On the Parikh-de-Bruijn grid

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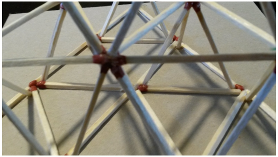
McMaster U (Canada) & Murdoch U (Australia)



Words & Complexity 2018
Lyon, 19-23 Feb. 2018

Alphabet Σ: finite, ordered, constant size σ

- Given string s, the Parikh vector of s $\mathbf{pv}(s)$ is vector of multiplicities of characters
- Given a Pv p, its order is the sum of its entries = length of a string with Pv p

Ex. $s = \mathtt{aabaccba}$ over $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, then $\mathbf{pv}(s) = (4, 2, 2)$, order 8.

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Abelian stringology

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- Jumbled Pattern Matching
- abelian borders
- abelian periods
- abelian squares, repetitions, runs
- abelian pattern avoidance
- abelian reconstruction
- abelian problems on run-length encoded strings
- abelian complexity
- ...

Abelian stringology

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But first: What's so different about abelian problems?

An example problem

Parikh-de-Bruijn strings

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We define the abelian analogue:

Def. A Parikh-de-Bruijn string of order k (a (k, σ)-PdB-string) is a string s over an alphabet of size σ s.t.

$$\forall\ p \text{ Parikh vector of order } k\ \exists!(i,j) \text{ s.t. } \mathbf{pv}(s_i \cdots s_j) = p$$

Ex. `aabbcca` is a $(\overset{k}{2}, \overset{\sigma}{3})$-PdB-string

Parikh-de-Bruijn strings

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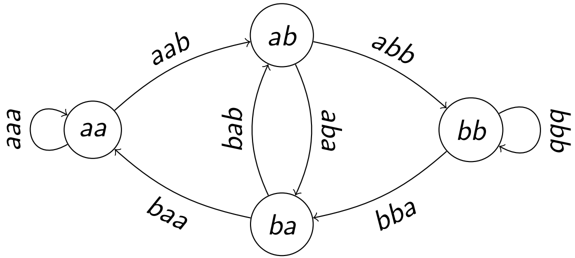

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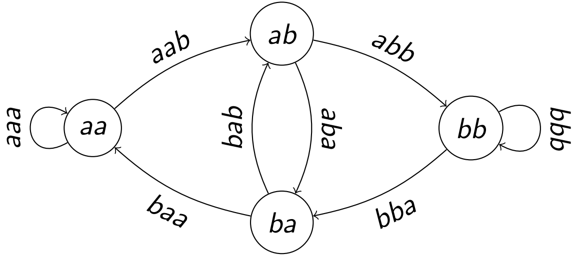

order 2: aabba

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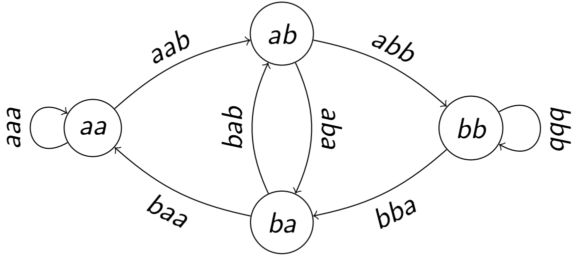

order 2: aabba

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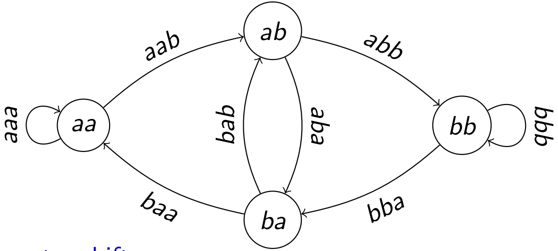
- abbbcccaaabc is a $(3, 3)$-PdB-string
- but no $(4, 3)$-PdB-string exists
- and no $(2, 4)$-PdB-string exists

