# Bioinformatics Algorithms

## (Fundamental Algorithms, module 2)

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Suffix Trees 2

# Pattern matching with the suffix tree

# Recall: Pattern matching

## Pattern matching

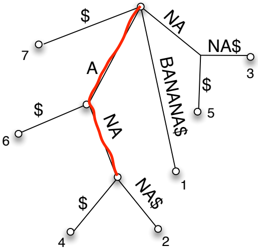
Given a string $T$ of length $n$ (the text), and a string $P$ of length $m$ (the pattern), find all occurrences of $P$ as substring of $T$.

## Variants:

- all-occurrences version: output all occurrences of $P$ in $T$
- decision version: decide whether $P$ occurs in $T$ (yes - no)
- counting version: output $occ_P$, the number of occurrences of $P$ in $T$

# Pattern matching with suffix tree

Let text $T = \text{BANANA}$ and pattern $P = \text{ANA}$. We try to match the pattern starting from the root and following the labels on the edges; when we encounter a node, we have at most one possible edge which to follow[1]:

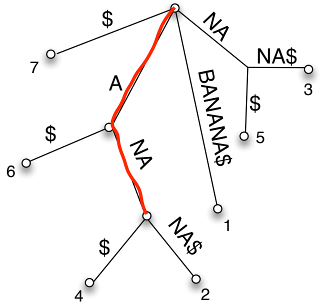



Since we have matched all of the pattern, we now know that $P = \text{ANA}$ occurs in $T$ (decision).

[1] recall that every outgoing edge from an inner node starts with a different character

# Pattern matching with suffix tree

Moreover, the occurrences of $P$ are exactly the numbers of the leaves in the subtree below *locus*($P$) (the position where we finished matching $P$).



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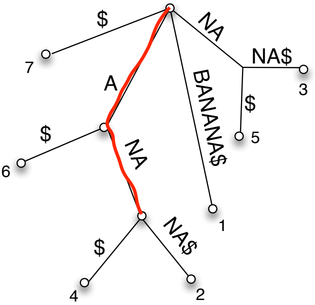


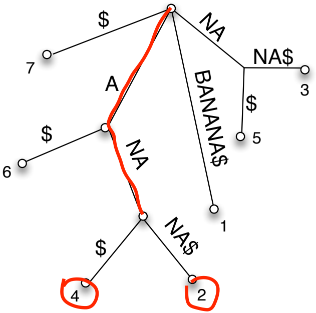

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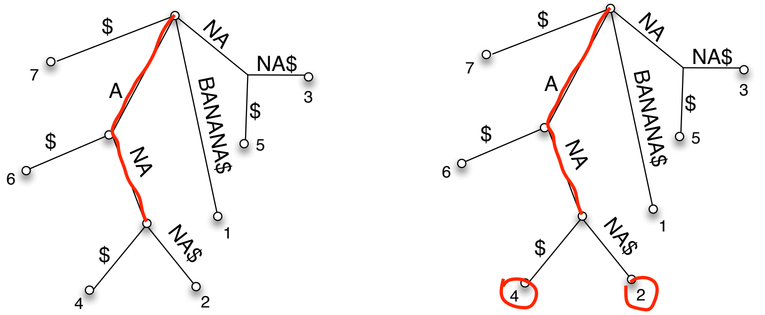


Why is this? Because $P$ occurs in position $i$ iff $P$ is a prefix of $Suf_i$. As we have seen, the path from the root to leaf number $i$ spells exactly $Suf_i$.

# Pattern matching with suffix tree

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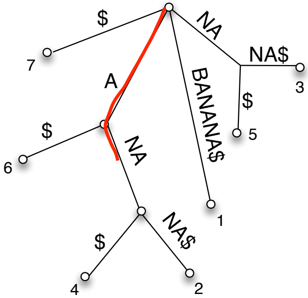

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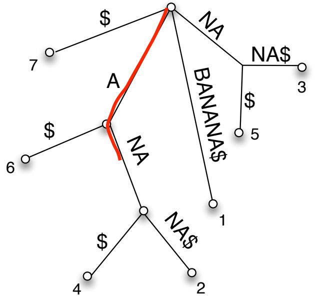


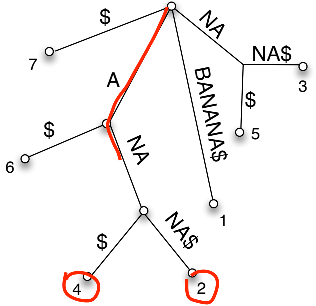

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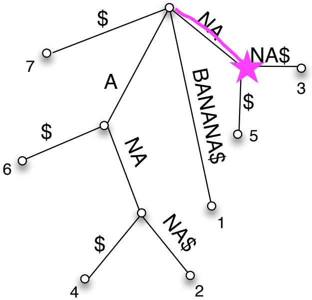

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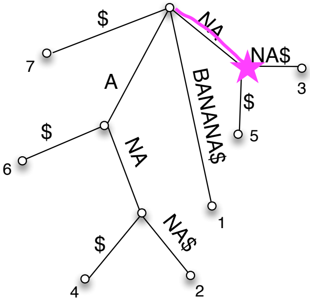


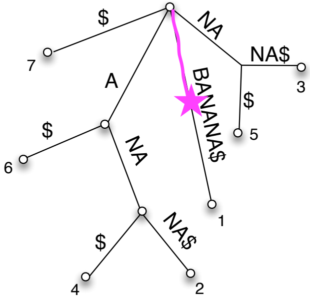

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**Note that all these times are independent of the size $n$ of the text.**

# Suffix tree construction

# Construction of suffix trees

Theorem:
$ST(T)$ can be constructed in $O(n)$ time.

