergent logics: K, D, T, $\mathrm{K4}$, $\mathrm{KD4}$, $\mathrm{S4}$, $\mathrm{B}[\rightarrow, \wedge]$, B^{+}, ... (not $\mathrm{S5}$!)

  $\Rightarrow$ Only remains to analyze ClL in each logic.

Modular analysis of ClL (proof of $\vdash x_1{:}D$)

- ClL is eliminable in $\mathrm{S(K)}$.

- ClL is not eliminable in $\mathrm{S(T)}$, e.g. $\vdash x{:}\sim\Box\sim(B \supset \Box B)$, but we need at most $O(n)$ applications of ClL in each branch, with $n = |\vdash x_1{:}D|$

$$\frac{x{:}\Box A, x{:}\Box A, \Gamma, \Delta \vdash^{s-1} \Gamma'}{x{:}\Box A, \Gamma, \Delta \vdash^{s} \Gamma'}\,\mathrm{ClL_s}$$

- ClL is not eliminable in $\mathrm{S(K4)}$ and $\mathrm{S(S4)}$, but we need at most $O(n^3)$ applications in each branch.

- ClL is eliminable in $\mathrm{B}[\rightarrow, \wedge]$ and B^{+}.

Summary

- A proof-theoretic recipe for bounding the complexity of non-classical logics:

  (1) logics presented as cut-free labelled sequent systems,

  (2) combination of bounds on non-simplifying rules.

- Examples: K, T, $\mathrm{K4}$, $\mathrm{S4}$, $\mathrm{B}[\rightarrow, \wedge]$ and B^{+} are decidable in PSPACE (bounds new/compare well with best currently known).

Let $n = |\vdash x_1{:}D|$

	ClL	generated sequent	proof depth	stack entry	space
$\mathrm{S(K)}$	none	$O(n\, log\, n)$	$O(n)$	$O(log\, n)$	$O(n\, log\, n)$
$\mathrm{S(T)}$	$\mathrm{ClL_s}$	$O(n^2\, log\, n)$	$O(n^2)$	$O(log\, n)$	$O(n^2\, log\, n)$
$\mathrm{S(K4)}$; $\mathrm{S(S4)}$	$\mathrm{ClL_s}$	$O(n^4\, log\, n)$	$O(n^4)$	$O(log\, n)$	$O(n^4\, log\, n)$
$\mathrm{S(B[\rightarrow, \wedge])}$; $\mathrm{S(B^+)}$	none	$O(n\, log\, n)$	$O(n)$	$O(log\, n)$	$O(n\, log\, n)$

Standard sequent systems for modal logics

- $\mathrm{SS(K)}$ is

$$\frac{}{A, \Sigma \vdash \Sigma', A}\,(\mathrm{AX}) \qquad \frac{\Sigma \vdash \Sigma', A}{\sim A, \Sigma \vdash \Sigma'}\,(\sim\mathrm{L}) \qquad \frac{A, \Sigma \vdash \Sigma'}{\Sigma \vdash \Sigma', \sim A}\,(\sim\mathrm{R})$$

$$\frac{\Sigma \vdash \Sigma', A \quad B, \Sigma \vdash \Sigma'}{A \supset B, \Sigma \vdash \Sigma'}\,(\supset\mathrm{L}) \qquad \frac{A, \Sigma \vdash \Sigma', B}{\Sigma \vdash \Sigma', A \supset B}\,(\supset\mathrm{R}) \qquad \frac{\Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A, \Sigma'}\,(\mathrm{K})$$

- $\mathrm{SS(T)} = \mathrm{SS(K)} + \dfrac{A, \Box A, \Sigma \vdash \Sigma'}{\Box A, \Sigma \vdash \Sigma'}\,(\mathrm{T})$

- $\mathrm{SS(K4)} = \mathrm{SS(K)} + \dfrac{\Gamma, \Box\Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A, \Sigma'}\,(\mathrm{K4})$

- $\mathrm{SS(S4)} = \mathrm{SS(T)} + \dfrac{\Box\Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A, \Sigma'}\,(\mathrm{S4})$

where the Σ's are multisets of formulas.