The Parikh-de-Bruijn grid

Recall: de Bruijn graph $dB_k = (V, E)$, where $V = \Sigma^k$ and $(\mathtt{x}u, u\mathtt{y}) \in E$ for all $\mathtt{x}, \mathtt{y} \in \Sigma$ and $u \in \Sigma^{k-1}$

N.B. edges $\hat{=}$ $(k+1)$-length strings



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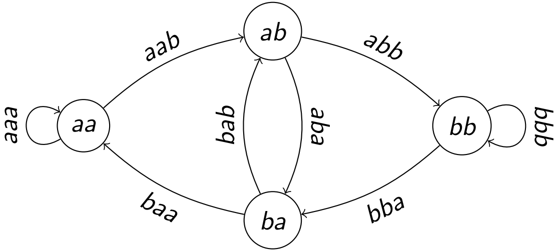

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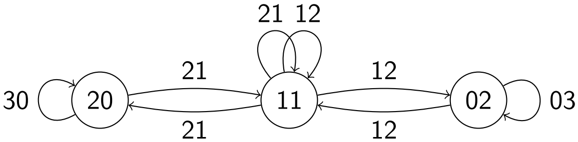
A straightforward generalization to Pv's gives:



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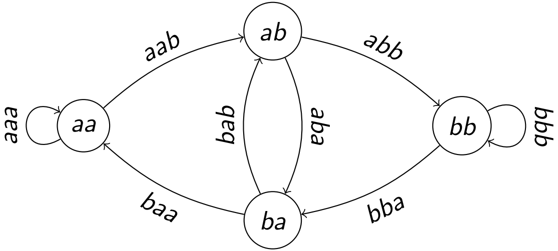

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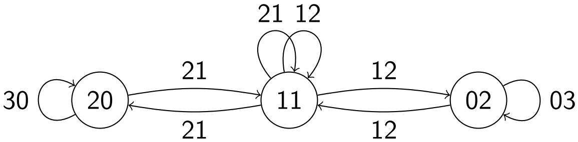

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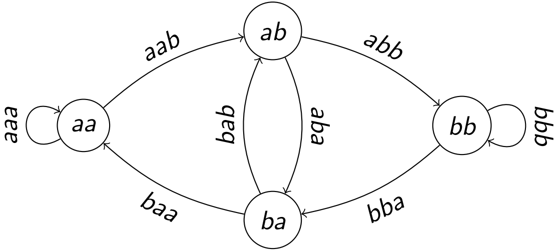

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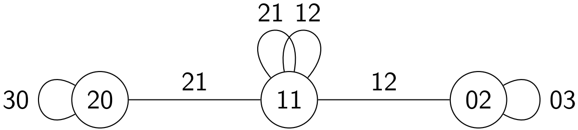

edges $\hat{=}$ one-character shifts, undirected edges

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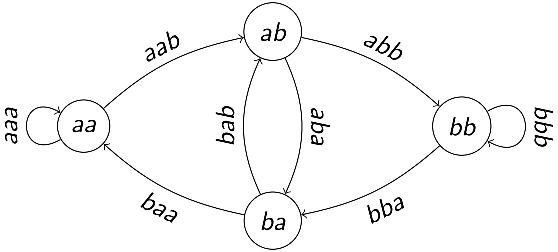

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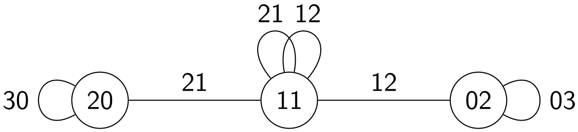
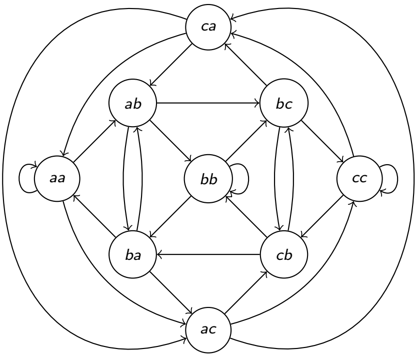
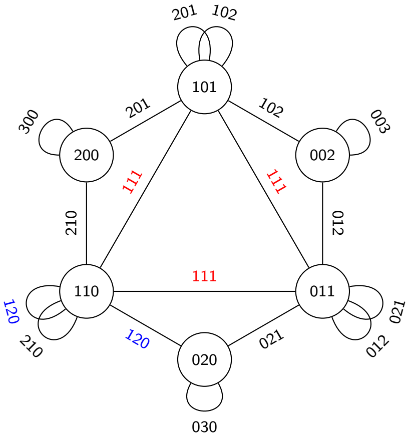
A straightforward generalization to Pv's gives: **NO:** edges $\hat{=}$ $(k+1)$-order Pv's!