Several linear time algorithms exist (beyond the scope of this course). We will see two simple quadratic-time construction algorithms.

# Simple ST construction algorithm 1

## Simple suffix insertion algorithm

1. start with tree $\mathcal{T}$ with one node (the root)
2. for $i = 1, \ldots, n+1$: insert $Suf_i$ into $\mathcal{T}$

## Insert string $S$ into $\mathcal{T}$

1. $\ell \leftarrow |S|$
2. start matching $S$ (as for pattern matching) in $\mathcal{T}$, starting from root
3. at first mismatch $j$ in $S$:
   - if currently in node $u$, add new child $v$ to $u$
   - otherwise, create new node $u$ at current locus with new child $v$
4. add edge label $L(u, v) = S_j \ldots S_\ell$

Note that there is always a mismatch, because no suffix is the prefix of another suffix (that's why we chose \$ as a new character!)

## Simple ST construction algorithm 2

Another simple algorithm is the following recursive algorithm (Giegerich & Kurtz, 1995):

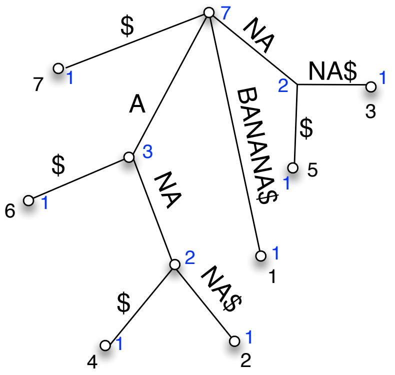
### WOTD algorithm (write-only, top-down)

1. Let $X$ be the set of all suffixes of $T\$$.
2. Sort the suffixes in $T$ according to their first character;
   for $c \in \Sigma \cup \{\$\}$: $X_c =$ suffixes starting with character $c$.
3. For each group $X_c$:
   (i) if $X_c$ is a singleton, create a leaf;
   (ii) otherwise, find the longest common prefix of the suffixes in $X_c$, create an internal node, and recursively continue with Step 2, $X$ being the set of remaining suffixes from $X_c$ after splitting off the longest common prefix.

**N.B.:** Both of these algorithms have worst-case running time $O(n^2)$ (without proof).

# Storing addition information in the suffix tree

Recall the pattern matching problem, counting variant: Return the number of occurrences of pattern $P$. Let $g(u)$ = number of leaves in subtree rooted in $u$.




If we store $g(u)$ in $u$, then we can solve the counting problem in $O(m)$ time: match $P$ in ST, if found in $locus(P) = (u, d)$, then return $g(u)$. E.g. the number of occurrences of $P =$ AN is 2, as can be seen immediately in ST.

# Postorder traversal of ST

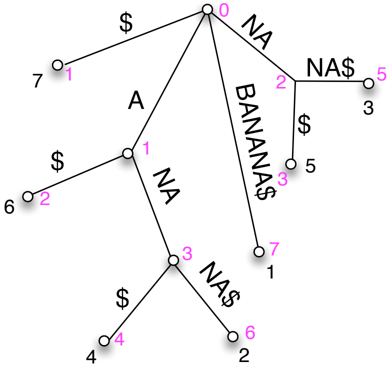
Note that the number of leaves in subtree rooted in $u$, where $u$ has children $v_1, \ldots, v_k$, equals the sum of the leaves in the subtrees of the $v_i$.

Compute the number of leaves in subtree, $g(u)$, via post-order traversal of the ST (bottom-up):

1. if $u$ leaf: $g(u) \leftarrow 1$
2. if $u$ inner node: $g(u) = \sum_{v \text{ child of } u} g(v)$

This takes linear time in the size of the tree, i.e. $O(n)$ time. Moreover, the information stored is constant per node, so the space needed for the ST is still $O(n)$.

Another piece of information we often need is the stringdepth $sd(u)$ of a node $u$ (the length of its label).



# Preorder traversal of ST

Note that the stringdepth of a node $u$ with parent $v$ equals the stringdepth of $v$ plus the length of the label of the edge connecting $v$ and $u$.

Compute the stringdepth of a node, $sd(u)$, via pre-order traversal of the ST (top-down):

1. for the root: $sd(root) = 0$
2. for all other nodes $u$: Let $v = parent(u)$.
   Then $sd(u) = sd(v) + |L(v, u)|$.

Again, this takes linear time $O(n)$ and total space $O(n)$ (since we store constant amount per node).

# Summary

- The suffix tree is an extremely versatile data structure for solving problems on strings/sequences.
- It takes linear storage space in the size of the text $O(n)$. (Remember: edge labels are stored as two pointers into $T$.)
- It can be constructed in linear time $O(n)$ (not studied in this course).
- Leaves of ST correspond to suffixes of $T$.
- Loci (inner nodes or "positions on edges") corr. to substrings of $T$.
- Leaves in subtree rooted in $u$ correspond to occurrences of substrings whose locus is on edge leading to $u$.
- The ST can be used to solve pattern matching queries in time independent of the text size: $O(m)$ for decision, $O(m + occ_P)$ for all-occurrences, $O(m)$ for counting (after linear time preproc.)
- The ST can be used to solve many many other types of queries on strings efficiently.