Justification (and refinement) of standard rules

- Theorem: Our labelled sequent systems provide proof-theoretical justifications (and in some case refinements) of the rules of standard modal sequent systems.

  Intuition:

  1. Derive labelled equivalents of standard rules.
  2. Transform $\mathrm{S}(\mathcal{L})$-proofs into $\mathrm{SS}(\mathcal{L})$-proofs and vice versa (by transforming $\mathrm{S}(\mathcal{L})$-proofs into a block form $\Rightarrow$ sequences of local and transitional reasoning).

- For (K):

$$\frac{y{:}\Gamma \vdash y{:}A}{x{:}\Sigma, x{:}\Box\Gamma \vdash x{:}\Box A, x{:}\Sigma'}\,\Box\mathrm{LR}_{\mathrm{K}} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{y{:}\Gamma \vdash y{:}A}{y{:}\Gamma, x\,R\,y \vdash y{:}A}\,\mathrm{WrL} \\ \vdots\ \Box\mathrm{L}\ (\text{all with active rwff } x\,R\,y) \\ \dfrac{x{:}\Box\Gamma, x\,R\,y \vdash y{:}A}{x{:}\Box\Gamma \vdash x{:}\Box A}\,\Box\mathrm{R} \\ \vdots\ \mathrm{W} \\ x{:}\Sigma, x{:}\Box\Gamma \vdash x{:}\Box A, x{:}\Sigma' \end{array}$$

Justification (and refinement) of standard rules

- For (T):

$$\frac{x{:}A, x{:}\Box A, x{:}\Sigma \vdash x{:}\Sigma'}{x{:}\Box A, x{:}\Sigma \vdash x{:}\Sigma'}\,\Box\mathrm{L}_{\mathrm{T}} \quad \rightsquigarrow \quad \cfrac{\cfrac{\overline{\vdash x\,R\,x}\,\mathit{refl} \qquad x{:}A, x{:}\Box A, x{:}\Sigma \vdash x{:}\Sigma'}{x{:}\Box A, x{:}\Box A, x{:}\Sigma \vdash x{:}\Sigma'}\,\Box\mathrm{L}}{x{:}\Box A, x{:}\Sigma \vdash x{:}\Sigma'}\,\mathrm{ClL}$$

Exploiting our results we can refine $\mathrm{SS}(\mathrm{T})$ by replacing

$$\frac{A, \Box A, \Sigma \vdash \Sigma'}{\Box A, \Sigma \vdash \Sigma'}\,(\mathrm{T})$$

with

$$\frac{\Box A, \Box A, \Sigma \vdash^{s-1} \Sigma'}{\Box A, \Sigma \vdash^{s} \Sigma'}\,(\mathrm{ClLs}) \qquad \frac{A, \Sigma \vdash^{s} \Sigma'}{\Box A, \Sigma \vdash^{s} \Sigma'}\,(\mathrm{T2})$$

