STOCHASTIC QUANTIZATION OF FINITE DIMENSIONAL SYSTEMS WITH ELECTROMAGNETIC INTERACTIONS

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Abstract. We study the stochastic quantization of finite dimensional systems via path-wise calculus of variations with the mean discretized classical action in the general case of electromagnetic interactions. We show that there exists a unique choice of the mean discretized action corresponding to the minimal classical magnetic coupling and derive the general equations of motion by means of a path-wise stochastic calculus of variations.

In the case of purely scalar interactions the total mean energy of the system (which gives the usual quantum mechanical expectation of the Hamiltonian in the canonical limit) works as Lyapunov functional and the system relaxes on the canonical solutions, represented by Nelson's diffusions, which act as an attracting set. We show that, in presence of a minimal magnetic coupling, the mean energy is no longer a Lyapunov functional. We construct for a simple example a new Lyapunov functional and we show that the system can reach the dynamical equilibrium also by absorbing energy from the external magnetic field.

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1. **Introduction**

Stochastic Quantization by means of variational principles with the classical lagrangian was introduced in [8] (surveys covering also related approaches can be found in [17] and in [9]).

One can show that the procedure in [8] is in fact the analogous of that for a classical ideal fluid in the Euler picture, and it shares with this scheme the non rotational character of solutions. To get all solutions of Euler equations one needs a specific constraint called Lin's constraint [10]. For a generalization of these concepts in the framework of Stochastic Mechanics we refer the reader to [13].

Since the classical limit of Madelung equations gives Euler equations, a natural question is whether it is possible or not to reformulate the stationary action principle with the classical action in order to get equations which lead to the unrestricted Euler equations in the classical limit. This was done in [14] and [12], exploiting a path-wise stochastic calculus of variations, extending to a stochastic system the Lagrangian method for an ideal fluid (for an extension to curved manifolds see [1]).

The new equations, which reduce to the unrestricted Euler equations in the classical limit, exhibit a dissipation induced by the vorticity of the current velocity field, so that rotational solutions relax on the usual conservative solutions of Schroedinger Equation, in some analogy with the Parisi-Wu stochastic quantization [18]. A non trivial feature of the rotational solutions is that the vorticity, which does not go to zero monotonically, can concentrate in the nodes of the wave function, approaching vortex line solutions [3]. As a consequence, the new equations give the correct quantization, possibly with a smooth approximation of the singularities of the Madelung solutions. We also refer the reader to [11] and [2] for applications to a system of identical pair interacting Bosons.

In this work we face the problem of considering a minimal magnetic coupling : this can be easily managed in the Euler picture (see for example [17]), leading to the canonical dynamics, while it is open for the Lagrangian picture, due to a non uniqueness of the discretized action functional in this case, and difficulties in describing the relaxation on canonical solutions.

In particular we show that there is a unique choice of the discretization of the path-wise minimal coupling which is consistent with the Eulerian picture and we derive equations of motions in Madelung variables by means of the path-wise stochastic calculus of variations introduced in [14].

It was proved in [12] that, in the case of purely scalar interaction, the total mean energy of the system (which gives the usual quantum mechanical expectation of the Hamiltonian in the canonical limit) works as a Lyapunov functional, and the system relaxes on the canonical solutions, represented by Nelson's diffusions, which act as an attracting set (see also [16]). We show that, in presence of a minimal magnetic coupling, the mean energy is no longer a Lyapunov functional. A Lyapunov functional is constructed for a simple gaussian example.We show that in this case the system can reach the dynamical equilibrium also by absorbing energy from the external magnetic field.

2. **Basics and Euler picture**

We consider a classical N-body system described by the following Lagrangian

$$L := \sum_{j=1}^{N}(\frac{1}{2}m_j(\dot{q}_j)^2 + \mathcal{A}_j \cdot \dot{q}_j) - \Phi$$

where m_j, $j = 1, \ldots, N$ denotes the mass of the j-th particle, q_j is the position of the j-th particle in $\mathbb{R}^3$, $\mathcal{A}_j$ and Φ are, respectively, a three-dimensional time dependent magnetic vector and a scalar potential depending on the configuration q in $\mathbb{R}^{3N}$.

