Worksheet: Arithmetic Expressions – *Some Answers*

(1) Using the rules from the lectures:

$$\dfrac{\dfrac{}{4 \Downarrow 4}\,(\text{B-NUM}) \qquad \dfrac{}{1 \Downarrow 1}\,(\text{B-NUM})}{(4+1) \Downarrow 5}\,(\text{B-ADD}) \qquad \dfrac{\dfrac{}{2 \Downarrow 2}\,(\text{B-NUM}) \qquad \dfrac{}{2 \Downarrow 2}\,(\text{B-NUM})}{(2+2) \Downarrow 4}\,(\text{B-ADD})$$
$$\overline{((4+1)+(2+2)) \Downarrow 9}\,(\text{B-ADD})$$

(2) To handle multiplication, we simply add a single rule to our existing system. In addition to the axiom and the rule for + we have the rule

$$(\text{B-MULT})\ \dfrac{E_1 \Downarrow \mathtt{n}_1 \quad E_2 \Downarrow \mathtt{n}_2}{(E_1 \times E_2) \Downarrow \mathtt{n}_3}\ n_3 = \mathsf{mult}(n_1, n_2)$$

(3) The proof consists of four uses of the axiom, two uses of the rule for + and one use of the new rule for ×:

$$\dfrac{\dfrac{}{3 \Downarrow 3}\,(\text{B-NUM}) \qquad \dfrac{}{2 \Downarrow 2}\,(\text{B-NUM})}{3+2 \Downarrow 5}\,(\text{B-ADD}) \qquad \dfrac{\dfrac{}{1 \Downarrow 1}\,(\text{B-NUM}) \qquad \dfrac{}{4 \Downarrow 4}\,(\text{B-NUM})}{1+4 \Downarrow 5}\,(\text{B-ADD})$$
$$\overline{((3+2) \times (1+4)) \Downarrow 25}\,(\text{B-MULT})$$

(4) The obvious rule to add for subtraction is

$$(\text{B-SUB})\ \dfrac{E_1 \Downarrow \mathtt{n}_1 \quad E_2 \Downarrow \mathtt{n}_2}{(E_1 - E_2) \Downarrow \mathtt{n}_3}\ n_3 = \mathsf{minus}(n_1, n_2)$$

However, in general this won't work, because if $\mathtt{n}_1$ is $\mathtt{3}$ and $\mathtt{n}_2$ is $\mathtt{7}$, then $n_3 = -4$, and we do not have a corresponding numeral $\mathtt{n}_3$.

One solution is to say that subtraction "gets stuck" when a negative value is needed. You can do this by making no rule available in the nasty case; this is done by adding a side condition.

$$(\text{B-SUB})\ \dfrac{E_1 \Downarrow \mathtt{n}_1 \quad E_2 \Downarrow \mathtt{n}_2}{(E_1 - E_2) \Downarrow \mathtt{n}_3}\ n_3 = \mathsf{minus}(n_1, n_2) \text{ and } n_1 \geq n_2$$

With this approach, there is no numeral $\mathtt{n}$ for which $(3 - 7) \Downarrow \mathtt{n}$, so this expression has *no* final answer at all.

Other solutions are possible: one could extend the semantics to allow expressions to signal errors, and then a "bad" subtraction would signal an error. If you choose this route, every rule of the semantics needs to be reconsidered in case errors might make a difference.

You might also decide that such subtractions default to returning $\mathtt{0}$, but then you have of beware of strange things like

$$(3 - 7) + 4 \Downarrow 4$$

while

$$(3 + 4) - 7 \Downarrow \mathtt{0}.$$

This is probably not a good idea.

(5) One possible E is $3 + (4 + 3)$. The axiom (s-Add) for the small-step semantics gives us

$$1 + 2 \rightarrow_{\mathrm{lr}} 3$$

and the rule (s-left) lets us use this on the left hand side of the expression in question, so the full derivation is

$$\dfrac{\dfrac{}{1 + 2 \rightarrow_{\mathrm{lr}} 3}\ (\text{s-add})}{((1 + 2) + (4 + 3)) \rightarrow_{\mathrm{lr}} (3 + (4 + 3))}\ (\text{s-left})$$

Question: Are there any other expression E diffferent from $3 + (4 + 3)$ for which a left-to-right derivation can be found ?