Example of transformation $\mathrm{S(T)} \rightsquigarrow \mathrm{SS(T)}$

We transform previous proof into block form:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{refl}
\UnaryInfC{$\vdash x\,R\,x$}
\AxiomC{}
\RightLabel{AXr}
\UnaryInfC{$x\,R\,y \vdash x\,R\,y$}
\AxiomC{}
\RightLabel{AXl}
\UnaryInfC{$y{:}B \vdash y{:}B$}
\RightLabel{WlR}
\UnaryInfC{$y{:}B \vdash y{:}B, y{:}\Box B$}
\RightLabel{$\supset$R}
\UnaryInfC{$\vdash y{:}B, y{:}B \supset \Box B$}
\RightLabel{$\sim$L}
\UnaryInfC{$y{:} \sim(B \supset \Box B) \vdash y{:}B$}
\RightLabel{WrL}
\UnaryInfC{$y{:} \sim(B \supset \Box B), x\,R\,y \vdash y{:}B$}
\RightLabel{$\Box$L}
\BinaryInfC{$x{:}\Box \sim(B \supset \Box B), x\,R\,y \vdash y{:}B$}
\RightLabel{$\Box$R}
\UnaryInfC{$x{:}\Box \sim(B \supset \Box B) \vdash x{:}\Box B$}
\RightLabel{WlL}
\UnaryInfC{$x{:}\Box \sim(B \supset \Box B), x{:}B \vdash x{:}\Box B$}
\RightLabel{$\supset$R}
\UnaryInfC{$x{:}\Box \sim(B \supset \Box B) \vdash x{:}B \supset \Box B$}
\RightLabel{$\sim$L}
\UnaryInfC{$x{:} \sim(B \supset \Box B), x{:}\Box \sim(B \supset \Box B) \vdash$}
\RightLabel{$\Box$L}
\BinaryInfC{$x{:}\Box \sim(B \supset \Box B), x{:}\Box \sim(B \supset \Box B) \vdash$}
\RightLabel{ClL}
\UnaryInfC{$x{:}\Box \sim(B \supset \Box B) \vdash$}
\RightLabel{$\sim$R}
\UnaryInfC{$\vdash x{:} \sim\Box \sim(B \supset \Box B)$}
\end{prooftree}
$$

Example of transformation $\mathrm{S(T)} \leadsto \mathrm{SS(T)}$

Then into

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y{:}B \vdash y{:}B, y{:}\Box B}\,\mathrm{AXl}}{\vdash y{:}B, y{:}B \supset \Box B}\,\supset\mathrm{R}}{y{:}\sim(B \supset \Box B) \vdash y{:}B}\,\sim\mathrm{L}}{x{:}\Box\sim(B \supset \Box B), x{:}B \vdash x{:}\Box B}\,\Box\mathrm{LR_K}}{x{:}\Box\sim(B \supset \Box B) \vdash x{:}B \supset \Box B}\,\supset\mathrm{R}}{x{:}\sim(B \supset \Box B), x{:}\Box\sim(B \supset \Box B) \vdash}\,\sim\mathrm{L}}{x{:}\Box\sim(B \supset \Box B) \vdash}\,\Box\mathrm{L_T}}{\vdash x{:}\sim\Box\sim(B \supset \Box B)}\,\sim\mathrm{R}$$

so that SS(T)-proof is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B, \Box B}\,(\mathrm{AX})}{\vdash B, B \supset \Box B}\,(\supset\mathrm{R})}{\sim(B \supset \Box B) \vdash B}\,(\sim\mathrm{L})}{\Box\sim(B \supset \Box B), B \vdash \Box B}\,(\mathrm{K})}{\Box\sim(B \supset \Box B) \vdash B \supset \Box B}\,(\supset\mathrm{R})}{\sim(B \supset \Box B), \Box\sim(B \supset \Box B) \vdash}\,(\sim\mathrm{L})}{\Box\sim(B \supset \Box B) \vdash}\,(\mathrm{T})}{\vdash\sim\Box\sim(B \supset \Box B)}\,(\sim\mathrm{R})$$

For $(\mathrm{K4})$ and $(\mathrm{S4})$

$$\dfrac{x_{i+1}{:}\Gamma, x_{i+1}{:}\Box\Gamma \vdash x_{i+1}{:}A}{x_i{:}\Sigma, x_i{:}\Box\Gamma \vdash x_i{:}\Box A, x_i{:}\Sigma'}\ \Box\mathrm{LR}_{\mathrm{K4}} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{x_{i+1}{:}\Gamma, x_{i+1}{:}\Box\Gamma \vdash x_{i+1}{:}A}{x_{i+1}{:}\Gamma, x_{i+1}{:}\Box\Gamma, x_i\,R\,x_{i+1} \vdash x_{i+1}{:}A}\ \mathrm{WrL} \\ \vdots\ \Box\mathrm{L}_{\mathrm{K4}}\ (\text{all with active rwff } x_i\,R\,x_{i+1}) \\ \dfrac{x_i{:}\Box\Gamma, x_i\,R\,x_{i+1} \vdash x_{i+1}{:}A}{x_i{:}\Box\Gamma \vdash x_i{:}\Box A}\ \Box\mathrm{R} \\ \vdots\ \mathrm{W} \\ x_i{:}\Sigma, x_i{:}\Box\Gamma \vdash x_i{:}\Box A, x_i{:}\Sigma' \end{array}$$