Stochastic quantization, in the Nelson's sense, basically comes from requiring that the configuration performs a Markov diffusion in $\mathbb{R}^{3N}$.

The natural class of diffusions in this setting was introduced by Carlen [4] in 1984 : it is given by all diffusions with constant diffusion coefficient (assumed to be equal to $\frac{\hbar}{m}$ in this setting) and such that there exist a time dependent density ρ, with finite second moment, and two vector fields u and v which determine the drift by the equality $v_+ = v + u$, satisfying the following assumptions

i) $u = \frac{\hbar}{2m}\nabla ln\rho$

ii)The couple (ρ, v) satisfies, at least in a weak sense, the continuity equation

$$\partial_t\rho + \nabla \cdot (\rho v) = 0 \tag{2.1}$$

iii) The condition of finite energy holds, that is

$$\int_0^t \int_{\mathbb{R}^d} (u^2 + v^2)\rho dx dt < \infty$$

Notice that the drift v_+ can be unbounded. The existence of diffusions of this type is proved in [4] (th. 4.1).

In particular, (see also [6] and [7]), there exists on the canonical space $C([0, T] \to \mathbb{R}^d), \mathcal{B}(C([0, T] \to \mathbb{R}^d))$, $T > 0$, a probability measure $\mathbb{P}$ such that, denoting by W and W^* respectively a forward and a backward $\mathbb{P}$- Brownian motions, the coordinate process q satisfies $\mathbb{P}$-a.s. the equalities

$$q(t) = q(0) + \int_0^t v_+ \left(q(s), s\right) ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} W(t) \tag{2.2}$$

$$q(t) = q(0) + \int_0^t v_-\left(q(s), s\right) ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} W^*(t) \tag{2.3}$$

In 1985 Carlen proved that all Nelson diffusions, that is all diffusions which are associated to solutions of Schrödinger equation in Stochastic Mechanics, belong to the above defined class in great generality (it is sufficient to assume that the scalar potential belongs to the Rellich class and has finite initial kinetic energy)[4]. For the case with a magnetic potential it is sufficient to assume that the scalar potential is in the Rellich class and that the vector potential is in $L^2_{loc}(\mathbb{R}^3)$ (see [19]).

To avoid notational complications, in the following we will assume $N = 1$. The generalization to any finite N does not give any additional difficulty.

To introduce the dynamics we consider the mean of a discretized version of the classical action functional: for a given finite time interval $[t_a, t_b]$ we consider the equipartition $\{t_i\}_0^n$ and make use of the following notations

$\Delta := \frac{t_b - t_a}{n}$

$\Delta^+ q(t_i) := q(t_{i+1}) - q(t_i)$ future increment

$\Delta^- q(t_i) := q(t_i) - q(t_{i-1})$ past increment

Taking into account all discretized versions which are equivalent in the classical case, we write the generic mean discretized actions as

$$\begin{aligned} A^{(n)}[q] := \mathbb{E} \sum_{i=1}^{n} \Bigg[\frac{1}{2} m \frac{\Delta^{+(-)} q(t_i) \cdot \Delta^{+(-)} q(t_i)}{\Delta^2} + \\ + \frac{1}{2} \mathcal{A}(q(t_i), t_i) \cdot \frac{(\Delta^{+(-)} q(t_i) + \Delta^{+(-)} q(t_i))}{\Delta} - \Phi(q(t_i)) \Bigg] \Delta \end{aligned} \tag{2.4}$$

where q is the triple $[W, v_+, q_o]$

Remark 1

We recall that the following equality holds, with $u := \frac{\hbar}{2m} \nabla ln\rho$ denoting the **osmotic velocity**:

$$
\begin{aligned}
lim_{n\to\infty}\ \mathbb{E}\sum_{i=1}^{n}\frac{1}{2}m\frac{\Delta^{+}q(t_i)\cdot\Delta^{-}q(t_i)}{\Delta^{2}} &=\\
=&\mathbb{E}\int_{t_a}^{t_b}\frac{1}{2}mv_{+}(q(s),s)\cdot v_{-}(q(s),s)=\\
=&\mathbb{E}\int_{t_a}^{t_b}\left[\frac{1}{2}mv^{2}-\frac{1}{2}mu^{2}\right]ds
\end{aligned}
\tag{2.5}
$$

On the other side one can notice (see [17]) that, by exploiting the backward representation (2.2) and estimating $\Delta^{+}q(t_i)$ to the order $\Delta^{\frac{3}{2}}$ in the Ito calculus, we get

$$
\begin{aligned}
\Delta^{+}q(t)=&\left(\frac{\hbar}{m}\right)^{\frac{1}{2}}\Delta W(t)+v_{+}\left(q(t),t\right)\Delta+\\
&+\left(\frac{\hbar}{m}\right)^{\frac{1}{2}}\sum_{k=1}^{3}\left[\partial_k v_{+}\left(q(t),t\right)\int_{t}^{t+\Delta}\left(W_k(s)-W_k(t)\right)ds\right]+\\
&+o(\Delta^{\frac{3}{2}})
\end{aligned}
\tag{2.6}
$$

so that

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^{N}\frac{1}{2}m\frac{\Delta^{+}q(t_i)\cdot\Delta^{+}q(t_i)}{\Delta^{2}}&=\\
&=\mathbb{E}\sum_{i=1}^{N}\left[\frac{1}{2}m\frac{\Delta^{+}q(t_i)\cdot\Delta^{-}q(t_i)}{\Delta^{2}}+\frac{3}{2}\frac{\hbar}{\Delta}+o(\Delta)\right]\Delta
\end{aligned}
\tag{2.7}
$$

As a consequence, a first possibility of extending the classical variational procedure is to go to the continuum limit and to drop the possible divergent terms, which are of zero variation if only the drift of the test diffusion is varied (or, equivalently, the current velocity and the density are varied). In the gauge $\nabla\cdot\mathcal{A}=0$, one easily gets the following action functional

$$
I[\rho,v]:=\int_{\mathbb{R}^3}\int_{t_a}^{t_b}\left\{\frac{1}{2}mv^{2}+\mathcal{A}\cdot v-\frac{1}{2}mu^{2}-\Phi\right\}\rho dxdt \tag{2.8}
$$

For $\mathcal{A}=0$, this coincides with the one considered in Guerra-Morato [8]. The case with electromagnetic interaction in the Euler picture is strictly analogous (see for example [17]). The class of test diffusions considered in this approach is that of diffusions in the Carlen's class with smooth drift.

The equations of motion are found to be

$$
mv(x,t)=\nabla S(x,t)-\mathcal{A}(x,t) \tag{2.9}
$$

$$\left[\partial_t S + \frac{1}{2m}(\nabla S - \mathcal{A})^2 - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} + \Phi\right](x,t) = 0 \tag{2.10}$$

With a change of variables the system of (2.10) and the continuity equation reads

$$i\hbar\partial_t\psi = \frac{1}{2m}\left[(i\hbar\nabla + \mathcal{A})^2 + \Phi\right]\psi \tag{2.11}$$

where

$$\psi := \rho^{\frac{1}{2}}\exp(\frac{i}{\hbar}S) \tag{2.12}$$

We recall that the extension of this result to test diffusions belonging to the whole Carlen's class is an open problem.

3. Path-wise (or Lagrangian) picture

The Lagrangian, or path-wise, approach consists in taking path-wise variations of q for fixed W in $A^n_{[t_a,t_b]}[q]$. This eliminates the divergent term.The limit for n going to infinity is taken only at the end of the calculus of variations (see [14] and [12]). Singularities of the drift can develop asymptotically in time if the initial current velocity field is not curl free [3].

