Worksheet 3: Mathematical Induction

(1) Use mathematical induction to prove that $2^n > n$, for every natural number n.

(2) In the lectures we showed that for every natural number n, the number $8^n - 2^n$ is divisible by 6. Using the same techniques show that $n^3 - n$ is also divisible by 6 for every natural number n.

No doubt you already know that $(k+1)^3$ can be expanded to k^3+3k^2+3k+1. You will also have to know that $0^k = 0$ for every natural number k.

(3) Show that for every natural number greater than 0, the n^{th} odd natural number is $2n - 1$.
In order to formulate the formal proposition to be proved, first let $Odd(n)$ denote the n^{th} odd natural number, where $n \geq 1$; it does not make sense to talk about the 0^{th} add natural number. Then let $P(n)$ be defined by

$n = 0$ or else $Odd(n) = 2n - 1$.

Then you have to prove $P(n)$ is true for every natural number n.

(4) Prove that $n! > 2^n$ whenever n is greater than or equal to 4.

(5) In the lectures we saw that every natural number greater than or equal to 14 could be written as a sum of 3s and/or 8s. Use a similar technique to prove that every natural number greater than or equal to 4 can be written as a sum of 2s or 5s.

(6) In the lectures we showed that

$$\sum_{i=0}^{n} i = \frac{n^2 + n}{2}.$$

Using a similar technique, show that

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

In the inductive step you will have to carry out some algebraic manipulations. These will use general laws from high-school maths such as

$$\begin{aligned} X &= \frac{XY}{Y} \\ \frac{X}{Z} + \frac{Y}{Z} &= \frac{X+Y}{Z} \end{aligned}$$

(7) Prove that for all natural numbers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

In this instance, your base case will be the case where $n = 1$. The rest is the same as usual. Again you will have to recall your algebraic manipulation skills.

(8) A *prime number* is a natural number n which cannot be written as a product $a \times b$ of natural numbers unless one of a and b is 1.

Attempt to prove by induction that for all natural numbers $n \geq 2$, either n is prime, or $n = a_1 \times a_2 \times \cdots \times a_k$ for some $k > 1$ and prime numbers $a_1, \ldots, a_k$. That is to say, any natural number greater than 1 can be written as a product of prime numbers.

Here are two hints. First, for the inductive step, suppose that $n + 1$ is not prime, which implies that $n + 1 = m_1 \times m_2$ for some numbers $m_1, m_2 \leq n$. Now try to finish the proof. Hint number two is that you can't do this using ordinary induction! See the next exercise.

(9) The principle of *strong induction* says that to prove that a property $P(-)$ holds of all natural numbers, it is sufficient to show that for all n,

if $P(k)$ is true for all $k < n$, then $P(n)$ is true.

This means that a strong-induction proof looks like:

Base case: Prove that $P(0)$ holds.

Inductive step: For any n, assuming that $P(0), P(1), \ldots, P(n)$ all hold, prove that $P(n + 1)$ holds.

Give an argument that justifies this principle in terms of the usual (weak) induction principle. Use strong induction to complete your proof from the previous exercise.