Algoritmi di Bioinformatica

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Computational efficiency I

Computational Efficiency

As we will see later in more detail, the efficiency of algorithms is measured w.r.t.

- running time
- storage space

We will make these concepts more concrete later on, but for now want to give some intuition, using an example.

Example: Computation of nth Fibonacci number

Fibonacci numbers: model for growth of populations (simplified model)

- Start with 1 pair of rabbits in a field
- each pair becomes mature at age of 1 month and mates
- after gestation period of 1 month, a female gives birth to 1 new pair
- rabbits never die[1]

Definition
$F(n)$ = number of pairs of rabbits in field at the beginning of the n'th month.

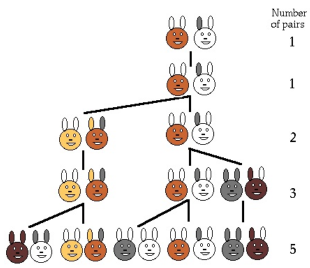
[1] This unrealistic assumption simplifies the mathematics; however, it turns out that adding a certain age at which rabbits die does not significantly change the behaviour of the sequence, so it makes sense to simplify.

Computation of nth Fibonacci number

- month 1: there is 1 pair of rabbits in the field $F(1) = 1$
- month 2: there is still 1 pair of rabbits in the field $F(2) = 1$
- month 3: there is the old pair and 1 new pair $F(3) = 1 + 1 = 2$
- month 4: the 2 pairs from previous month, plus the old pair has had another new pair $F(4) = 2 + 1 = 3$
- month 5: the 3 from previous month, plus the 2 from month 3 have each had a new pair $F(5) = 3 + 2 = 5$

Recursion for Fibonacci numbers
$F(1) = F(2) = 1$
for $n > 2$: $F(n) = F(n-1) + F(n-2)$.

Computation of nth Fibonacci number



source: Fibonacci numbers and nature (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html)

Computation of nth Fibonacci number

The first few terms of the Fibonacci sequence are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$F(n)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377

n	15	16	17	18	19	20	21	22	23
$F(n)$	610	987	1 597	2 584	4 181	6 765	10 946	17 711	28 657

Fibonacci numbers in nature

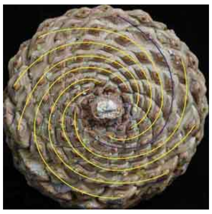

21 spirals left

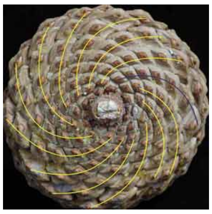

34 spirals right

source: Plant Spiral Exhibit
(http://cs.smith.edu/ phyllo/Assets/Images/ExpoImages/ExpoTour/index.htm)

On these pages it is explained how these plants develop. Very interesting!

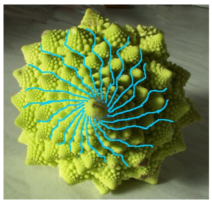
Fibonacci numbers in nature

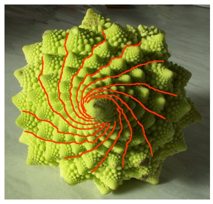

8 spirals left


13 spirals right

source: Plant Spiral Exhibit
(http://cs.smith.edu/ phyllo/Assets/Images/ExpoImages/ExpoTour/index.htm)

Fibonacci numbers in nature


21 spirals left


13 spirals right

source: Fibonacci numbers and nature
(http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html)

very nice page! recommended!

Growth of Fibonacci numbers

Theorem
For $n > 6$: $F(n) > (1.5)^{n-1}$.

Proof:
Note that from $n = 3$ on, $F(n)$ strictly increases, so for $n \geq 4$, we have $F(n-1) > F(n-2)$. Therefore, $F(n-1) > \frac{1}{2}F(n)$.

We prove the theorem by induction:
Base: For $n = 6$, we have $F(6) = 8 > 7.59\ldots = (1.5)^5$.
Step: Now we want to show that $F(n+1) > (1.5)^n$. By the I.H. (induction hypothesis), we have that $F(n) > (1.5)^{n-1}$. Since $F(n-1) > 0.5F(n)$, it follows that $F(n+1) = F(n) + F(n-1) > 1.5 \cdot F(n) > (1.5) \cdot (1.5)^{n-1} = (1.5)^n$.

Computation of nth Fibonacci number

Algorithm 1 (let's call it fib1) works exactly along the recursive definition:

Algorithm *fib1(n)*
1. **if** $n = 1$ or $n = 2$
2. **then return** 1
3. **else**
4. **return** $fib1(n-1) + fib1(n-2)$

Computation of nth Fibonacci number

Analysis

(sketch) Looking at the computation tree, we can see that the tree for computing $F(n)$ has $F(n)$ many leaves (show by induction), where we have a lookup for $F(2)$ or $F(1)$. A binary rooted tree has one fewer internal nodes than leaves (see second part of course, or show by induction), so this tree has $F(n) - 1$ internal nodes, each of which entails an addition. So for computing $F(n)$, we need $F(n)$ lookups and $F(n) - 1$ additions, altogether $2F(n) - 1$ operations (additions, lookups etc.).

The algorithm has exponential running time, since it makes $2F(n) - 1$, i.e. at least $2 \cdot (1.5)^{n-1} - 1$ steps (operations).

Computation of nth Fibonacci number

Algorithm 2 (let's call it fib2) computes every $F(k)$, for $k = 1 \dots n$, iteratively (one after another), until we get to $F(n)$.

Algorithm *fib2(n)*
array of int $F[1 \dots n]$;
$F[1] \leftarrow 1$; $F[2] \leftarrow 1$;
for $k = 3 \dots n$
    do $F[k] \leftarrow F[k-1] + F[k-2]$;
return $F[n]$;

Analysis

(sketch) One addition for every $k = 1, \dots, n$. Uses an array of integers of length n.—The algorithm has linear running time and linear storage space.

Computation of nth Fibonacci number

Algorithm 3 (let's call it fib3) computes $F(n)$ iteratively, like Algorithm 2, but using only 3 units of storage space.

Algorithm *fib3(n)*
int a, b, c;
$a \leftarrow 1$; $b \leftarrow 1$; $c \leftarrow 1$;
for $k = 3 \dots n$
    do $c \leftarrow a + b$;
        $a \leftarrow b$; $b \leftarrow c$;
return c;

Analysis

(sketch) Time: same as Algo 2. Uses 3 units of storage (called a, b, and c).—The algorithm has linear running time and constant storage space.

Comparison of running times

n	1	2	3	4	5	6	7	10	20	30	40
$F(n)$	1	1	2	3	5	8	13	55	6 765	832 040	102 334 155
fib1	1	1	3	5	9	15	25	109	13 529	1 664 079	204 668 309
fib2	1	2	3	4	5	6	7	10	20	30	40
fib3	1	2	3	4	5	6	7	10	20	30	40

The number of steps each algorithm makes to compute $F(n)$.

Summary

- We saw 3 different algorithms for the same problem (computing the nth Fibonacci number).
- They differ greatly in their efficiency:
  - Algo fib1 has exponential running time.
  - Algo fib2 has linear running time and linear storage space.
  - Algo fib3 has linear running time and constanct storage space.
- We saw on an example computation (during class) that exponential running time is not practicable.

Summary (2)

Take-home message

- There may be more than one way of computing something.
- It is very important to use efficient algorithms.
- Efficiency is measured in terms of running time and storage space.
- Computation time is important for obvious reasons: the faster the algorithm, the more problems we can solve in the same amount of time.
- In computational biology, inputs are often very large, therefore storage space is at least as important as running time.