by a suitable number of applications of

$$\dfrac{\Delta \vdash x_i\,R\,x_j \quad x_j{:}A, x_j{:}\Box A, \Gamma, \Delta \vdash \Gamma'}{x_i{:}\Box A, \Gamma, \Delta \vdash \Gamma'}\ \Box\mathrm{L}_{\mathrm{K4}}$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\Delta \vdash x_i\,R\,x_j \qquad \dfrac{x_i{:}\Box A \vdash x_i{:}\Box\Box A \qquad \dfrac{\Delta \vdash x_i\,R\,x_j \quad x_j{:}A, x_j{:}\Box A, \Gamma, \Delta \vdash \Gamma'}{x_j{:}A, x_i{:}\Box\Box A, \Gamma, \Delta \vdash \Gamma'}\ \Box\mathrm{L}}{x_j{:}A, x_i{:}\Box A, \Gamma, \Delta \vdash \Gamma'}\ cut}{x_i{:}\Box A, x_i{:}\Box A, \Gamma, \Delta \vdash \Gamma'}\ \Box\mathrm{L}}{x_i{:}\Box A, \Gamma, \Delta \vdash \Gamma'}\ \mathrm{ClL}$$

Yields justification of $\mathrm{SS}(\mathrm{K4})$, but no immediate refinement because of cut.

Analogous for $(\mathrm{S4})$ and $\mathrm{SS}(\mathrm{S4})$.

Road Map

- Introduction: A framework for non-classical logics.
- Labelled deduction for modal logics.
- Labelled deduction for non-classical logics.
- Encoding non-classical logics in Isabelle.
- Substructural and complexity analysis of labelled non-classical logics.
- Conclusions and outlook.

