LOCALIZATION OF RELATIVE ENTROPY IN BOSE-EINSTEIN CONDENSATION OF TRAPPED INTERACTING BOSONS

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Abstract. We consider a system of interacting diffusions which is naturally associated to the ground state of the Hamiltonian of a system of N pair-interacting bosons and we give a detailed description of the phenomenon of the "localization of the relative entropy".The method is based on peculiar rescaling properties of the mean energy functional.

1. **Introduction**

The new state of matter known as Bose Einstein Condensate was predicted from a theoretical point of view by Bose and Einstein in 1925 and it was confirmed in the experiment only in 1995. To be precise, it was observed that if a large number of identical pair interacting bosons is confined in a trap of macroscopic size at very low temperature, then almost all particles belong to a "condensed" cloud, where every particle is in the same macroscopic one-body quantum state, called the "wave function of the condensate". As far as the interacting case is concerned, this phenomenon was firstly investigated by Bogolubov [4] and later by Gross [5] and Pitaevskii [28]. In the Gross-Pitaevskii theory the wave function of the condensate satisfies a cubic nonlinear Schrödinger equation, in this context called Gross-Pitaevskii equation, where the effect of the interactions gives rise to the non linear term. This model has been widely confirmed by experimental results.

A completely rigorous derivation of the Bose-Einsten condensation, for the case of the ground state of a diluted pair-interacting Bose gas in a trap, was done quite recently by Lieb and Sieringer [22] , exploiting a suitable scaling limit, consistent with the Gross-Pitaewskii theory, with the number of particles going to infinity. In particular they can prove that any finite order reduced density matrix converges in the trace norm to the factorized one.

Stochastic tools have also been considered and in particular the interest in stochastic descriptions has increased during the last decade.

For example boson random point processes (fields or general Cox processes), have been exploited by many authors. For the ideal case we quote [13], [12], [11],[7],[6] and [8]. In particular in [8] the random point field describing the position distribution of the ideal boson gas in a state of Bose-Einstein Condensation is obtained in the thermodynamic limit. Limit theorems for this field, including a large deviation principle, are established in [10].

For the interacting case interesting results were obtained in [14] and in [9].

We also quote the work [3], where the authors exploit a model of spatial random permutations, finding the occurrence of infinite cycles and [2] where large deviation principles are obtained for a model consisting of N mutually repellent Brownian Motions confined in a bounded region.

The possibility offered by Nelson processes, that can be rigorously associated to the quantum N-body Hamiltonians, was considered only very recently[15] [32]. In this approach the N-body system is described by a system of N interacting diffusions, the interaction being described by the structure of the Mean Energy Functional. Under the assumption of strictly positivity and continous differentiability of the many-body ground state wave function, all one-particle diffusions have the same law. This allows to consider a generic one-particle process and to show that, in a proper scaling limit, such a process continuously remains outside a time dependent random *interaction set* with probability one and that its stopped version converges, in a relative entropy sense, toward a Markov diffusion whose drift is uniquely determined by wave function of the condensate [32].

In this paper we focus our attention on the scaling properties of the Mean Energy Functional which is associated to the system of the N interacting diffusions and we describe in detail the phenomenon of the concentration of relative entropy, which plays a fundamental role in understanding the peculiarities of the stochastic motion of a particle in the condensate.

2. **Basics**

We start by considering a single spinless quantum particle of mass m in a potential V. Denoting by ψ its wave function, we know that it is a solution of the Schrödinger equation

$$i\hbar\partial_t\psi = (-\frac{\hbar^2}{2m}\triangle + V)\psi \tag{2.1}$$

We also know that, if V is of Rellich class and the initial kinetic energy is finite [15], then there exists a weak solution X to the three-dimensional Stochastic Differential Equation

$$dX_t = \frac{\hbar}{m}(Re\frac{\nabla\psi}{\psi} + Im\frac{\nabla\psi}{\psi})(X_t,t)dt + (\frac{\hbar}{m})^{\frac{1}{2}}dW_t \tag{2.2}$$

where dW_t denotes the increment of a standard Brownian Motion.

Notably, the diffusion X satisfies the stochastic version of the second Newton's law

$$a_N(X_t,t) = -\frac{1}{m}\nabla V(X_t.t) \tag{2.3}$$

where a_N denotes the natural mean stochastic acceleration as introduced by Nelson in 1966 [27]. In addition, up to regularity assumptions, X is critical for the mean classical action functional [18] (see also [16] for a recent review).

