Worksheet 4: Structural induction

Unless otherwise states you may assume that expressions only use numerals, and the operation $+$.

1. Define inductively a function $\mathsf{plusses}$ from expressions to numbers such that $\mathsf{plusses}(E)$ is the number of $+$ symbols in the expression E.
2. Let nums be the function from expressions to numbers such that $\mathsf{nums}(E)$ is the number of numerals in E. Give an inductive definition of nums.

   Then prove, by **structural** induction, that $\mathsf{plusses}(E) < \mathsf{nums}(E)$.
3. Prove that for every expression E if both $E \Downarrow \mathtt{n}$ and $E \Downarrow \mathtt{n}'$ then $n = n'$.

   Use induction on the **structure** of the expression E and lay out your proof so that the inductive hypothesis is clear. In other words the statement to be proved should be expressed in the form $P(E)$ where $P(-)$ is the property to be proved by structural induction on expressions.
4. Prove that for all expressions $E,\ F$, if $E \rightarrow^n E'$ then $E + F \rightarrow^n E' + F$. You should prove this by *mathematical induction* on n. In other words let $P(n)$ be the property

   $$E \rightarrow^n E' \ \ \textit{implies} \ \ E + F \rightarrow^n E' + F$$

   You have to show that $P(n)$ is true for every natural number n.
5. Prove that for all expressions E and numerals $\mathtt{m}$, if $E \rightarrow^n E'$ then $\mathtt{m} + E \rightarrow^n \mathtt{m} + E'$.
   Follow the instructions for the previous question. Lay out the proof so that the property being proved, and and applications of the inductive hypothesis is perfectly clear.
6. Use the previous two results to show that $E_1 \rightarrow^* \mathtt{n_1}$ and $E_2 \rightarrow^* \mathtt{n_2}$ implies $E_1 + E_2 \rightarrow^* \mathtt{n}$, where $n = n_1 + n_2$.
7. Prove that $E \Downarrow \mathtt{n}$ implies $E \rightarrow^* \mathtt{n}$.
   Here you should use *structural induction* on E.
8. Show that the small-step semantics we gave in our lectures has the property that whenever $E \rightarrow E'$, $\mathsf{plusses}(E) = \mathsf{plusses}(E') + 1$. This shows that each step of our semantics deals with exactly one $+$ operation.
9. Prove, by induction on the **structure** of expressions, that for any expression E,

   if E is not a numeral, then $E \rightarrow E'$ for some E'.

   Again you should consider the language with numerals and $+$ only.

10. Combine the above two observations to argue that for any expression E, there is a numeral $\mathtt{n}$ such that $E \rightarrow^* \mathtt{n}$.
    Note that this gives an alternative proof to Lemma 8 in the course notes.

11. Give a similar argument for the larger language incorporating $\times$ as well as $+$.