Normal, Abby Normal, Prefix Normal

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FUN 2014
Lipari, 1-3 July 2014

Prefix normal words

Fici, Lipták (DLT 2011)

Definition

A binary word w is prefix normal (w.r.t. 1) if no substring has more 1s than the prefix of the same length.

Example

$$w = 1011000100110111 0010$$

$$w' = 11101001011001010010$$

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Example

$$w = 1011000100110111 0010 \quad \textit{NO}$$

$$w' = 1110100101100101 0010 \quad \textit{YES}$$

all p.n. words of length 4:
$1111, 1110, 1101, 1100, 1010, 1001, 1000, 0000.$

Prefix normal games

Let's play a game.

Alice and Bob are constructing a binary string of length n together.
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Alice has won: $10000, 10100, 10101$ are all p.n.

Prefix normal games

Exercise

Find n_0 s.t.

- Alice has a winning strategy for all $n < n_0$, and
- Bob has a winning strategy for all $n \geq n_0$.

Some properties of prefix normal words

- w is p.n. $\Rightarrow$ $w0$ is p.n.
- w is p.n. $\Rightarrow$ every prefix of w is p.n.
- for any w of length n, $v = 1^n w$ is p.n.
- ...

Prefix normal forms

- Let $F(w, k)$ = max #1s in a substring of length k

  Ex. $\mathbf{w} = \mathbf{11010010101}$, $F(w, 1) = 1, F(w, 2) = 2, F(w, 3) = 2, \ldots$

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- **prefix normal form**:
  For every w exists unique p.n. word w' s.t.

  $$\forall k : F(w,k) = F(w',k)$$

  $w' = \mathrm{PNF}_1(w)$. (next slides)

Where do prefix normal words come from?

Binary Jumbled Pattern Matching

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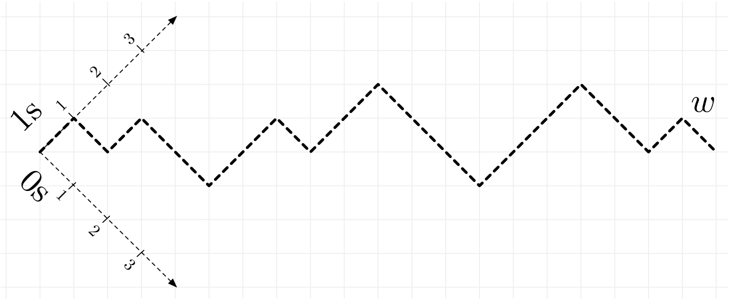
Recent papers on this problem:
PSC 2009, FUN 2010, IPL 2010, JDA 2012, ToCS 2012, IJFCS 2012, CPM 2012, SPIRE 2012, IPL 2013, IPL 2013, ESA 2013 x 2, SPIRE 2013, TCS 2014, PhTRS-A 2014, CPM 2014, CPM 2014, ISIT 2014, ICALP 2014, ...

(red ones have an intersection with the authors of this paper)

BJPM with prefix normal forms

Does $\mathbf{w} = \mathbf{1010011011000111 0010}$ have a substring of length 11 containing exactly 5 ones?



$\diagup = 1$, $\diagdown = 0$,

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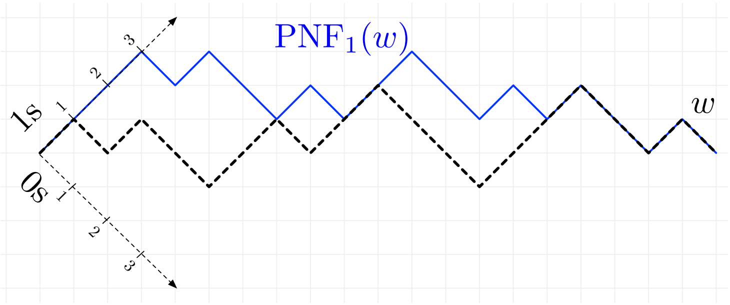

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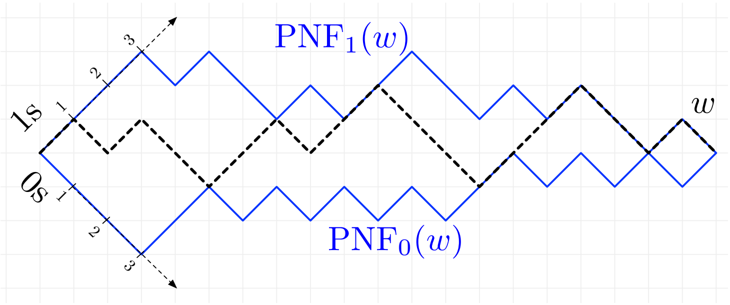

╱ = 1, ╲ = 0, area between blue lines: all substrings of *w*

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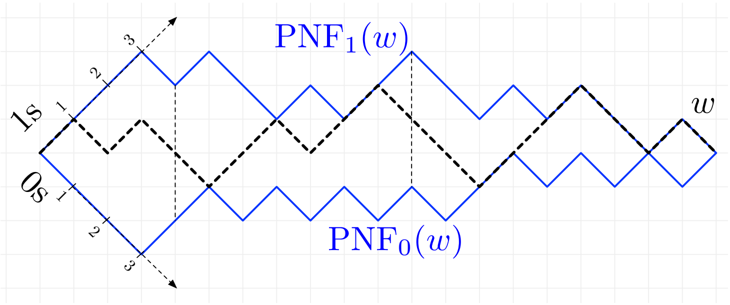
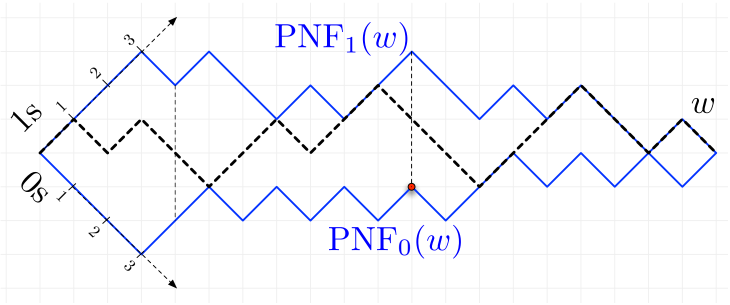


$\diagup = 1$, $\diagdown = 0$, area between blue lines: all substrings of w

verticals: fixed length substrings $k = 4, 11$.