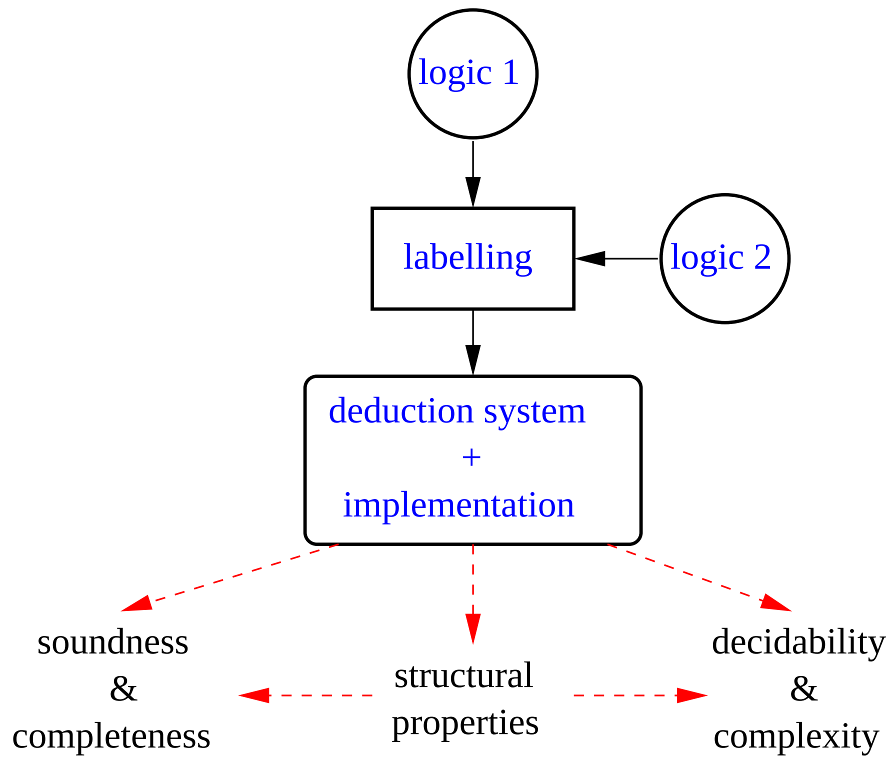
Conclusions and outlook

- A framework for non-classical logics.
  - Labelled ‘natural’ deduction systems.
  - Structural properties vs. generality.
  - Structure
    $\Rightarrow$ implementation, decidability, complexity, justification of standard rules.

- Outlook:
  - Decidability and complexity of relevance logics?
  - Other logics?
  - Increase automation for applications in ‘real’ world.

Conclusions and outlook: combination of logics

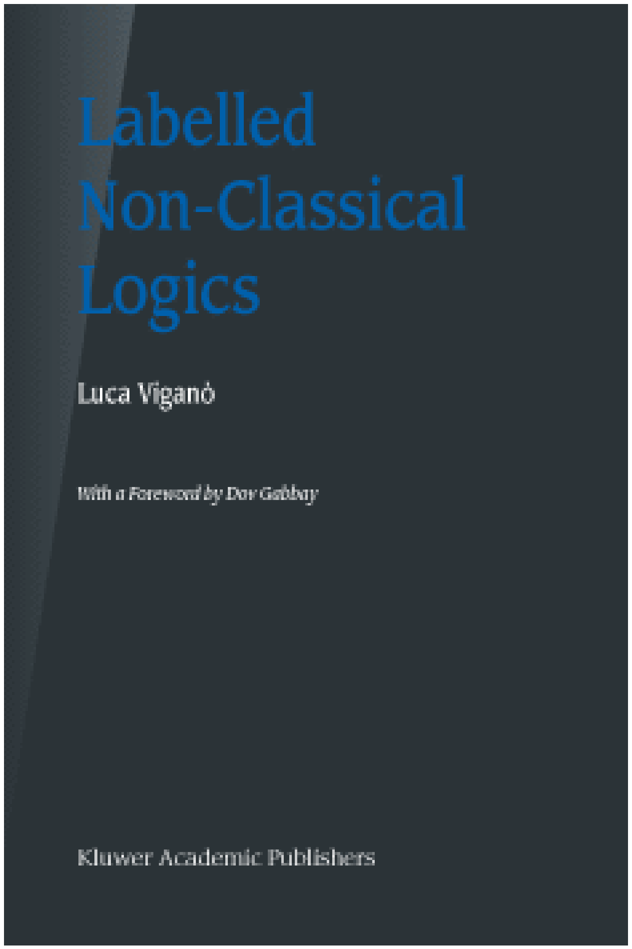
Labelled deductive systems provide a suitable basis for combination/fibring of logics (see papers by D. Gabbay, A. Sernadas, C. Sernadas, and many many many others):