The system we are considering consists of N pair interacting copies of such a particle, with Hamiltonian

$$H_N = \sum_{i=1}^{N}(-\frac{\hbar^2}{2m}\triangle_i + V(\mathbf{r_i})) + \sum_{1\le i<j\le N} v(\mathbf{r_i}-\mathbf{r_j}) \tag{2.4}$$

We adopt the following notations: bold letters denote vectors in $\mathbb{R}^3$, capital letters stochastic processes and $\hat{X} = (X_1, ..., X_N)$ arrays in $\mathbb{R}^{3N}$.

Under suitable assumptions on V and v one can prove the existence of the ground state Ψ_N of (2.4), which is unique up to a phase coefficient. We also assume that it is strictly positive and continuously differentiable (see [29], Thm.XIII.46 and XIII.47, for the regularity conditions on the potentials V and v implying the strictly positivity, and (XIII.11) for those implying the differentiability of the ground state wave function).

We denote by $\hat{X}$ the corresponding $3N$-dimensional Nelson's diffusion, whose generator is related to H_N by a Doob's transformation [17] [30] (see also [31] for extensions).

$\hat{X}$ is the N-body ground state process and it consists of a family of N three dimensional one-particle interacting diffusions $(X_1, \dots, X_N)$.

It satisfies the SDE, written in compact form,

$$d\hat{X}_t = \hat{b}(\hat{X}_t)dt + (\frac{\hbar}{m})^{\frac{1}{2}} d\hat{W}_t \tag{2.5}$$

where $\hat{b} = (b_1, \dots, b_N)$, $\hat{b}_i(\hat{X}_t) = \frac{\nabla_i \Psi_N}{\Psi_N}(\hat{X}_t)$ for $i = 1, \dots, N$, and $\hat{W}$ is a $3N$-dimensional standard Brownian Motion.

If Bose-Einstein condensation occurs, the condensate is usually described by the order parameter $\phi_{GP} \in L^2(\mathbb{R}^3)$, also called the wave function of the condensate, which is the minimizer of the Gross-Pitaevskii functional

$$E^{GP}[\phi] = \int (\frac{\hbar^2}{2m} |\nabla \phi(r)|^2 + V(r)|\phi(r)|^2 + g|\phi(r)|^4) d\mathbf{r} \tag{2.6}$$

under the L^2-normalization condition

$$\int_{\mathbb{R}^3} |\phi^{GP}|^2 d\mathbf{r} = 1$$

and where $g > 0$ is a parameter depending on the interaction potential v (see also next assumption h2)). Therefore ϕ_{GP} solves the stationary cubic non-linear equation (in this context called Gross-Pitaevskii equation)

$$-\frac{\hbar^2}{2m} \triangle \phi + V\phi + 2g|\phi|^2 \phi = \lambda \phi \tag{2.7}$$

λ denoting the chemical potential.

3. **Mean energy and rescaling**

The basic mathematical object which contains all elements necessary to prove, from first principles, the existence of BEC and its proper stochastic description, is the quantum mechanical energy of the N-body system in the ground state.

Its explicit expression, with $\hbar = m = 1$, is

$$E(\Psi_N) = \langle \Psi_N, H_N \Psi_N \rangle =$$
$$= \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} \frac{1}{2} |\nabla_i \Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N +$$
$$+ \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} V(\mathbf{r}_i) |\Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N + \sum_{1 \leq i < j \leq N}^{N} \int v(\mathbf{r}_j - \mathbf{r}_i) |\Psi_N|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N \tag{3.1}$$

Exploiting the $3N$-dimensional ground state process $\hat{X}$, the kinetic quantum mechanical energy turns to be the sum of the expectation of the kinetic energies of the single particles at any time t and the quantum energy takes the more compact form

$$E(\Psi_N) = \mathbb{E}\{ \sum_{i=1}^{N} [\frac{1}{2} b_i^2(\hat{X}(t)) + V(X_i(t))] + \sum_{1 \leq i < j \leq N}^{N} [v(X_j(t) - X_i(t))] \} \tag{3.2}$$

b_i being the drift of the interacting i-th particle.