BJPM with prefix normal forms

Does $\mathbf{w} = \mathbf{10100110110001110010}$ have a substring of length 11 containing exactly 5 ones? YES



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BJPM with prefix normal forms

Does $\mathbf{w} = \mathbf{10100110110001110010}$ have a substring of length 11 containing exactly 5 ones?

$\#$ 1s in $\text{pref}(\mathrm{PNF}_1(w), 11) = 7$
$\#$ 1s in $\text{pref}(\mathrm{PNF}_0(w), 11) = 5$

$$7 \geq 5 \geq 5 \quad \leadsto \text{YES}$$

Thus, fast computation of PNFs yields fast solution to BJPM.

BJPM with prefix normal forms

Does $\mathbf{w} = \mathbf{1010011011000111001 0}$ have a substring of length 11 containing exactly 5 ones?

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Thus, fast computation of PNFs yields fast solution to BJPM.
Better understanding p.n. words might be useful for BJPM.

BJPM with prefix normal forms

Does $\mathbf{w} = \mathbf{101001101100011100010}$ have a substring of length 11 containing exactly 5 ones?

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Thus, fast computation of PNFs yields fast solution to BJPM.
Better understanding p.n. words might be useful for BJPM.
Or might just be fun.

Some questions about prefix normal words

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5. In general, properties of p.n. words. – ongoing

Computing the PNF

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Analysis

Uses a quadratic amount of sand but can be faster than other algorithms if implemented by a very fast person.

Enumerating p.n. words

Let $pnw(n) = \#$ prefix normal words of length n.

Easy:

$pnw(n)$ grows exponentially.
Proof: $\forall w : 1^{|w|}w$ is p.n. $\quad \Rightarrow \quad pnw(2n) \geq 2^n$.

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Theorem

Exists $c > 0$*:* $pnw(n) = \Omega(2^{c\sqrt{n \ln n}}) = \Omega((2-\epsilon)^n) \quad \forall \epsilon > 0$.

(Proof uses a variant of the prefix normal game.)

Theorem

$pnw(n) = O\left(\frac{2^n(\ln n)^2}{n}\right) = o(2^n)$.

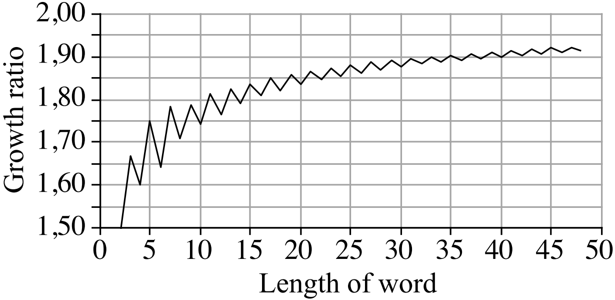
Enumerating p.n. words

Thanks to the new generating algorithm, we can now count p.n. words up to $n = 50$, and also special subsets, e.g. fixed-density, fixed-prefix, fixed-suffix, ... And make intelligent guesses.

Enumerating p.n. words

OEIS sequence no. A194850

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$pnw(n)$	2	3	5	8	14	23	41	70	125	218	395	697	1273	2279	4185	7568



Fixed-density p.n. words

For a word w, its density d is the number of 1s in w,
$pnw(n,d)$ = no. of p.n. words of length n and density d.

Let $f_d(x) = \sum_{n \geq 0} pnw(n,d) x^n$, the generating function of $pnw(n,d)$.

gen.func.	coefficients	closed form
$f_0(x) = \frac{1}{1-x}$	$1,1,1,1,1,1,1,1\ldots$	1
$f_1(x) = \frac{x}{1-x}$	$0,1,1,1,1,1,1,1\ldots$	$[n > 0]$
$f_2(x) = \frac{x^2}{(1-x)^2}$	$0,0,1,2,3,4,5,6\ldots$	$(n-1)$
$f_3(x) = \frac{x^3}{(1-x^2)(1-x)^2}$	$0,0,0,1,2,4,6,9,\ldots$	$pnw(2n) = n(n-1)$, $pnw(2n+1) = (n-1)^2$

...and more (but for larger d becomes less and less manageable) ...

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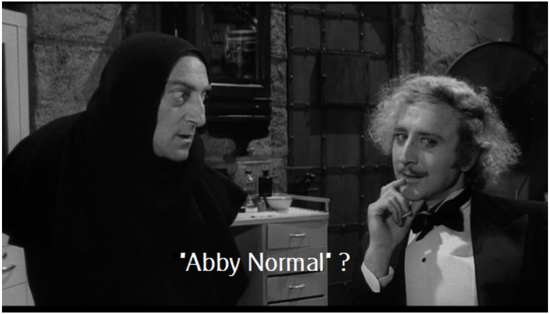

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Mel Brooks: Young Frankenstein (1974)

THANK YOU!