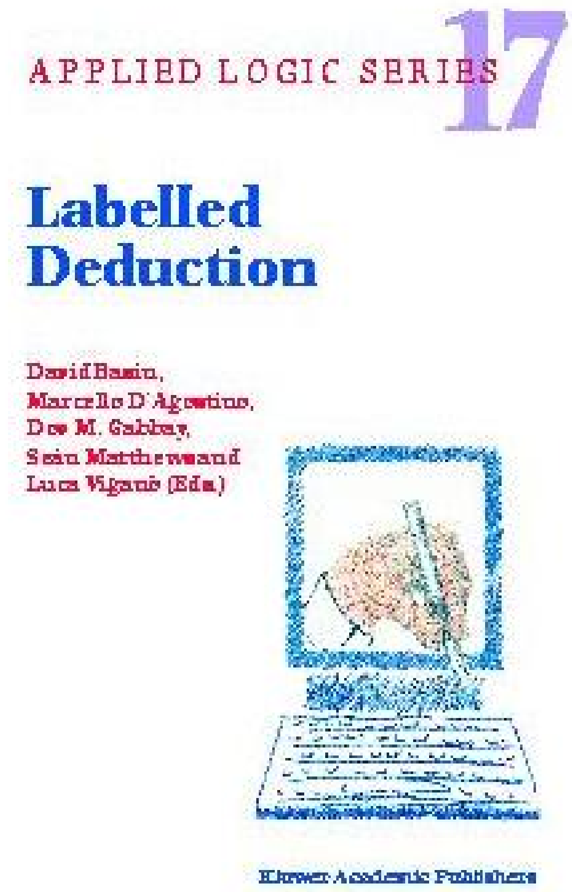


See also "translations", "hybrid logics", "substructural logics", ...

(Labelled non-classical logics, Labelled Deductive Systems, Labelled Deduction, ...)

See also www.inf.ethz.ch/~vigano





Eliminating ClL in $\mathrm{S(K)}$

- Theorem: ClL is eliminable in $\mathrm{S(K)}$, i.e. every $\vdash x_1{:}D$ provable in $\mathrm{S(K)}$ has a proof with no applications of ClL.

  By 3 nested inductions (number, grade, rank of contractions).

  $\Rightarrow \mathtt{m}$ is $O(n)$

  $\Delta \vdash x\,R\,y$ is provable iff $x\,R\,y \in \Delta \Rightarrow \mathtt{r}$ is $O(n)$

- Theorem: Overall space required $O(\mathtt{m}\log\mathtt{m} + \mathtt{r})$ is $O(n\,log\,n)$.

Eliminating ClL in $\mathrm{S(K)}$ (cont.)

Example of a case:

$$
\begin{array}{c}
\vdots \\
\dfrac{\cdots \quad z{:}B, \ldots, x\,R\,z \vdash \ldots}{x{:}\Box B, \ldots, x\,R\,z \vdash \ldots}\ \Box\mathrm{L} \\
\vdots \\
\dfrac{x{:}\Box B, \ldots, x\,R\,z \vdash \ldots, z{:}C}{x{:}\Box B, \ldots \vdash \ldots, x{:}\Box C}\ \Box\mathrm{R} \\
\vdots \\
\dfrac{\dfrac{\cdots \quad y{:}B, x{:}\Box B, \ldots, x\,R\,y \vdash \ldots}{x{:}\Box B, x{:}\Box B, \ldots, x\,R\,y \vdash \ldots}\ \Box\mathrm{L}}{x{:}\Box B, \ldots, x\,R\,y \vdash \ldots}\ \mathrm{ClL} \\
\vdots \\
\vdash x_1{:}D
\end{array}
$$

Permutations:

$$
\begin{array}{c}
\vdots \\
\dfrac{\dfrac{\cdots \quad \dfrac{\cdots \quad y{:}B, z{:}B, \ldots, x\,R\,y, x\,R\,z \vdash \ldots, z{:}C}{y{:}B, x{:}\Box B, \ldots, x\,R\,y, x\,R\,z \vdash \ldots, z{:}C}\ \Box\mathrm{L}}{x{:}\Box B, x{:}\Box B, \ldots, x\,R\,y, x\,R\,z \vdash \ldots, z{:}C}\ \Box\mathrm{L}}{x{:}\Box B, x{:}\Box B, \ldots, x\,R\,y \vdash \ldots, x{:}\Box C}\ \Box\mathrm{R} \\
\vdots \\
\dfrac{x{:}\Box B, x{:}\Box B, \ldots, x\,R\,y \vdash \ldots}{x{:}\Box B, \ldots, x\,R\,y \vdash \ldots}\ \mathrm{ClL} \\
\vdots \\
\vdash x_1{:}D
\end{array}
$$

