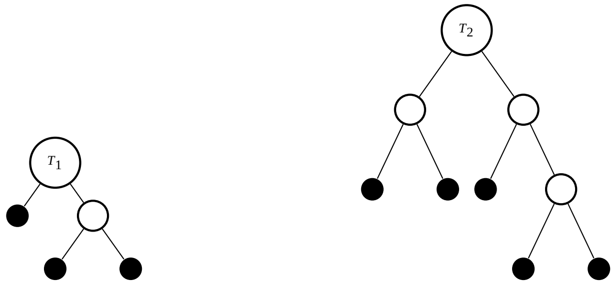

Figure 1: Some binary trees

Worksheet: Structural induction

Unless otherwise states you may assume that expressions only use numerals, and the operation $+$.

(1) Consider the set of binary trees BN discussed in the lectures:

$$\textbf{bTree} \quad ::= \quad \textbf{leaf} \;\mid\; \textbf{Branch}(\textbf{bTree}, \textbf{bTree})$$

(a) Use structural induction to define a function $\mathsf{nodes} : BN \to \mathbb{N}$ which counts the (total) number of nodes in a binary tree.
This function should be defined so that $\mathsf{nodes}(T_1) = 5$ and $\mathsf{nodes}(T_2) = 9$.

(b) Use structural induction to define a function $\mathsf{height} : BN \to \mathbb{N}$ which returns the *height* of a binary tree.
The *height* of a binary tree is the longest path from the root to a leaf. So $\mathsf{height}(T_1)$ should be 2 while $\mathsf{height}(T_2)$ should be 3.
Note: The binary tree with only one node has height 0.

(c) Use structural induction to prove that $\mathsf{nodes}(T) \le 2^{\mathsf{height}(T)+1} - 1$ for every binary tree T.

(2) Define inductively a function $\mathsf{plusses}$ from expressions to numbers such that $\mathsf{plusses}(E)$ is the number of $+$ symbols in the expression E.

(3) Let nums be the function from expressions to numbers such that $\mathsf{nums}(E)$ is the number of numerals in E. Give an inductive definition of nums.

Then prove, by **structural** induction, that $\mathsf{plusses}(E) < \mathsf{nums}(E)$.

(4) Consider the following grammar for *binary numerals BinNum*:

$$b \quad ::= \quad \mathtt{0} \mid \mathtt{1} \mid b\mathtt{0} \mid b\mathtt{1}$$

(a) Explain the way in which functions over binary numerals can be defined by structural induction.

Use this principle to define the function $\mathsf{number} : \mathit{BinNum} \rightarrow \mathbb{N}$ which returns the natural number which a binary numeral represents. For example you should have

$$\begin{aligned} \mathsf{number}(\mathtt{101}) &= 5 \\ \mathsf{number}(\mathtt{001111}) &= 15 \end{aligned}$$

Also define the function $\mathsf{sum} : \mathit{BinNum} \rightarrow \mathbb{N}$ which simply sums up the value of all the digits in a binary numeral. For example

$$\begin{aligned} \mathsf{sum}(\mathtt{101}) &= 2 \\ \mathsf{sum}(\mathtt{001111}) &= 4 \end{aligned}$$

(b) Explain the principle of structural induction for binary numerals.

Use structural induction to prove that $\mathsf{sum}(b) \leq \mathsf{number}(b)$ for every binary numeral b.

(5) Prove that for every expression E if both $E \Downarrow \mathtt{n}$ and $E \Downarrow \mathtt{n}'$ then $n = n'$.

Use induction on the **structure** of the expression E and lay out your proof so that the inductive hypothesis is clear. In other words the statement to be proved should be expressed in the form $P(E)$ where $P(-)$ is the property to be proved by structural induction on expressions.

(6) Prove that for all expressions $E,\ F$, if $E \rightarrow_{\mathrm{lr}}{}^{n} E'$ then $E + F \rightarrow_{\mathrm{lr}}{}^{n} E' + F$.
You should prove this by *mathematical induction* on n. In other words let $P(n)$ be the property

$$E \rightarrow_{\mathrm{lr}}{}^{n} E' \ \ \textit{implies} \ \ E + F \rightarrow_{\mathrm{lr}}{}^{n} E' + F$$

You have to show that $P(n)$ is true for every natural number n.

(7) Prove that for all expressions E and numerals $\mathtt{m}$, if $E \rightarrow_{\mathrm{lr}}{}^{n} E'$ then $\mathtt{m} + E \rightarrow_{\mathrm{lr}}{}^{n} \mathtt{m} + E'$.
Follow the instructions for the previous question. Lay out the proof so that the property being proved, and and applications of the inductive hypothesis is perfectly clear.

(8) Use the previous two results to show that $E_1 \rightarrow_{\mathrm{lr}}{}^{*} \mathtt{n}_1$ and $E_2 \rightarrow_{\mathrm{lr}}{}^{*} \mathtt{n}_2$ implies $E_1 + E_2 \rightarrow_{\mathrm{lr}}{}^{*} \mathtt{n}$, where $n = n_1 + n_2$.

(9) Prove that $E \Downarrow \mathtt{n}$ implies $E \rightarrow_{\mathrm{lr}}{}^{*} \mathtt{n}$.
Here you should use *structural induction* on E.

(10) Show that the small-step semantics we gave in our lectures has the property that whenever $E \rightarrow E'$, $\mathsf{plusses}(E) = \mathsf{plusses}(E') + 1$. This shows that each step of our semantics deals with exactly one + operation.

(11) Prove, by induction on the **structure** of expressions, that for any expression E,

$$\text{if } E \text{ is not a numeral, then } E \rightarrow E' \text{ for some } E'.$$

Again you should consider the language with numerals and + only.

Questions

(12) Combine the above two observations to argue that for any expression E, there is a numeral $\mathtt{n}$ such that $E \rightarrow^* \mathtt{n}$.

(13) Give a similar argument for the larger language incorporating $\times$ as well as $+$.

(14) (Rule induction) Consider the inductive system defined by the following three rules:

$$\text{(AX)}\ \frac{}{(n+1, 0, n+1) \in \text{GCD}} \qquad \text{(LESS)}\ \frac{(m, n-m, k) \in \text{GCD}}{(m, n, k) \in \text{GCD}}\ m < n \qquad \text{(SW)}\ \frac{(n, m, k) \in \text{GCD}}{(m, n, k) \in \text{GCD}}$$

- Prove, using Rule induction, that if $(m, n, k) \in \text{GCD}$ then at least one of m and n is non-zero.
- Prove, using Rule induction, that if $(m, n, k) \in \text{GCD}$ then k divides m and k divides n exactly.