A possible rescaling which leads to the Gross-Pitaevskii description of the condensate is defined as follows [21]

h1) V is locally bounded, positive and going to infinity when $|\mathbf{r_i}|$ goes to infinity. The interaction potential v is smooth, compactly supported, non negative, spherically symmetric, with finite *scattering length* a ([23] Appendix C).

h2) $N \longrightarrow \infty$ and the interaction potential v satisfies the Gross-Pitaevskii scaling [21], that is

$$v(r) = v_1(\frac{r}{a})/a^2$$

$$a = \frac{g}{4\pi N}$$

where v_1 has scattering length equal to 1. We notice that g is positive as a consequence of our assumptions on v (see h1)) and it is kept constant in the rescaling.

For given N and a we denote by $E_o(N, a)$ the ground state energy $E(\Psi_N)$ of the N-body system and by E_{GP} the minimum value of the Gross-Pitaevskii functional (2.6).

The following two therems, proved in [21] and [22], clarify the two main properties of the rescaling procedure. The first is the important

Theorem 1. *(Energy Theorem)* [21] *If $N \uparrow +\infty$ with Na fixed, then*

$$\lim_{N\to\infty} \frac{E_0(N,a)}{N} = E_{GP} \tag{3.3}$$

and

$$\lim_{N\to\infty} \int |\Psi_N|^2 d\mathbf{r}_2 \cdots \mathbf{r}_N = |\phi^{GP}|^2 \tag{3.4}$$

Moreover there exists $s \in (0,1]$, depending on the interaction potential v through the solution of the zero-energy scattering equation, such that

$$\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3N}} |\nabla_1 \Psi_N(\mathbf{r}_1,...,\mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int_{\mathbb{R}^3} |\nabla \phi^{GP}(\mathbf{r})|^2 d\mathbf{r} + \\ + gs \int_{\mathbb{R}^3} (\phi^{GP})^4 d\mathbf{r} \tag{3.5}$$

,

$$\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3N}} V(\mathbf{r}) |\Psi_N(\mathbf{r}_1, X)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int V(\mathbf{r}) |\phi_{GP}|^2 d\mathbf{r} \tag{3.6}$$

and

$$\lim_{N\uparrow\infty} \frac{1}{2} \sum_{j=2}^{N} \int_{\mathbb{R}^{3N}} v(|\mathbf{r}_1 - \mathbf{r}_j|) |\Psi_N(\mathbf{r}_1,...,\mathbf{r}_N)|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N = (1-s) g \int |\phi_{GP}|^4 d\mathbf{r} \tag{3.7}$$

A second fundamental tool, originally called "Localization of energy" is the following

Theorem 2. *(Localization Theorem)* [22]. *Defining*

$$F^N(\mathbf{r}_2, \ldots, \mathbf{r}_N) := (\bigcup_{i=2}^{N} B^N(\mathbf{r}_i))^c \tag{3.8}$$

where $B^N(\mathbf{r})$ denotes the open ball centered in $\mathbf{r}$ with radius $N^{-\frac{1}{3}-\delta}$ where $0 < \delta \leq \frac{4}{51}$,

$$\lim_{N\uparrow\infty} \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \ldots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N)} (\nabla_1 \frac{\Psi_N}{\phi_{GP}})^2 (\phi_{GP})^2 d\mathbf{r}_1 = 0 \tag{3.9}$$

These "quantum mechanical" theorems allow to prove that, in the limit of N going to infinity, one has a complete condensation, in the sense that any finite order reduced density matrix converges in the trace norm to the factorized one [22]. Moreover they can be seen as analytical tools which are crucial in understanding the scaling properties of the mean energy of the N-body interacting system represented by the system of interacting diffusions (2.5), the interaction being defined by the Mean Energy Functional $E(\Psi_N)$ by (3.2). They also allow to study the limit stochastic behavior of a single generic particle [32].

In particular the Localization Theorem is very interesting and gains a clear meaning in the stochastic framework.

For this reason we report in the Appendix a synthesis of the main analytical steps which lead to its proof.

4. Localization of relative entropy and the BEC process

We now turn to the stochastic description and notice that the fixed time joint probability density of the N-body ground state process $\hat{X} = (X_1, \dots, X_N)$ is given by $\rho_N := |\Psi_N|^2$, which is invariant under spatial permutations. Moreover, as expected, if some smoothness conditions are assumed for Ψ_N, the processes $\{X_i\}_{i=1,\dots,N}$ are equal in law. To be more precise (see [32]), one can say that, if Ψ_N is the ground state of H_N and it is strictly positive and of class C^1, then the three-dimensional one-particle interacting diffusions $\{X_i\}_{i=1,\dots,N}$ are equal in law.