edges $\hat{=}$ one-character shifts, undirected edges

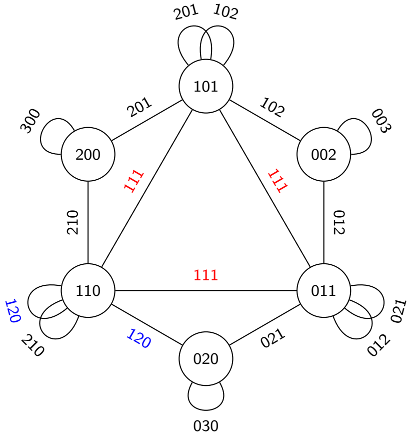
Let's look at another example: Here, $\sigma = 3, k = 2$.





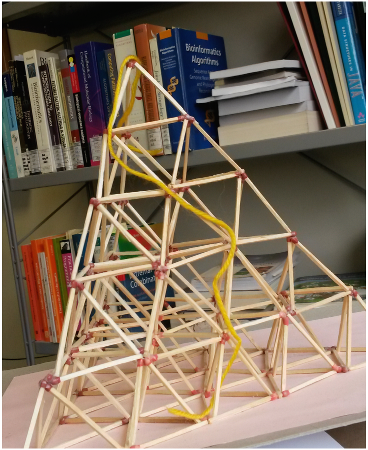
In the abelian version, several edges have the same label (i.e. here: 3-order Pv).

Turns out the right way to generalize de Bruijn graphs is the Parikh-de-Bruijn grid:



?

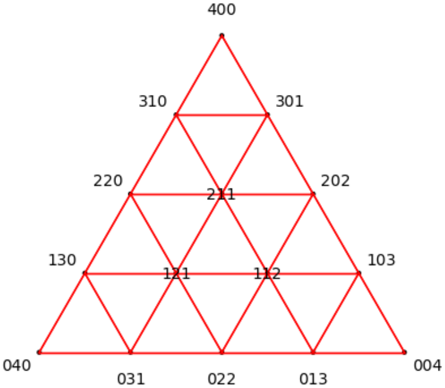
For $\sigma = 4, k = 5$, the Parikh-de-Bruijn grid looks like this:



The Parikh-de-Bruijn grid

PdB-grid:

- $V = k$-order Pv's

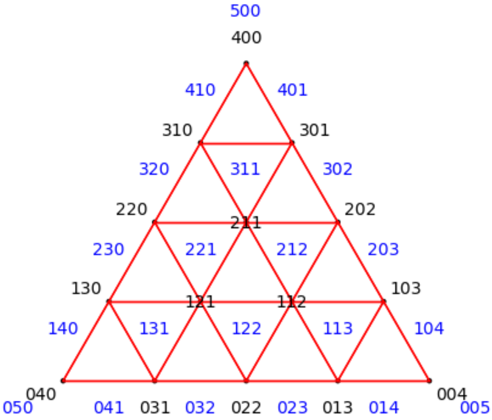


(4, 3)-grid; loops not included in figure

The Parikh-de-Bruijn grid

PdB-grid:

- $V = k$-order Pv's
- $pq \in E$ iff exist $\mathtt{x}, \mathtt{y} \in \Sigma$ s.t. $p = q - \mathtt{x} + \mathtt{y}$



