Commutative Hearts: the Gabriel Spectrum

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on ongoing joint work with Michal Hrbek and Sergio Pavon

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§1. Compactly generated t-structures in $\mathcal{D}(R)$

R commutative noetherian, $\operatorname{Spec}^H R$ with Hochster dual topology:

$X \subseteq \operatorname{Spec}^H R$ open $\iff$ X is union of Zariski-closed

THEOREM [Alonso-Jeremías-Saorín '10, Hrbek-Nakamura '21]

$$\left\{\begin{array}{c}\text{Compactly generated intermediate}\\ t\text{-structures in } \mathcal{D}(R)\end{array}\right\} \xrightarrow{\ 1:1\ } \left\{\begin{array}{l}\Phi : \mathbb{Z} \longrightarrow \operatorname{Op}(\operatorname{Spec}^H R)\\ \bullet \text{ decreasing}\\ \bullet\ \exists a : \Phi(a) = \operatorname{Spec} R\\ \bullet\ \exists b : \Phi(b) = \emptyset\end{array}\right\}$$

$$\left(\mathcal{U}_\Phi := \{x \in \mathcal{D}(R) : \operatorname{Supp} H^k(x) \subseteq \Phi(k)\ \forall k \in \mathbb{Z}\}, \mathcal{V}_\Phi\right) \longleftarrow\!\!\shortmid\ \Phi$$

$\updownarrow$ 1:1 $\qquad\qquad\qquad\qquad$ $\updownarrow$ 1:1

$$\left\{\begin{array}{c}\text{Grothendieck hearts of}\\ \text{Smashing } t\text{-structures}\\ \mathcal{H} \hookrightarrow \mathcal{D}^{[a,b]}(R)\end{array}\right\} \overset{?}{\rightsquigarrow} \left\{ f : \operatorname{Spec} R \longrightarrow \mathbb{Z} : \begin{array}{l}\bullet \text{ increasing}\\ \bullet \operatorname{Im}(f) \text{ bounded}\end{array}\right\}$$

$$\{k(\mathfrak{p})[-f(\mathfrak{p})] : \mathfrak{p} \in \operatorname{Spec} R\} \subseteq \mathcal{H} \longleftarrow\!\!\shortmid\ f$$

$[f_\Phi(\mathfrak{p}) = n \iff \mathfrak{p} \in \Phi(n) \setminus \Phi(n+1)]$

What is the role of these objects?

§2. The Gabriel filtration

DEFINITION $\mathcal{G}$ Grothendieck category. The Gabriel filtration of $\mathcal{G}$ is:

$$0 = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 \subseteq \cdots \subseteq \mathcal{G}_\alpha \subseteq \mathcal{G}_{\alpha+1} \subseteq \cdots \subseteq \mathcal{G}$$

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\pi_\alpha"] & \mathcal{G}/\bigcup_{\beta<\alpha} \mathcal{G}_\beta \arrow[l, bend left, "\rho_\alpha"] \end{tikzcd} \quad \text{and} \quad \mathcal{G}_\alpha = \pi_\alpha^{-1}\Big(\mathrm{Loc}\big(\mathrm{simp}\, \mathcal{G}/\bigcup_{\beta<\alpha}\mathcal{G}_\beta\big)\Big)$$

$\mathcal{G}$ is Seminoetherian if $\exists d : \mathcal{G}_d = \mathcal{G}$. Write $d = \mathrm{Gdim}\, \mathcal{G}$.

If $\mathcal{G}$ is seminoetherian, write $\mathrm{GSimp}(\mathcal{G}) = \{\rho_\alpha(\mathrm{simp}\, \mathcal{G}/\bigcup_{\beta<\alpha}\mathcal{G}_\beta) : \alpha < d\}$

Example: R commutative noetherian, $\mathcal{G} = \mathrm{Mod}\, R$, $\mathrm{kdim}\, R < \infty$.
Then $\mathrm{Mod}\, R$ is Seminoetherian, $\mathrm{Gdim}(\mathrm{Mod}\, R) = \mathrm{kdim}\, R$, and

$$\mathrm{GSimp}(\mathrm{Mod}\, R) = \{ k(\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}\, R \}$$

§3. The Gabriel Spectrum

DEFINITION $\mathrm{GSpec}\,\mathcal{G} := \mathrm{Ind}(\mathrm{Inj}\,\mathcal{G})$

$V \subseteq \mathrm{GSpec}\,\mathcal{G}$ closed if $\exists (\mathcal{T},\mathcal{F})$ hereditary torsion pair: $V = \mathrm{GSpec}\,\mathcal{G} \cap \mathcal{F}$

Example: R commutative noetherian, $\mathcal{G} = \mathrm{Mod}\,R$

$\mathrm{GSpec}(\mathrm{Mod}\,R) \cong \mathrm{Spec}^H R$

[Matlis] $\mathrm{GSpec}\,\mathcal{G} = \{E_\mathfrak{p} := E(R/\mathfrak{p}) = E(k(\mathfrak{p})) : \mathfrak{p} \in \mathrm{Spec}(R)\}$