Definition 1. A diffusion is said to be admissible for the Lagrangian variational principle if it belongs to the Carlen's class and has smooth drift.

We now recall some details: for the test diffusion $q(t)$ at time t let $q'(t) := q(t) + \delta q(t)$ denote the varied diffusion. We require that this is still an admissible diffusion with the same W.Thus there must exist a smooth drift field v'_+ such that

$$q(t) = q(0) + \int_0^t v_+\left(q(s), s\right) ds + W(t) \tag{3.1}$$

$$q'(t) = q'(0) + \int_0^t v'_+\left(q(s), s\right) ds + W(t) \tag{3.2}$$

We introduce the variation process h and the variation of the drift f by putting, for $\epsilon > 0$,

$$\begin{cases} \varepsilon h(t) := \delta q(t) & \varepsilon > 0 \\ \varepsilon f := v'_+ - v_+ \end{cases} \tag{3.3}$$

Then one finds

$$\dot{h}(t) = \sum_{j=1}^{3} \partial_j v_+\left(q(t), t\right) h_j(t) + f\left(q(t), t\right) \tag{3.4}$$

that is $h(t)$ is a differentiable stochastic process. It satisfies a first order ODE for every realization of q. As a consequence h cannot be fixed both in t_a and t_b, at variance with the classical case.

Definition 2. A process h will be said to be an"admissible variation process " for the test diffusion q if it is a solution of (3.4) for some smooth f and q' belongs to the Carlen class.

We want now to characterize the motions which are represented by "critical diffusions " :

Definition 3. An admissible diffusion q^* is critical with fixed initial position if, $\forall\, h$ admissible and $\epsilon > 0$

$$\lim_{n\uparrow\infty} \left\{ A^n_{[t_a,t_b]}\left[q^* + \varepsilon h\right] - A^n_{[t_a,t_b]}\left[q^*\right] - \varepsilon p_{t_b} h_{t_b} \right\} = o(\varepsilon), \quad h(t_a) = 0 \tag{3.5}$$

and an admissible diffusion q^* is critical with fixed final position if, $\forall\, h$ admissible,

$$\lim_{n\uparrow\infty} \left\{ A^n_{[t_a,t_b]}\left[q^* + \varepsilon h\right] - A^n_{[t_a,t_b]}\left[q^*\right] + \varepsilon p_{t_a} h_{t_a} \right\} = o(\varepsilon), \quad h(t_b) = 0 \tag{3.6}$$

where p_{t_a} and p_{t_b} are fixed random variables playing the role of the classical initial and final "momentum ".

Before stating our main result we study in detail the path-wise variation of the term with the magnetic coupling in A^n.

Proposition 1. *The only discretized mean classical action with minimal magnetic coupling for which the path-wise picture is consistent with the Euler picture is given by*

$$\begin{aligned} \bar{A}^n[q] := \mathbb{E} \sum_{i=0}^{n} \Bigg\{ \frac{1}{2} m \frac{\Delta^+ q(t_i) \cdot \Delta^- q(t_i)}{\Delta^2} + \left[\frac{3}{2} \frac{\hbar}{\Delta} + o(\Delta) \right] + \\ + \mathcal{A}(q(t_i), t_i) \cdot \frac{\Delta^+ q(t_i) + \Delta^- q(t_i)}{2\Delta} - \Phi(q(t_i), t_i) \Bigg\} \Delta \end{aligned} \tag{3.7}$$

Proof. Exploiting a discrete "integration by parts" and Ito's rule, for any admissible test diffusion and variation process (with simplified notations), we get the contribution