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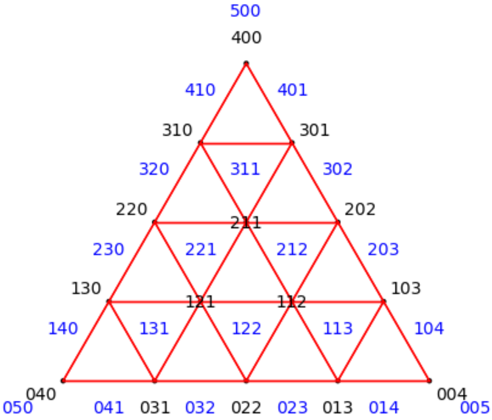

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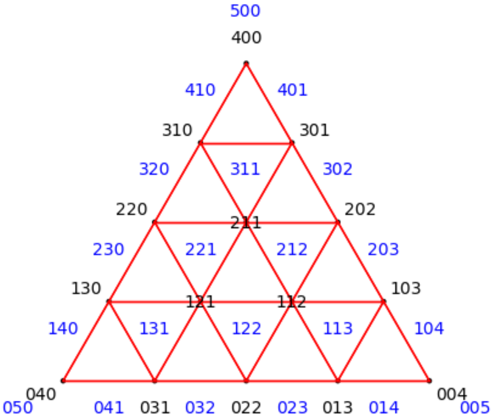

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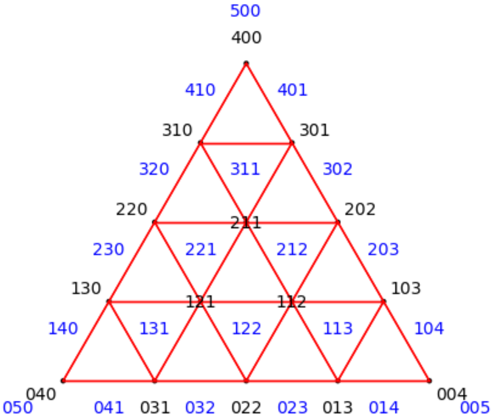

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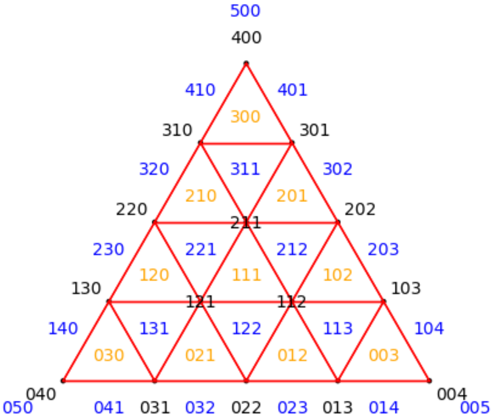

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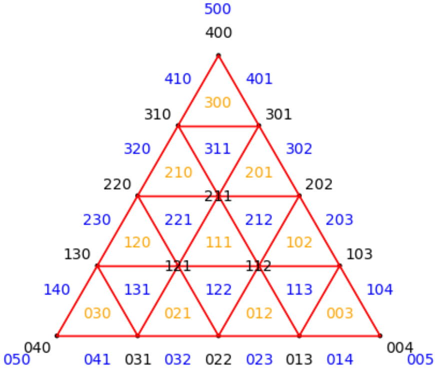
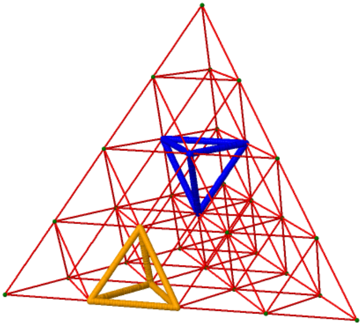
The Parikh-de-Bruijn grid

The $(4, 3)$-PdB-grid

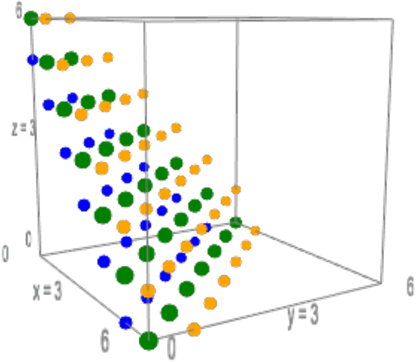


The $(4, 4)$-PdB-grid



vertices: k-order Pv's (vertices), downward triangles/tetrahedra: $(k+1)$-order Pv's, (upward triangles/tetrahedra: $(k-1)$-order Pv's.