[Gabriel] $\mathcal{U} \subseteq \mathrm{GSpec}\,\mathcal{G}$ open $\iff \forall \mathfrak{p} \subseteq \mathfrak{q},\ E_\mathfrak{p} \in \mathcal{U} \Rightarrow E_\mathfrak{q} \in \mathcal{U}$

$$\mathrm{GSupp} : \left\{\begin{array}{c}\text{hereditary torsion}\\ \text{classes in } \mathcal{G}\end{array}\right\} \twoheadrightarrow \mathcal{O}_p(\mathrm{GSpec}\,\mathcal{G})$$

$$\mathcal{T} \longmapsto \mathrm{GSpec}\,\mathcal{G} \setminus (\mathrm{GSpec}\,\mathcal{G} \cap \mathcal{T}^\perp)$$

$$\mathcal{T}_\mathcal{U} := {}^\perp\mathcal{U} \longleftarrow\!\shortmid\ \mathcal{U}$$

THEOREM [Burke '94] [Popescu '73] If $\mathcal{G}$ is Seminoetherian, then:

(1) $\mathrm{GSimp}\,\mathcal{G} \longrightarrow \mathrm{GSpec}\,\mathcal{G}$ is a bijection.
$$S \longmapsto E(S)$$

(2) $E(S)$ is isolated $\iff$ S is simple.

(3) $\mathrm{Gdim}\,\mathcal{G} = \text{CB-rank}\left(\mathrm{GSpec}\,\mathcal{G}\right)$

(4) $\mathrm{GSpec}\,\mathcal{G}$ is T_0.

$\pi_\gamma : \mathcal{G} \longrightarrow \mathcal{G}/\bigcup_{\beta<\gamma} \mathcal{G}_\beta$

$\mathcal{G}_\gamma = \pi_\gamma^{-1}(\mathrm{Loc}(\text{simples}))$

$$0 = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_\alpha \subseteq \mathcal{G}_{\alpha+1} \subseteq \cdots \subseteq \mathcal{G} = \mathcal{G}_d$$

$\Big\downarrow$ Gsupp

$X = \mathrm{GSpec}\,\mathcal{G}$

$X_\gamma = \bigcup_{\beta<\gamma} X_\beta \cup \mathrm{isol}\left(X \setminus \bigcup_{\beta<\gamma} X_\beta\right)$

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\alpha \subseteq X_{\alpha+1} \subseteq \cdots \subseteq X = X_d$$

Example: $\mathrm{GSimp}(\mathcal{G}) = \{\mathbb{Z}/p\mathbb{Z} : p \text{ prime}, \mathbb{Q}\} \xrightarrow[1:1]{} \mathrm{GSpec}(\mathcal{G}) = \{\mathbb{Z}(p^\infty) : p \text{ prime}, \mathbb{Q}\}$

$\mathcal{G} = \mathrm{Mod}\,\mathbb{Z}$

$$0 = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 = \mathrm{Loc}(\mathbb{Z}/p\mathbb{Z} : p \text{ prime}) = \mathrm{Tors} \subseteq \mathrm{Mod}\,\mathbb{Z}$$

Gsupp

$$\emptyset = X_{-1} \subseteq \{\mathbb{Z}(p^\infty) : p \text{ prime}\} \subseteq X = \mathrm{GSpec}\,\mathcal{G}$$

§3. The Gabriel Spectrum of $\mathcal{H}_\Phi$

$(\mathcal{U},\mathcal{V}) = (\mathcal{U}_\Phi, \mathcal{V}_\Phi)$
intermediate compactly generated in $\mathcal{D}(R)$

$\Phi : \mathrm{Spec}(R) = \Phi(a) \supseteq \Phi(a+1) \supseteq \cdots \supseteq \Phi(b) = \emptyset$

$\mathcal{H}_\Phi$ Grothendieck heart
$\{ k(\mathfrak{p})[-f(\mathfrak{p})] : \mathfrak{p} \in \mathrm{Spec}\, R \} \subseteq \mathcal{H}_\Phi$

$$f_\Phi : \mathrm{Spec}(R) \longrightarrow \mathbb{Z}$$
$$\mathfrak{p} \longmapsto n : \mathfrak{p} \in \Phi(n) \setminus \Phi(n+1)$$

<u>THEOREM</u> [Hrbek - Pavon - V'24] R commutative noetherian, $\mathrm{kdim}\, R < \infty$

(1) $\mathcal{H}_\Phi$ is seminoetherian ($\Rightarrow \mathrm{GSpec}\, \mathcal{H}_\Phi$ is T_0)