Motivated by the fact that the first part of Energy Theorem claims that the one-particle marginal density of ρ_N converges to $|\phi_{GP}|^2$ in $L^1(\mathbb{R}^3)$, we consider a three-dimensional process X^{GP} with invariant density $\rho_{GP} := |\phi_{GP}|^2$ and we compare it with the generic interacting non markovian diffusion X_1.

We assume that X^{GP} is a solution of the SDE

$$dX_t^{GP} := u_{GP}(X_t^{GP})dt + (\frac{\hbar}{m})^{\frac{1}{2}} dW_t$$

where,

$$u_{GP} := \frac{\nabla \phi_{GP}}{\phi_{GP}}$$

Then, since ϕ_{GP} is a solution to the stationary Gross-Pitaevkii equation (2.7), a standard calculation in Stochastic Mechanics shows that Nelson acceleration of X^{GP} reads, quite reasonably,

$$a_N(X_t^{GP}) = -\frac{1}{m}\nabla \left\{ V(X_t^{GP}) + g \mid \phi_{GP}(X_t^{GP}) \mid^2 \right\} \tag{4.1}$$

(On the other side one could observe that now, by the non-linearity of (2.7), Doob's transformation is not expected to play any role.)

By the equality

$$\mid \phi_{GP} \mid^2 (\nabla \frac{\Psi_N}{\phi_{GP}})^2 = \mid \Psi_N \mid^2 (\frac{\nabla \Psi_N}{\Psi_N} - \frac{\nabla \phi_{GP}}{\phi_{GP}})^2$$

we see that the L^2 distance between the two drifts b_1^N and u_{GP}, is given by

$$\int_{\mathbb{R}^{3N}} \|b_1^N - u_{GP}\|^2 \rho_N d\mathbf{r}_1 \cdots d\mathbf{r}_N = \int_{\mathbb{R}^{3N}} (\nabla_1 \frac{\Psi_N}{\phi_{GP}})^2 \mid \phi_{GP} \mid^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N \tag{4.2}$$

This can be exploited to show that the localization theorem is related to the localization of the relative entropy between the generic one particle non markovian interacting diffusion and the process X^{GP}.

To do this we introduce a $3N$-dimensional process $\hat{X}^{GP}$ which satisfies a stochastic differential equation with the same diffusion coefficient as $\hat{X}$ and drift $\hat{u}_{GP}$, defined by

$$\hat{u}_{GP}(\mathbf{r}_1, \cdots, \mathbf{r}_N) = (u_{GP}(\mathbf{r}_1), \cdots, u_{GP}(\mathbf{r}_N))$$

We consider the measurable space $(\Omega^N, \mathcal{F}^N)$ where Ω^N is $C(\mathbb{R}^+ \to \mathbb{R}^{3N})$, and $\mathcal{F}^N$ is its Borel sigma-algebra. We denote by $\hat{Y} := (Y_1, \ldots, Y_N)$ the coordinate process and by $\mathcal{F}_t^N$ the natural filtration.

We denote by $\mathbb{P}_N$ and $\mathbb{P}_{GP}$ the measures corresponding to the weak solutions of the $3N$- dimensional stochastic differential equations

$$\hat{Y}_t - \hat{X}_0 = \int_0^t \hat{b}^N(\hat{Y}_s) ds + \hat{W}_t \tag{4.3}$$

$$\hat{Y}_t - \hat{X}_0 = \int_0^t \hat{u}_{GP}(\hat{Y}_s) ds + \hat{W}'_t \tag{4.4}$$

where $\hat{X}_0$ is a random variable with probability density equal to $\mid \Psi_N \mid^2$ while $\hat{W}_t$ and $\hat{W}'_t$ are $3N$-dimensional $\mathbb{P}_N$ and $\mathbb{P}_{GP}$ standard Brownian Motions, respectively.

We will assume that u_{GP} is bounded.