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{1}{\epsilon} \delta\, \mathbb{E} \left\{ \sum_{i=0}^{n} \mathcal{A}(q(t_i), t_i) \cdot \Delta^+ q(t_i) \right\} = \\ &= \mathbb{E} \left\{ \sum_k h_k \left[\sum_j (\partial_k \mathcal{A}_j - \partial_j \mathcal{A}_k) b_j^+ - \partial_t \mathcal{A}_k \right] \Delta + o(\Delta) \right\} \end{aligned} \tag{3.8}$$

For the term involving the backward increment $\Delta^- q(t_i)$ we make use of (2.3). As a consequence in applying Ito's rule we get terms involving backward increments of W^*, which are not independent of the past of the test diffusion.Then, with a method already exploited in [12] we make use of the equality

$$\Delta^- W_*(t) = 2(\frac{m}{\hbar})^{\frac{1}{2}} u(q(t),t)\Delta + \Delta^+ W(t-\Delta) + o(\Delta) \tag{3.9}$$

and iterate Ito's rule. We find

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\delta\,\mathbb{E}\left\{\sum_{i=0}^{n}\mathcal{A}(q(t_i),t_i)\cdot\Delta^- q(t_i)\right\} = \\
&=\mathbb{E}\left\{\sum_k h_k\left[\sum_j (2u_j)(\partial_k\mathcal{A}_j-\partial_j\mathcal{A}_k)+\sum_j\frac{\hbar}{m}\partial_j(\partial_k\mathcal{A}_j-\partial_j\mathcal{A}_k)\right]\Delta+\right.\\
&\left.+\sum_k h_k\left[\sum_j(\partial_k\mathcal{A}_j-\partial_j\mathcal{A}_k)b_j^- -\partial_t\mathcal{A}_k\right]\Delta+o(\Delta)\right\}
\end{aligned} \tag{3.10}$$

In the Madelung variables (ρ, v), with $v := \frac{1}{m}(\nabla S - \mathcal{A})$, the equations of motion in the Euler picture (2.10) read

$$\begin{cases}\partial_t\rho = -\nabla\cdot(\rho v)\\ \partial_t v + (v\cdot\nabla)\,v - \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) = -\frac{1}{m}\left(\nabla\Phi + \partial_t\mathcal{A} - v\wedge(\nabla\wedge\mathcal{A})\right)\end{cases} \tag{3.11}$$

We notice that the terms involving the external magnetic field are of zero order in $\frac{\hbar}{m}$. It is immediate to see from (3.11) and (3.10) the correct term of zero order in $\frac{\hbar}{m}$ is reproduced only if we discretize the classical coupling term as $\mathcal{A}(q(t_i),t_i)\cdot\frac{\Delta^+ q(t_i)+\Delta^- q(t_i)}{2\Delta}$ □

We are now in a position to generalize to the case with magnetic coupling the stochastic quantization procedure via the path-wise variational principle.

Theorem 1. *A sufficient condition in order an admissible diffusion q^* to be critical with fixed initial condition is*

$$q^*(t) = q^*(0) + \int_0^t v_+\left(q^*(s),s\right)ds + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} W(t)$$

where:

$$v_+ = v + \frac{\hbar}{2m}\nabla\ln\rho$$

and , if the initial position is fixed as a random variable, the couple (ρ, v) is a solution of the following system of partial differential equations

$$\begin{cases} \partial_t \rho = -\nabla \cdot (\rho v) \\ \partial_t v + (v\cdot\nabla)\, v - \dfrac{\hbar^2}{2m^2}\nabla\left(\dfrac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) + \\ \qquad - \dfrac{\hbar}{m}\left(\nabla \ln \rho + \nabla\right)\wedge\left(\nabla\wedge\left(v+\dfrac{1}{m}\mathcal{A}\right)\right) = \\ \qquad = -\dfrac{1}{m}\left(\nabla\Phi + \dfrac{1}{m}\partial_t\mathcal{A} - v\wedge\left(\nabla\wedge\dfrac{1}{m}\mathcal{A}\right)\right) \end{cases} \tag{3.12}$$