Then:

$$
\begin{array}{c}
\vdots \\
\dfrac{\dfrac{\cdots \quad y{:}B, \ldots, x\,R\,y, x\,R\,z \vdash \ldots, z{:}C}{x{:}\Box B, \ldots, x\,R\,y, x\,R\,z \vdash \ldots, z{:}C}\ \Box\mathrm{L}}{x{:}\Box B, \ldots, x\,R\,y \vdash \ldots, x{:}\Box C}\ \Box\mathrm{R} \\
\vdots \\
x{:}\Box B, \ldots, x\,R\,y \vdash \ldots \\
\vdots \\
\vdash x_1{:}D
\end{array}
$$

Bounding ClL in $\mathrm{S(T)}$

- ClL is not eliminable in $\mathrm{S(T)}$.

$$\vdash x_1{:}\sim\Box\sim(B \supset \Box B)$$

requires 1 application of ClL.

$$\vdash x_1{:}\Box^p((C \supset\sim\Box\sim D) \wedge (D \supset\sim\Box\sim E)\wedge\sim E) \supset \Box\sim C \qquad (p \geq 3)$$

requires 2 applications of ClL, but can be instantiated to require more, e.g. by replacing '$\wedge\sim E$' with '$\wedge(E \supset\sim\Box\sim F)\wedge\sim F$' and requiring that $p \geq 4$.

- Lemma: At most one left contraction of each $x{:}\Box A$ in each branch.
  Intuition: in each branch we need at most two instances of each $x{:}\Box A$ in the antecedent of a sequent: one for $x{:}A$ and one for $z{:}A$ for a new world z that is a successor of x.

- Lemma: ClL only if A contains a negative subformula of the form $\Box B$, i.e. we only contract of the form $x{:}\Box A[\Box B]_-$.
  Intuition: we create a new world.

Bounding ClL in $\mathrm{S(T)}$ (cont.)

- Given $S = \Gamma, \Delta \vdash \Gamma'$, $pbs(S)$ and $nbs(S)$ are the number of positive and negative boxed subformulas of S.

- Lemma: At most $pbs(S)$ contractions in each branch.

- Theorem: Every sequent $S = \vdash x_1{:}D$ provable in $\mathrm{S(T)}$ has a proof in which there are no contractions, except for applications of ClL with principal formula of the form $x_i{:}\Box A[\Box B]_-$. However, ClL need not be applied more than $pbs(S)$ times in each branch. Hence, we can restrict ClL to be $\mathrm{ClL_s}$ with s set to $pbs(S)$ at the start of the backwards proof, i.e. $\vdash^{pbs(\vdash x_1:D)} x_1{:}D$.

  $\Rightarrow$ Measure (s, Σ) is $O(n^2)$, since $pbs(S)$ and size Σ of S are both $O(n)$.

  $\Delta \vdash x\,R\,y$ is provable iff $x\,R\,y \in \Delta$ or y is $x \Rightarrow \mathtt{r}$ is $O(n)$

- Theorem: Overall space required $O(\mathtt{m} \log \mathtt{m} + \mathtt{r})$ is $O(n^2\, log\, n)$.

Bounding ClL in $\mathrm{S(K4)}$ and $\mathrm{S(S4)}$

- ClL is not eliminable in $\mathrm{S(K4)}$ and $\mathrm{S(S4)}$.

$$\vdash x_1{:}\Box(\bigwedge_{i=1}^{n}(C_i \supset \sim\Box\sim C_{i+1}) \wedge \sim C_n) \supset \Box\sim C_1$$

requires i contractions of

$$x_1{:}\Box(\bigwedge_{i=1}^{n}(C_i \supset \sim\Box\sim C_{i+1}) \wedge \sim C_n)$$

namely, one contraction for each $\Box$ that occurs negative in it (i.e. one for each of its subformulas $\Box\sim C_{i+1}$).

Moreover, it can be modified to require more contractions.