We recall that under our hypothesis on the potentials v and V, ϕ_{GP} is strictly positive and in $C^1(\mathbb{R}^3) \bigcap L^\infty(\mathbb{R}^3)$ and therefore $u_{GP} \in L^2(\mathbb{R}^3)$ (see [21], Thm 2.1). Then, since Ψ_N is the minimizer of $E^N[\Psi]$ and u_{GP} is bounded, the following finite energy conditions hold (with the shorthand notation $\hat{b}_s^N =: \hat{b}^N(\hat{Y}_s)$ and $\hat{u}_s^N =: \hat{u}_{GP}(\hat{Y}_s)$):

$$E_{\mathbb{P}_N} \int_0^t \|\hat{b}_s^N\|^2 ds < \infty \tag{4.5}$$

$$E_{\mathbb{P}_N} \int_0^t \| \hat{u}_s^{GP}\|^2 ds < \infty, \tag{4.6}$$

Then, by Girsanov theorem, we have, for all $t > 0$,

$$\frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}}|_{\mathcal{F}_t} = \exp\{-\int_0^t (\hat{b}_s^N - \hat{u}_s^{GP}) \cdot d\hat{W}_s + \frac{1}{2}\int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds\} \tag{4.7}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^{3N}$. The relative entropy restricted to $\mathcal{F}_t$ reads

$$\mathcal{H}(\mathbb{P}_N,\mathbb{P}_{GP})|_{\mathcal{F}_t} =: \mathbb{E}_{\mathbb{P}_N}[\log \frac{d\mathbb{P}_N}{d\mathbb{P}_{GP}} \;|_{\mathcal{F}_t}] = \frac{1}{2}E_{\mathbb{P}_N}\int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds \quad (4.8)$$

Since under $\mathbb{P}_N$ the $3N$-dimensional process $\hat{Y}$ is a solution of (4.3) with invariant probability density $|\Psi_N|^2$, we can write, recalling also (4.5) and (4.6),

$$\frac{1}{2}E_{\mathbb{P}^N}\int_0^t \|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds =$$
$$= \frac{1}{2}\int_0^t E_{\mathbb{P}_N}\|\hat{b}_s^N - \hat{u}_s^{GP}\|^2 ds =$$
$$= \frac{1}{2}t\int_{\mathbb{R}^{3N}} \|\hat{b}^N(\mathbf{r}_1,\dots,\mathbf{r}_N) - \hat{u}_{GP}(\mathbf{r}_1,\dots,\mathbf{r}_N)\|^2 \rho_N d\mathbf{r}_1\dots d\mathbf{r}_N \quad (4.9)$$

so that, the symbol $\| \,.\, \|$ now denoting the euclidean norm in $\mathbb{R}^3$, we get

$$\mathcal{H}(\mathbb{P}_N,\mathbb{P}_{GP})|_{\mathcal{F}_t} =$$
$$= \frac{1}{2}t\int_{\mathbb{R}^{3N}}\sum_{i=1}^N \|b_i^N(\mathbf{r}_1,\dots,\mathbf{r}_N) - u_{GP}(\mathbf{r}_i)\|^2 \rho_N d\mathbf{r}_1\dots d\mathbf{r}_N =$$
$$= \frac{1}{2}Nt\int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1,\dots,\mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 \rho_N d\mathbf{r}_1\dots d\mathbf{r}_N =$$
$$= \frac{1}{2}NE_{\mathbb{P}_N}\int_0^t \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds \quad (4.10)$$

where the symmetry of $\hat{b}^N$ and Ψ_N has been exploited.

Finally we get the sum of N identical *one-particle relative entropies*, each of them being defined by

$$\bar{\mathcal{H}}(\mathbb{P}_N,\mathbb{P}_{GP})|_{\mathcal{F}_t} =: \frac{1}{N}\mathcal{H}(\mathbb{P}_N,\mathbb{P}_{GP})|_{\mathcal{F}_t} =$$
$$= \frac{1}{2}E_{\mathbb{P}_N}\int_0^t \|b_1^N(\hat{Y}_s) - u^{GP}(Y_1(s))\|^2 ds \quad (4.11)$$

Recalling (3.5) in *Energy Theorem* and (4.2), we can write

$$\lim_{N\uparrow\infty}\int_{\mathbb{R}^{3N}} \|b_1^N - u_{GP}\|^2 \rho_N d\mathbf{r}_1\cdots d\mathbf{r}_N = gs\int_{\mathbb{R}^3}(\phi^{GP})^4 d\mathbf{r} \quad (4.12)$$

As a consequence, for all $t > 0$, the one particle relative entropy is asymptotically finite but it does not go to zero in the scaling limit. This means that the process X^{GP} is not trivially the stochastic description of the generic particle in the condensate.

