Tutorial 10: Laziness and Rule Induction

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1. Consider a simple *let* language:

$$A ::= x \mid n \mid A_1 + A_2 \mid \texttt{let}\ x = A_1\ \texttt{in}\ A_2$$

We can give the following big-step semantics for this language:

$$\frac{}{n \Downarrow_s n} \qquad \frac{A_1 \Downarrow_s n_1 \qquad A_2 \Downarrow_s n_2}{A_1 + A_2 \Downarrow_s n}\, n = n_1 + n_2 \qquad \frac{A_1 \Downarrow_s n_1 \qquad A_2({}^{n_1}\!/\!_x) \Downarrow_s n_2}{\texttt{let}\ x = A_1\ \texttt{in}\ A_2 \Downarrow_s n_2}$$

We have annotated the downwards arrow to indicate that this semantics is *strict*: this means that in `let` expressions we first evaluate A_1 before substituting it for x. We can also give a *lazy* semantics:

$$\frac{}{n \Downarrow_\ell n} \qquad \frac{A_1 \Downarrow_\ell n_1 \qquad A_2 \Downarrow_\ell n_2}{A_1 + A_2 \Downarrow_\ell n}\, n = n_1 + n_2 \qquad \frac{A_2({}^{A_1}\!/\!_x) \Downarrow_\ell n}{\texttt{let}\ x = A_1\ \texttt{in}\ A_2 \Downarrow_\ell n}$$

For this language, the only difference between the two semantics is "efficiency": some expressions evaluate in fewer steps with the strict semantics than with the lazy semantics, and some expressions evaluate in fewer steps with the lazy semantics with the strict semantics. Give an example of both.

2. Prove that the strict semantics implies the lazy semantics. We will need dan auxiliary lemma about the lazy semantics:

Lemma 1 *For all A_1, A_2, n, if $A_1 \Downarrow_\ell n_1$ and $A_2({}^{n_1}\!/\!_x) \Downarrow_\ell n_2$ then $A_2({}^{A_1}\!/\!_x) \Downarrow_\ell n_2$.*

This lemma is a little tricky to prove (good exercise though!) so for now you can just assume it given.

With this lemma you should be able to prove:

Lemma 2 *For all A, n, if $A \Downarrow_s n$ then $A \Downarrow_\ell n$.*

3. To prove that the strict semantics implies the lazy semantics, we will need two auxiliary lemmas.

   (a) The first is very similar (but opposite) to the auxiliary lemma we needed above:

   Lemma 3 *For all A_1, A_2, n, if $A_1 \Downarrow_s n_1$ and $A_2({}^{A_1}\!/\!_x) \Downarrow_s n_2$ then $A_2({}^{n_1}\!/\!_x) \Downarrow_s n_2$.*

   Prove this lemma.

   (b) We will also need this lemma:

   Lemma 4 *For all A there exist an n such that $A \Downarrow_s n$.*

   Prove this lemma.

   (c) You should now be able to prove:

   Lemma 5 *For all A, n, if $A \Downarrow_\ell n$ then $A \Downarrow_s n$.*