The Parikh-de-Bruijn grid



The diagonal section of the integer grid with the hyperplanes $\mathcal{H}_k$ (green), $\mathcal{H}_{k+1}$ (blue), and $\mathcal{H}_{k-1}$ (yellow), for $k = 6$ and $\sigma = 3$.

Back to the example problem

More differences

Classical case: (dB_k) One-to-one correspondence: walks and strings.
Abelian case: Every string corresponds to a walk in the PdB-grid, but not every walk corresponds to a string.

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Classical case: De Bruijn sequences exist for every k and σ.
Abelian case: PdB-strings do not exist for every k and σ.

(**N.B.** Not all PdB-strings come from circular strings = universal cycles!)

Back to Parikh-de-Bruijn strings

Theorem 1

No $(k, 3)$-PdB strings exist for $k \geq 4$.

Theorem 2

A $(2, \sigma)$-PdB string exists if and only if σ is odd.

Theorem 3

A $(3, \sigma)$-PdB string exists if and only if $\sigma = 3$ or σ not a multiple of 3.

Theorem 1

No $(k, 3)$-PdB strings exist for $k \geq 4$.

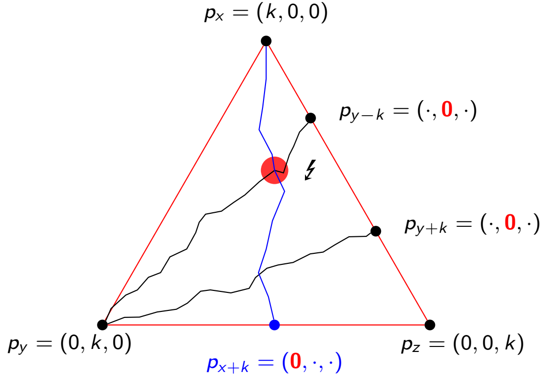
Lemma 1

If the walk induced by string w does not use any loops, then for all i: $w_i \neq w_{i+k}$.

Proof

Otherwise we would have two consecutive occurrences of the same Pv p, thus using a loop at p.

Theorem 1: No $(k,3)$-PdB strings exists for $k \geq 4$. (Proof uses Lemma 1.)



$$w = \cdots \quad \underbrace{aaa\cdots a}_{k}\cdots \quad \underbrace{bbb\cdots b}_{k}\cdots \quad \underbrace{ccc\cdots c}_{k}\cdots$$

$\uparrow$ position x

$\uparrow$ position y

$\uparrow$ position z

Example 2: Covering strings

Classical case: If s is a (classical) de Bruijn sequence of order k, then it also contains all $(k-1)$-length strings as substrings.

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Abelian case: not always true, e.g.

`aaaaabbbbbcaaaadbbbcccccddddaaaccdbcbaccaccddbddbadacddbbbb`

is a $(5,4)$-PdB-string but is not $(4,4)$-covering: no substring with Pv $(1,1,1,1)$.