with the boundary constraint

$$mv(q_{t_b}, t_b) = p_{t_b}$$

or, if the final position is fixed as a random variable,

$$\begin{cases} \partial_t \rho = -\nabla \cdot (\rho v) \\ \partial_t v + (v\cdot\nabla)\, v - \dfrac{\hbar^2}{2m^2}\nabla\left(\dfrac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) + \\ \qquad + \dfrac{\hbar}{m}\left(\nabla \ln \rho + \nabla\right)\wedge\left(\nabla\wedge\left(v+\dfrac{1}{m}\mathcal{A}\right)\right) \\ \qquad = -\dfrac{1}{m}\left(\nabla\Phi + \dfrac{1}{m}\partial_t\mathcal{A} - v\wedge\left(\nabla\wedge\dfrac{1}{m}\mathcal{A}\right)\right) \end{cases} \tag{3.13}$$

with the boundary constraint

$$mv(q_{t_a}, t_a) = p_{t_a}$$

Proof. The proof follows from that given in detail in [12],[14] and Proposition 1. We recall that the differentiable process h is such that for any $t \in [0,T]$, $h(t)$ is measurable with respect to the σ-algebra generated by $(q(s))_{s\in[0,t]}$ if $h_a = 0$, while it is measurable with respect to the σ-algebra generated by $(q(s))_{s\in[t,T]}$ if $h_b = 0$.

With the correct choice of the discretized action the term with the magnetic coupling gives, by the proof of Proposition 1, if $h_a = 0$,

$$\begin{aligned} &\lim_{\epsilon\to 0}\delta\left(\mathcal{A}(q(t_i),t_i)\cdot\frac{\Delta^+ q(t_i)+\Delta^- q(t_i)}{2}\right) = \\ &\qquad = \mathbb{E}\left\{\sum_{k,j} h_k\left[(\partial_k\mathcal{A}_j - \partial_j\mathcal{A}_k)\frac{b_j^+ + b_j^-}{2} - \partial_t\mathcal{A}_k + \right.\right. \\ &\qquad \left.\left. +\ u_j(\partial_k\mathcal{A}_j - \partial_j\mathcal{A}_k) + \frac{\hbar}{2m}\partial_j(\partial_k\mathcal{A}_j - \partial_j\mathcal{A}_k)\right]\Delta + o(\Delta)\right\} \end{aligned} \tag{3.14}$$

For all other terms we refer to [12], proof af Th.1. Going to the continuum limit we get, if the initial position is fixed as a random variable,

$$\lim_{N\to\infty} \delta A^N_{[t_a,t_b]}[q] = \epsilon\,\mathbb{E}\int_{t_a}^{t_b}\left[-\partial_t v - (v\cdot\nabla)\,v + \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)\right.$$
$$+\frac{\hbar}{m}\left(\nabla\ln\rho + \nabla\right)\wedge\left(\nabla\wedge(v+\mathcal{A}\right)-$$
$$\left.-\frac{1}{m}(\nabla\Phi + \partial_t\mathcal{A} - v\wedge(\nabla\wedge\mathcal{A}))\right](q(t),t)\cdot h(t)dt$$
$$+\epsilon\,\mathbb{E}\left[mv(q_{t_b},t_b) - p_{t_b}\right] \tag{3.15}$$

If the final position is fixed as a random variable, i.e. $h_b = 0$, the only difference comes from the terms of first order in $\frac{\hbar}{m}$, where the role of W^* and W must to be interchanged. This leads to the same contribution as in the previous case, with the opposite sign. We find

$$\lim_{N\to\infty} \delta A^N_{[t_a,t_b]}[q] = \epsilon\,\mathbb{E}\int_{t_a}^{t_b}\left[-\partial_t v - (v\cdot\nabla)\,v + \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)\right.$$
$$-\frac{\hbar}{m}\left(\nabla\ln\rho + \nabla\right)\wedge\left(\nabla\wedge(v+\mathcal{A}\right)-$$
$$\left.-\frac{1}{m}(\nabla\Phi + \partial_t\mathcal{A} - v\wedge(\nabla\wedge\mathcal{A}))\right](q(t),t)\cdot h(t)dt$$
$$+\epsilon\,\mathbb{E}\left[mv(q_{t_a},t_a) - p_{t_a}\right] \tag{3.16}$$