$\Rightarrow$ We obtain a formula such that for each subformula that has a positive $\Box$ as its main operator we need at most as many contractions as there are $\Box$'s that occur negative in its scope. That is, $O(|\vdash x_1{:}D|^2)$ left contractions.

Bounding ClL in $\mathrm{S(K4)}$ and $\mathrm{S(S4)}$

- ClL is not eliminable in $\mathrm{S(K4)}$ and $\mathrm{S(S4)}$

  $\Rightarrow$ Infinite chains $x_1, x_2, x_3, x_4, \ldots$ may arise.

  $\Rightarrow$ Infinite branches.

  $\Rightarrow$ Proof search does not terminate.

- Possible solution: infinite chains are periodic:

  there exist worlds x_i and x_j in the chain such that x_j is accessible from x_i, and A holds at x_j iff A holds at x_i.

  - Dynamic loop checkers to truncate chains and branches:
    proof search terminates but requires history (computationally expensive).
  - Static counter-part: a-priori polynomial bounds on the number of applications of ClL in each branch.

Bounding ClL in $\mathrm{S(K4)}$ and $\mathrm{S(S4)}$

- Extend results for $\mathrm{S(K)}$ and $\mathrm{S(T)}$ and combine them with

- polynomial bound on length of branches.

- Lemma: There is a proof of $S = \vdash x_1{:}D$ such that in each branch $\Box\mathrm{R}$ is applied at most $pbs(S) + 1$ times with principal formula $\Box B$ labelled with increasing worlds in a chain.
  Intuition: consider set of positive boxed subformulas

- Lemma: In each branch there are at most $nbs(S) \times (pbs(S) + 1)$ applications of $\Box\mathrm{R}$, so that chains contain at most $1 + nbs(S) \times (pbs(S) + 1)$ worlds.
  Intuition: at most $nbs(S)$ negative boxed subformulas in S, and ClR eliminable by $\Box$-disjunction property.

Bounding ClL in $\mathrm{S(K4)}$ and $\mathrm{S(S4)}$

- Lemma: In each branch there are at most $(nbs(S) \times (pbs(S) + 1)) - 1$ applications of ClL with the same principal formula $x_i{:}\Box A$, so that there is one instance of $x_i{:}\Box A$ for each world accessible from x_i.

- Theorem: In each branch there are at most $((nbs(S) \times (pbs(S) + 1)) - 1) \times pbs(S)$ applications of ClL with principal formula of the form $x_i{:}\Box A[\Box B]_-$. Hence, we can restrict ClL to be $\mathrm{ClL_s}$.

  $\Rightarrow$ Chains may consist of $O(n^2)$ worlds.

  $\Rightarrow$ Branches may contain $O(n^3)$ applications of ClL.

  $\Rightarrow$ Measure (s, Σ) is $O(n^4)$.

  $\Delta \vdash x\,R\,y$ provable by (reflexive-)transitive closure of $\Delta \Rightarrow \mathtt{r}$ is $O(n)$.

- Theorem: Overall space required $O(\mathtt{m} \log \mathtt{m} + \mathtt{r})$ is $O(n^4\, log\, n)$.

A smaller bound?

Conjecture

chains contain at most $1 + nbs(S) \times (pbs(S) + 1)$ worlds
$\Rightarrow$ at most $((nbs(S) \times (pbs(S) + 1)) - 1)$ applications of ClL in each branch.

Intuition: transform branches of proofs so that

$$\text{ClL of } x_i{:}\Box A[\Box B]_{-}$$
$$\equiv$$
$$\text{append a new world to the chain that we are constructing}$$

Consequence Relations

Given a language $\mathcal{L}$, a consequence relation is a relation between *finite multisets* of formulas in $\mathcal{L}$ that is

- Reflexivity: $\{A\} \vdash \{A\}$
- Transitivity (cut): if $\{A\} \vdash \{B\}$ and $\{B\} \vdash \{C\}$, then $\{A\} \vdash \{C\}$

If we take *sets* of formulas, instead of *multisets* we call the relation 'regular'.