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Theorem 4

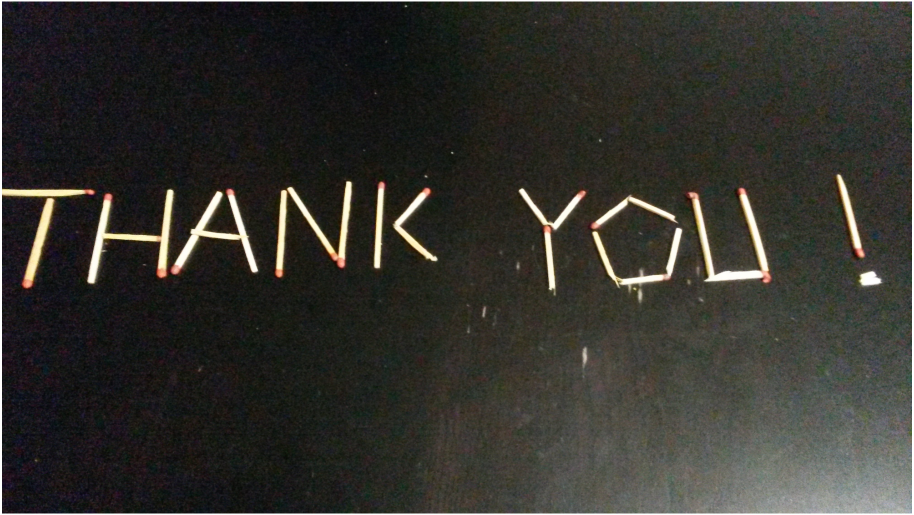
For every $\sigma \geq 3$ and $k \geq 4$, there exist (k,σ)-covering strings which are not $(k-1,\sigma)$-covering.

Experimental results

k	σ	*string*	*length (excess)*
2	3	aabbcca	7 (0)
3	3	abbbcccaaabc	12 (0)
4	3	aaaabbbbccccaacabcb	19 (1)
5	3	aaaaabbbacccccbbbbbaacaaccb	27 (2)
6	3	aaaabccccccaaaaaabbbbbbcccbbcabbaca	35 (2)
7	3	aabbbccbbcccabacaaabcbbbbbbbaaaaaaaccccccccba	44 (2)
2	4	aabbcadbccdd	12 (1)
3	4	aaabbbcaadbdbccadddccc	22 (0)
4	4	aabbbbcaacadbddbccacddddaaaabdbbccccdd	38 (0)
5	4	aaaaabbbbbcaaaadbbbcccccddddaaaccdbcbaccaccddbddbadacddbbbb	60 (0)
2	5	aabbcadbeccddeea	16 (0)
3	5	aaabbbcaadbbeaccbdddcccebededadceeeaa	37 (0)
4	5	aaaabbbbcaaadbbbeaaccbbddaaeaebcccadbeeeadddcccceeeedddd...	73 (0)

Conclusion and open problems

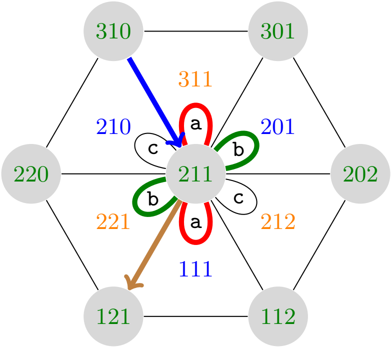
- new tool for modeling and solving abelian problems
- find good characterization for walks which correspond to strings
- many open problems on PdB- and covering strings, e.g.
  - For which σ and k do *PdB*-strings exist? (We answered this question only for some special cases.)
  - What is the length of a shortest covering string when no PdB-string exists, e.g. $k = 3$, $\sigma = 6$?
  - What is the minimum proportion of $(k - 1)$-order Pv's covered by a k-covering string?
- apply PdB-grid to other abelian problems
- paper on Arxiv


THANK YOU !

Appendix

The Parikh-de-Bruijn grid

$k = 4$, $\sigma = 3$



$(k+1)$	a	3	3	2	2				
	b	1	1	2	2				
	c	1	1	1	1				
		a	a	b	a	c	a	b	b
k	a	3	2	2	2	1			
	b	1	1	1	1	2			
	c	0	1	1	1	1			
$(k-1)$	a		2	1	2	1			
	b		1	1	0	1			
	c		0	1	1	1			

Walk corresponding to aabacabb. $(k+1)$- and $(k-1)$-order Pv's: triangles incident to the edges traversed by the walk. The $(k+1)$ and $(k-1)$-order Pv's for loops (same k-order Pv twice) lie in opposite direction, hence the name bow.