Then the assertion follows, recalling that the couple (ρ, v) satisfies the continuity equation by assumption. □

Following [12] one can easily show that the equations of motion (3.12) and (3.13) can be rewritten as nonlinear equations of Schrödinger type. Putting

$$\mathcal{A}_\psi := \nabla S - (m\mathbf{v} + \mathcal{A})$$

$$\psi := \rho^{\frac{1}{2}}\exp\frac{\imath}{\hbar}S$$

one gets the two systems

$$\begin{cases} \imath\hbar\partial_t\psi = \frac{1}{2m}(\imath\hbar\nabla + \mathcal{A} + \mathcal{A}_\psi)^2\psi + \Phi\psi \\ \partial_t\mathcal{A}_\psi = \mathbf{b}_\pm\wedge(\nabla\wedge\mathcal{A}_\psi) \pm \frac{\hbar}{2m}\nabla\wedge\nabla\wedge\mathcal{A}_\psi \end{cases} \tag{3.17}$$

where

$$\mathbf{b}_\pm := \frac{\nabla S - \mathcal{A} - \mathcal{A}_\psi}{m} \pm \frac{\hbar}{2m}\nabla\log(|\psi|^2) \tag{3.18}$$

It was proved in [12] that, if the external magnetic field $\mathcal{A}$ is zero, the second one, with the + sign in front of the term of first order in $\frac{\hbar}{m}$ and which corresponds to fix the end position as a random variable, is in fact dissipative, as far as $\rho(\nabla \wedge v)$ is different from zero dx-a.s. To be more precise, one can prove that, if (ρ, v) is a smooth solution of (3.13) and ρ has a good behaviour at infinity (a sufficient condition for this is $\frac{\nabla \rho}{\rho}$ going to zero at infinity), we have, introducing the energy functional

$$E[\rho, v] = \int_{\mathbb{R}^3} \left(\frac{1}{2} m v^2 + \frac{1}{2} m u^2 + \Phi \right) \rho d^3 x \tag{3.19}$$

the following equality

$$\frac{dE}{dt} = -\frac{\hbar}{2} \int_{\mathbb{R}^3} (\nabla \wedge v)^2 \rho d^3 x \tag{3.20}$$

Notice that a simple calculation shows that the energy functional reduces to the usual quantum mechanical energy in the case where $\rho(\nabla \wedge \; v)$ is equal to zero dx-a.s..

This means that, taking as physical the variational principle with final position fixed as a random variable, canonical quantization is asymptotically reached after a relaxation, in some analogy with Parisi-Wu stochastic quantization [18]

To generalize this result to the case with magnetic coupling is not trivial.

Let the external magnetic field be time indepedent and put $A_t = \mathcal{A}_\psi + \mathcal{A}$. Then the time derivative of E reads

$$\frac{dE}{dt} = -\int v \cdot \dot{A}_t \rho d^3 x = -\int v \cdot \dot{\mathcal{A}}_\psi \rho d^3 x \tag{3.21}$$

An easy calculation with an integration by parts gives

$$\frac{dE}{dt} = -\frac{\hbar}{2} \mathbb{E}\{(\nabla \wedge v)^2\} - \frac{\hbar}{2m} \mathbb{E}\{(\nabla \wedge v) \cdot (\nabla \wedge \mathcal{A})\} \tag{3.22}$$

It is clear that there exist initial conditions such that $\frac{dE}{dt} \;|_{t=0}$ is positive. As a consequence the functional E, which gives the usual quantum mechanical energy if $\mathcal{A}_\psi(0)$ is identically equal to zero, is no longer a Lyapunov functional for the system. Equation (3.22) shows that, in case a Lyapunov functional exists so that the system relaxes on the canonical solutions, the relaxation can happen by absorbing energy from the external magnetic field.