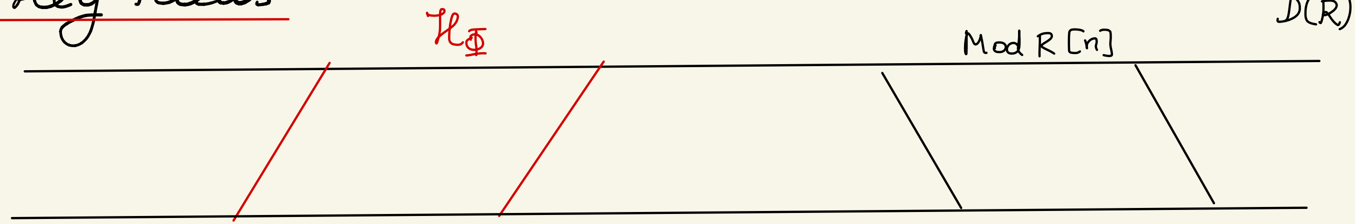
(2) $\mathrm{GSimp}\, \mathcal{H}_\Phi = \{ k(\mathfrak{p})[-f(\mathfrak{p})] : \mathfrak{p} \in \mathrm{Spec}(R) \}$

(3) $\sup_{n \in [a,b]} \{ \mathrm{kdim}\, \Phi(n) \setminus \Phi(n+1) \} \leq \mathrm{Gdim}\, \mathcal{H}_\Phi \leq \mathrm{Kdim}\, R$

(4) $\mathrm{GSpec}\, \mathcal{H}_\Phi$ is Alexandrov. (arbitrary intersection of open sets is open)

Four key ideas:

(I)



$\mu^-_{V_1}$, $\mu^-_{V_2}$, $\dots$, $\mu^-_{V_n}$

Right cosilting mutation

[Angeleri Hügel - Laking - Šťovíček - V'22]

[Pavon - V'21]

$$\mathrm{GSpec}\,\mathcal{H}_\Phi \xleftarrow{1:1} \xleftarrow{1:1} \cdots \xleftarrow{1:1} \mathrm{Spec}^H R \cong \mathrm{GSpec}(\mathrm{Mod}\,R[n])$$

(II)

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{hereditary torsion} \\ \text{classes in } \mathcal{H}_\Phi\end{array}\right\} \arrow[r, two heads, "\mathrm{Gsupp}", "1:1"'] \arrow[d, hook, "\mathrm{supp}"] & \mathrm{Op}(\mathrm{GSpec}\,\mathcal{H}_\Phi) \arrow[d, hook] \\
\mathcal{P}(\mathrm{Spec}\,R) \arrow[r, leftrightarrow, dashed, "1:1"] & \mathcal{P}(\mathrm{GSpec}\,\mathcal{H})
\end{tikzcd}$$

$\mathrm{Gspec}\,\mathcal{H}_\Phi \overset{1:1}{\longleftrightarrow} \mathrm{Spec}\,R$

$+$

$\overset{!}{\Longrightarrow}$

$$\mathrm{Op}(\mathrm{GSpec}\,\mathcal{H}_\Phi) = \left\{ \mathrm{supp}\,\mathcal{T} : \mathcal{T} \text{ hereditary torsion class in } \mathcal{H}_\Phi \right\}$$

$$T_0 \Longleftarrow \supseteq \mathrm{Op}(\mathrm{Spec}^H R)$$

III Seminoetherian

$\mathcal{T}$ hereditary torsion class $\quad P = \operatorname{supp}\mathcal{T} \subsetneq \operatorname{Spec}^H R$

$\Longrightarrow \mathcal{H}_\Phi/\mathcal{T}$ has a Simple: $\quad V = \mathring{P}$

$$\mathcal{H}_\Phi \longrightarrow \mathcal{H}_\Phi/\mathcal{T}_V \longrightarrow \mathcal{H}_\Phi/\mathcal{T}$$

$\operatorname{isol}(\operatorname{Spec}^H R \setminus V) \neq \emptyset$. $\quad \operatorname{isol}(\operatorname{Spec}^H R \setminus V) \cap P = \emptyset$

IV Alexandrov

$$\forall\, (\mathcal{T},\mathcal{F}) \text{ hereditary in } \mathcal{H}_\Phi \begin{cases} \mathfrak{p} \in \operatorname{supp}\mathcal{T} \iff k(\mathfrak{p})[-f(\mathfrak{p})] \in \mathcal{T} \\ \mathfrak{p} \notin \operatorname{supp}\mathcal{T} \Longrightarrow E(k(\mathfrak{p})[-f(\mathfrak{p})]) \in \mathcal{F} \\ \qquad\qquad\quad\;\; \Longrightarrow k(\mathfrak{p})[-f(\mathfrak{p})] \in \mathcal{F} \end{cases}$$

Every indecomposable injective has a hereditary-torsion-simple subobject!

[Pavon '24]