But the key point is that, for great N, the one particle process continously "lives" outside a properly defined "random interaction-set" $D_N(t)$.

We define it by the equality

$$D_N(t) := \bigcup_{i=2}^{N} B^N(X_i(t)) \tag{4.13}$$

where $B^N(\mathbf{r})$ is again the ball with radius $N^{-1/3-\delta}$, $\quad 0 < \delta \leq 4/51$ centered in $\mathbf{r}$. We also introduce the stopping time

$$\tau^N := inf\{t \geq 0 : X_1(t) \in D_N(t)\} \tag{4.14}$$

The following Proposition claims that, in the scaling limit, a generic particle remains outside the *interaction-set*, for any finite time interval, with probability one.

Notice that the result is not obvious: even in dimension $d = 3$, where the Lebesgue measure of $D_N(t)$ goes to zero for all t, it could happen that, asymptotically, such a set takes the form of a very complicated surface, dividing the physical three-dimensional space into smaller and smaller non connected regions.

Proposition 1. [32] *Let h1) and h2) hold and the ground state Ψ_N be of class C^1. Then in dimension $d = 3$, for all $t > 0$, we have*

$$\lim_{N\to\infty} \mathbb{P}(\tau^N > t \mid X_1(0) \notin D_N(0)) = 1 \tag{4.15}$$

and τ^N has an exponential distribution.

This allows to apply the *Localization Theorem* to the stopped one-particle process,

Theorem 3. *Let h_1) and h_2) hold. Assume also that Ψ_N is of class C^1 and that u_{GP} is bounded. Then, with τ^N defined as in* (4.14)*, we have*

$$\lim_{N\uparrow\infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP}) \mid_{\mathcal{F}_{t\wedge\tau^N}} = 0 \tag{4.16}$$

Proof. [32]

Recalling (4.5) and (4.6) we can write

$$\bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_{t\wedge\tau^N}} = \frac{1}{2}E_{\mathbb{P}_N}\int_0^{t\wedge\tau^N} \|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 ds \leq$$
$$\leq \frac{1}{2}\int_0^t E_{\mathbb{P}_N}\{\|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 I_{\{Y_1\notin D_s^N\}}\}ds =$$
$$= \frac{1}{2}tE_{\mathbb{P}_N}\{\|b_1^N(\hat{Y}_s) - u_{GP}(Y_1(s))\|^2 I_{\{Y_1\notin D_s^N\}}\} =$$
$$= \frac{1}{2}t\int_{\mathbb{R}^{3N}} \|b_1^N(\mathbf{r}_1,\ldots,\mathbf{r}_N) - u_{GP}(\mathbf{r}_1)\|^2 I_{F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N)}(\mathbf{r}_1)\rho_N^2 d\mathbf{r}_1\cdots d\mathbf{r}_N \tag{4.17}$$

Thus, by (4.2) and the Localization Theorem, we finally get

$$\lim_{N\uparrow\infty} \bar{\mathcal{H}}(\mathbb{P}_N, \mathbb{P}_{GP})|_{\mathcal{F}_{t\wedge\tau^N}} =$$
$$= \frac{1}{2}t\lim_{N\uparrow\infty}\int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2\ldots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N)} \|b_1^N - u_{GP}\|^2\rho_N^2 d\mathbf{r}_1\cdots d\mathbf{r}_N = 0 \tag{4.18}$$

□

Concluding, for great N, $X_{t\wedge\tau^N}$ is close in the sense of relative entropy to the "BEC process" $X^{GP}_{t\wedge\tau^N}$, whenever at an arbitrary time-origine the particle is not in the random interacting set, while the probability that the particle hits such a set in a finite time becomes negligible.