(6) The full evaluation sequence is

$$\begin{array}{rcl} ((1 + 2) + (4 + 3)) & \rightarrow_{\mathrm{lr}} & (3 + (4 + 3)) \\ & \rightarrow_{\mathrm{lr}} & (3 + 7) \\ & \rightarrow_{\mathrm{lr}} & \mathtt{10}. \end{array}$$

We have already seen the derivation of the first step. The axiom for the small step semantics (s-add) allows us to derive

$$\dfrac{}{4 + 3 \rightarrow_{\mathrm{lr}} 7}$$

Since 3 is a numeral this can be used in an application of the second rule, (s-n.right) to give the following derivation of the second step:

$$\dfrac{\dfrac{}{4 + 3 \rightarrow_{\mathrm{lr}} 7}\ (\text{s-add})}{3 + (4 + 3) \rightarrow_{\mathrm{lr}} (3 + 7)}\ (\text{s-n.right})$$

The derivation of the final step is simpler. It is an application of the axiom:

$$\dfrac{}{3 + 7 \rightarrow_{\mathrm{lr}} \mathtt{10}}\ (\text{s-add})$$

(7) Every derivation in the left-to-right semantics is also a derivation in the standard semantics. So from Question (5) we know

$$((1 + 2) + (4 + 3)) \rightarrow 3 + (4 + 3)$$

But using the more general rule (s-right) we can also derive

$$((1 + 2) + (4 + 3)) \rightarrow (1 + 2) + 7$$

Here is the derivation:

$$\dfrac{\dfrac{}{4 + 3 \to 7}\ (\textsc{s-add})}{((1 + 2) + (4 + 3)) \to (1 + 2) + 7}\ (\textsc{s-right})$$

It turns out that these are the only two possible E, namely $(1+2)+7$ and $3+(4+3)$.

(8) The left-to-right small step semantics uses only three rules, (s-left), (s-n.right) and (s-add). Therefore if an expression is to evaluate to $((1 + 2) + 7)$ in one step, the step must be derived from an instance of the first rule, (s-left), since the axiom (s-add) only lets us derive steps that end in a numeral, and the second rule (s-n.right) only lets us derive steps that lead to expressions of the form $\mathtt{n} + E$.

Therefore we know that any such expression must have the form $(E_1 + 7)$, where $E_1 \to_{lr} (1 + 2)$.

So we need to find such an E_1. Using the first rule (s-left) we can derive

$$((0 + 1) + 2) \to (1 + 2)$$

So one possible E is $(0 + 1) + 2) + 7$. The full derivation consists of one use of the axiom followed by two applications of the first rule:

$$\dfrac{\dfrac{\dfrac{}{0 + 1 \to 1}\ (\textsc{s-add})}{(0 + 1) + 2 \to_{lr} 1 + 2}\ (\textsc{s-left})}{((0 + 1) + 2) + 7 \to_{lr} (1 + 2) + 7}\ (\textsc{s-left})$$

(9) From the reasoning in the previous answer we know that they all have the form $(E_1 + 7)$, where $E_1 \to (1 + 2)$. So we need to know all the possible E_1 such that $E_1 \to (1 + 2)$.

This step can be derived using either the first rule (s-left) or the second rule (s-right). Here are all the suitable expressions E_1:

- $((0 + 1) + 2)$, using (s-left)
- $((1 + 0) + 2)$, using (s-left)
- $(1 + (0 + 2))$, using (s-n.right)
- $(1 + (1 + 1))$, using (s-n.right)
- $(1 + (2 + 0))$, using (s-n.right)

So the only expressions that can evaluate to $((1 + 2) + 7)$ in one step are

- $(((0 + 1) + 2) + 7)$
- $(((1 + 0) + 2) + 7)$
- $((1 + (0 + 2)) + 7)$
- $((1 + (1 + 1)) + 7)$

- $((1 + (2 + 0)) + 7)$.

(10) If $E \to ((1+2)+7)$ but $E \not\to_{\mathrm{lr}} ((1+2)+7)$ then the more general rule (S-RIGHT) must be used in the derivation of the first judgement. So one possible E is $((1+2)+(2+5))$. Here is a derivation:

$$\dfrac{\dfrac{}{2 + 5 \to 7}\ (\textsc{s-add})}{((1 + 2) + (2 + 5)) \to (1 + 2) + 7}\ (\textsc{s-right})$$

(11) $((1+1)+1)$ takes two steps to get to the final answer 3. The full evaluation sequence is

$$((1 + 1) + 1) \to_{\mathrm{lr}} (2 + 1) \to_{\mathrm{lr}} 3.$$

If we add another 1 to get $(((1 + 1) + 1) + 1)$ then three steps are needed.

In general, the number of steps needed is the same as the number of + symbols since each step reduces the number of + symbols in an expression by one. You will be asked to prove this fact in a later exercise class.