4. An example : the two-dimensional harmonic oscillator in a constant magnetic field

Let us consider the two-dimensional linear harmonic oscillator in central symmetry, with a constant magnetic field. We put, using polar coordinates (r, θ) (with $\hbar = m = 1$)

$$\Phi(r) = \frac{1}{2}k^2 r^2$$
$$\mathcal{A} = Br\hat{\theta}$$

where A, a, α are time-dependent parameters with $A > 0$, while k and B are time independent.

We look for particular solutions of the type

$$\rho(r,t) = \frac{A(t)}{\pi} exp(-A(t)r^2)$$

$$v(r,t) = a(t)r\hat{r} - \alpha(t)r\hat{\theta} - Br\hat{\theta} \tag{4.1}$$

With $u(r,t) := -A(t)r\hat{r}$, (3.13) becomes the following non linear system of ordinary differential equations

$$\begin{cases} \dot{A} = -2aA \\ \dot{a} = A^2 + \alpha^2 - a^2 - k^2 - B^2 \\ \dot{\alpha} = -2(a+A)\alpha \end{cases} \tag{4.2}$$

where α plays the role of the intrinsic vorticity parameter associated to the wave function.

The energy functional reads

$$\begin{aligned} E &:= \int_{\mathbb{R}^3} \left(\frac{1}{2}mv^2 + \frac{1}{2}mu^2 + \Phi \right) \rho d^3x = \\ &= \frac{1}{2A}(A^2 + a^2 + (\alpha + B)^2 + k^2) \end{aligned} \tag{4.3}$$

and its time derivative along the motion is

$$\frac{dE}{dt} = -2\alpha^2 - 2\alpha B$$

For $\alpha = 0$ we have the representation on the (a, A)-plane of the canonical gaussian solutions of the Schrödinger equation with the external magnetic field $Br\hat{\theta}$, which reads

$$\begin{cases} \dot{A} = -2aA \\ \dot{a} = A^2 - a^2 - k^2 - B^2 \end{cases} \tag{4.4}$$

The corresponding conserved quantum energy is

$$E_Q := \frac{1}{2A}(A^2 + a^2 + B^2 + k^2)$$

A Lyapunov functional for the system (4.2) is

$$W := \frac{1}{2A}(A^2 + a^2 + \alpha^2 + B^2 + k^2)$$

We can see that this functional is positive definite and its time derivative along the motion is

$$\frac{dW}{dt} = -2\alpha^2$$

The minimum value of W is $\sqrt{k^2 + B^2}$, which coincides with the quantum ground state energy of the harmonic oscillator in the external magnetic field $Br\hat{\theta}$. The ground state is represented by the point $(\sqrt{k^2 + B^2}, 0, 0)$ which is the unique fixed point of the three-dimensional system (4.2). Notice that the current velocity in the ground state is equal to $-Br\hat{\theta}$.

In analogy with the case of purely scalar interaction, one can then prove that any solution of the nonlinear dynamical system (4.2) can be continued for t going to ∞. Moreover the same qualitative study given in [15] allows to prove the existence of an unique invariant two-dimensional center manifold which coincides with the Schrödinger plane (A, a). The corresponding Nelson diffusions converge in the sense of the relative entropy [16]. One can also show that there exists a separation plane given by the equation

$$h(\xi) = 0$$

with

$$h(\xi) = h(A, a, \alpha) := (a + A)$$

such that the vorticity decreases when the state ξ remains in the region where $h > 0$ while it increases when $h < 0$. Thus the phenomenon of "increasing vorticity", as firstly described in the case of purely scalar potential [15], is then preserved in the case of this minimal magnetic coupling. It is also very easy to see that both relaxation towards the Schrödinger plane and the increasing vorticity phenomenon can happen by increasing the initial value of the total expected kinetic and potential energy E.

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