5. **Appendix**

In this section we put $\frac{\hbar^2}{2m} = 1$

The proof of *Localization Theorem* is essentially devoted to establish a proper lower bound for the Energy Functional (3.1) and it is based on two results concerning the following interacting Hamiltonian for the homogeneous case

$$H_N^I = -\sum_{i=1}^N \Delta_i + \sum_{1\leq i\leq j\leq N} v(|x_i - x_j|) \tag{5.1}$$

Lemma 1. *(Smoothing Lemma). Let v be non negative with finite range R_0 and let U be any non negative function satisfying,*

$$\int U(r)r^2 dr \leq 1 \qquad U(r) = 0 \quad r < R_0$$

then, a being the scattering length and $\epsilon \in (0,1)$,

$$H_N^I \geq \epsilon T_N + (1-\epsilon) a W_R \tag{5.2}$$

where

$$T_N = -\sum_i \Delta_i, \qquad W_R = \sum_i^N U(t_i)$$

with

$$t_i = t_i(x_1, x_2, ..., x_N) := \min_{j, j \neq i} |x_i - x_j| \tag{5.3}$$

Moreover one can take

$$U(r) = 3(R^3 - R_0^3)^{-1} \qquad R_0 < r < R$$

and otherwise equal to zero, where R represents the range of the potential U.

We can observe that in the lower bound operator, only a part of the kinetic energy survives and the interaction potential is softer than v, but with a larger range.

The proof is based on a generalization of a Dyson's Lemma [25] due to Lieb and Yngvason [24].

Lemma 2. *(Lower Bound Theorem in a finite box)* [24] *Let* (5.1) *the Hamiltonian for N interacting bosons in a cubic box Λ with side length L, where v is a spherically symmetric pair potential having finite scattering length a. Then there exists $\lambda > 0$ such that, denoting by $E_0(N, L)$ the ground state energy of H_N^I, with Neumann boundary conditions, one has*

$$\frac{E_0(N, L)}{N} \geq 4\pi\rho a(1 - CY^{1/17}) \tag{5.4}$$

where $\rho = \frac{N}{L^3}$ is the particle density, $Y = 4\pi\rho\frac{a^3}{3}$ is the number of particles in the ball of radius a and L is such that $Y < \lambda$ and $\frac{L}{a} > C_1 Y^{-\frac{6}{17}}$.

Moreover C and C_1 are positive constants independent of N and L.

For the proof see [23] Thm. 2.4.

Proof of the Localization Theorem ([23], Lemma 7.3 and [22])

It is sufficient to show that, when $N \uparrow \infty$

$$\int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int_{F_N^c(\mathbf{r}_2,\ldots,\mathbf{r}_N)} (\nabla_1 \frac{\Psi_N}{\phi_{GP}})^2 (\phi_{GP})^2 d\mathbf{r}_1 + \\ + \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int |\Psi_N|^2 [\frac{1}{2}\sum_{k\geq 2} v(|\mathbf{r}-\mathbf{r}_k|) - 2g{\phi_{GP}}^2] \\ \geq -g \int |\phi_{GP}|^4 d\mathbf{r} - o(1) \quad (5.5)$$

This implies the thesis because (5.5) can be written as

$$\int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int (\nabla_1 \frac{\Psi_N}{\phi_{GP}})^2 (\phi_{GP})^2 d\mathbf{r}_1 + \\ + \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int |\Psi_N|^2 [\frac{1}{2}\sum_{k\geq 2} v(|\mathbf{r}-\mathbf{r}_k|) - 2g{\phi_{GP}}^2] \\ - \int_{\mathbb{R}^{3(N-1)}} d\mathbf{r}_2 \cdots d\mathbf{r}_N \int_{F^N(\mathbf{r}_2,\ldots,\mathbf{r}_N)} (\nabla_1 \frac{\Psi_N}{\phi_{GP}})^2 (\phi_{GP})^2 d\mathbf{r}_1 \\ \geq -g \int |\phi_{GP}|^4 d\mathbf{r} - o(1) \quad (5.6)$$

and then, by (3.5), (3.6) and (3.7) in *Energy Theorem*, with the external potential V particularized to $2g|\phi_{GP}|^2$ in (3.6), one obtains the thesis (3.9).