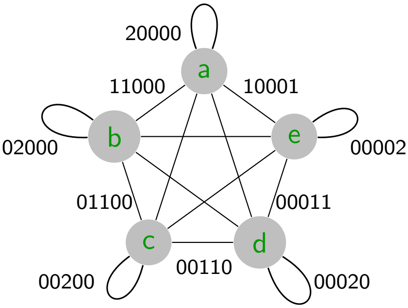
Parikh-de-Bruijn and covering strings

Theorem 2

A $(2, \sigma)$-PdB string exists if and only if σ is odd.

Proof

Pv's of order 2 either form $(0...0, \overset{i}{2}, 0..0)$ or $(0...0, \overset{i}{1}, 0...0, \overset{j}{1}, 0..0)$. So we need exactly one substring aa for all $\mathtt{a} \in \Sigma$, and either ab or ba for all $\mathtt{a}, \mathtt{b} \in \Sigma$. Consider the undirected complete graph $G = (V, E)$ with loops where $V = \Sigma$ (N.B.: not the PdB-grid!): an Euler path exists iff σ is odd.



Example 2: Covering strings

Next best thing: covering strings.

Def.

- We call a string s (k,σ)-covering if

$$\forall\ p \text{ Parikh vector of order } k\ \exists(i,j) \text{ s.t. } \mathbf{pv}(s_i \cdots s_j) = p$$

  (There is at least one substring in s which has Pv p.)
- The excess of s is: $|s| - \underbrace{\binom{\sigma+k-1}{k} + k - 1}_{\text{length of a PdB-string}}$.

Ex.

- `aaaabbbbccccaacabcb` is a shortest $(4,3)$-covering string, with excess 1.
- `aabbcadbccdd` is a shortest $(2,4)$-covering string, with excess 1.

Example 2: Covering strings

Classical case: If s is a (classical) de Bruijn sequence of order k, then it also contains all $(k-1)$-length strings as substrings.

For PdB-strings, this is not always true, e.g.

`aaaaabbbbbcaaaadbbbcccccddddaaaccdbcbaccaccddbddbadacddbbbb`

is a $(5,4)$-PdB-string but is not $(4,4)$-covering: no substring with Pv $(1,1,1,1)$.

Theorem 4

For every $\sigma \geq 3$ and $k \geq 4$, there exist (k,σ)-covering strings which are not $(k-1,\sigma)$-covering.

The Parikh-de-Bruijn grid

Lemma 2

A set of k-order Parikh vectors is realizable if and only if the induced subgraph in the k-PdB-grid is connected.

realizable = exists string with exactly these k-order sub-Pv's.

Proof sketch

$\Rightarrow$: clear.

$\Leftarrow$: Use loops until undesired character $\mathtt{x}$ exits, replace by new character $\mathtt{y}$.

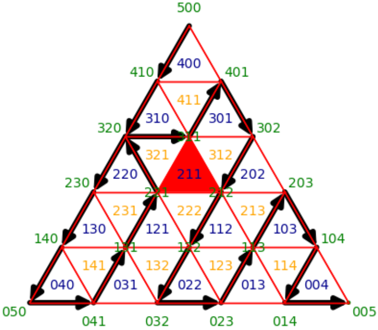
Parikh-de-Bruijn and covering strings

Theorem 4

For every $\sigma \geq 3$ and $k \geq 4$, there exist (k, σ)-covering strings which are not $(k-1, \sigma)$-covering.

Proof

$w =$ `aaaaabbbbbcabbaaacacbbcbccacaccccbccccc`



General construction:

- remove $(k-1)$-order Pv $p = (k-3, 1, 1, 0, \ldots, 0)$ with incident edges and vertices
- the rest is connected, hence a string exists (Lemma 2)
- add vertices of p without traversing edges incident to p
- can be done by detours from corners of PdB-grid