To prove (5.5) one introduces a function F such that

$$\frac{\Psi_N}{\phi_{GP}(\mathbf{r}_1)} := \prod_{k\geq 2} \phi_{GP}(\mathbf{r}_k) F(\mathbf{r}_1, \ldots, \mathbf{r}_N) \quad (5.7)$$

Using the fact that F is *symmetric* in the particle coordinates, one can see that (5.5) is equivalent to

$$\frac{Q_\delta(F)}{N} \geq -g \int |\phi_{GP}|^4 d\mathbf{r} - o(1) \quad (5.8)$$

where Q_δ has the following definition

$$Q_\delta := \sum_{i=1}^{N} \int_{\Gamma_i^c} |\nabla_i F|^2 \prod_{k=1}^{N} |\phi_{GP}(\mathbf{r}_k)|^2 d\mathbf{r}_k \\ + \sum_{1\leq i\leq j\leq N} \int v(|\mathbf{r}_i - \mathbf{r}_j|) |F|^2 \prod_{k=1}^{N} |\phi_{GP}(\mathbf{r}_k)|^2 d\mathbf{r}_k + \\ - 2g \sum_{i=1}^{N} \int |\phi_{GP}(\mathbf{r}_i)|^2 |F|^2 \prod_{k=1}^{N} |\phi_{GP}(\mathbf{r}_k)|^2 d\mathbf{r}_k \quad (5.9)$$

with

$$\Gamma_i^c = \{(\mathbf{r}_1, ..., \mathbf{r}_N) \in \mathbb{R}^{3N} | \min_{k \neq i} |\mathbf{r}_i - \mathbf{r}_k| \leq R'\}$$

where $R' = N^{-\frac{1}{3}-\delta}$.

To handle the expression of Q_δ one applies the "cell-method", considering the space as divided into cells of width L, and then minimizing over all possible distributions of the particles in the different cells. Since one is looking for a lower bound and v is positive, the interactions due to particles in different cells can be ignored. Finally one leaves the width of the cells going to zero.

Labeling cells with the index α, one has,

$$\inf_F Q_\delta(F) \geq \inf_{n_\alpha} \sum_\alpha \inf_{F_\alpha} Q_\delta^\alpha(F_\alpha)$$

where Q_δ^α is defined as Q_δ but with the integrations limited to the cell α. F_α is defined as F but with N replaced by n_α. The infimum is taken over all distributions such that $\sum_\alpha n_\alpha = N$.

One now fixes some $M > 0$ and considers only cells inside a cube Λ_M of side length M. For the cells belonging to Λ_M one can evaluate the maximum and minimum value of ρ_{GP}. For the cell α those are denoted by $\rho_{\alpha,max}$ and $\rho_{\alpha,min}$, respectively.

One then can observe that, if the range R of the smoothing potential U is sufficiently small, one can apply *Smoothing Lemma* "in the cell α" and restrict all integrations to Γ_i^c.

This at the end leads to the inequality

$$Q_\delta^\alpha(F_\alpha) \geq \frac{\rho_{\alpha,min}}{\rho_{\alpha,max}} E_0^U(n_\alpha, L) - 8\pi a N \rho_{\alpha,max} n_\alpha - \epsilon C_M n_\alpha \tag{5.10}$$

where $E_0^U(n_\alpha, L)$ is the ground state energy of

$$\sum_{i=1}^{n_\alpha} (-\frac{1}{2}\epsilon\triangle_i + (1-\epsilon)aU(t_i)) \tag{5.11}$$

with $C_M = \sup_{\mathbf{r}\in\Lambda_M} |\nabla\phi_{GP}(\mathbf{r})|^2$, independent of N.

To minimize (5.10) with respect to n_α one takes advantage of *Lower Bound Theorem* and of Lemma 6.4 in [23], p.55.

One finds after some manipulations that $\bar{n}_\alpha$ is at least of the order of NL^3.

If one takes $L \sim N^{-\frac{1}{10}}$, the range of smoothing potential U can be shown to be well estimated as $R \sim N^{-\frac{1}{17}}$ and the assumption on δ is sufficient to guarantee that R remains lower or equal to R', allowing the application of *Smoothing Lemma* in constructing the lower bound for Q_δ^α.

Further standard manipulations then give rise, C_o denoting a positive constant independent of N, to the following inequality:

$$Q_\delta(F) \geq$$
$$\geq 4\pi a N^2 \int |\phi_{GP}|^4[1+C_o \cdot N^{-1/10}] - Y^{1/17} N C_M - 8\pi a N^2 \sup_{\mathbf{r}\notin\Lambda_M} |\phi_{GP}|^2(\mathbf{r}) \tag{5.12}$$

Dividing by N, taking $N \uparrow \infty$ and then $M \uparrow \infty$, one obtains the result. In fact, since ϕ_{GP} decreases more than exponentially at infinity ([21], Lemma A.5), the last term is arbitrarily small for M large. This proves (5.8), which is equivalent to (5.5).

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