Approximations and Vopěnka's Principles

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Motivation: Salce's duality for approximations

Salce's Lemma (1979)

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in the category $\mathrm{Mod}\text{–}R$. Then the following conditions are equivalent:

(i) $\mathcal{A}$ is a special precovering class (*i.e., for each module M, there is a short exact sequence of the form $0 \to B \to A \to M \to 0$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$*);

(ii) $\mathcal{B}$ is a special preenveloping class (*i.e., for each module M', there is a short exact sequence of the form $0 \to M' \to B' \to A' \to 0$ for some $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$*).

Salce's proof even gives an easy way of constructing a special preenvelope from the dual notion of a special precover, and vice versa.

Set-theoretic tools: Vopěnka's Principles

The Weak Vopěnka's Principle (WVP)

There exists no proper class of graphs $(G_\alpha \mid \alpha \in \mathrm{Ord})$ such that for all ordinals α, β, $\mathrm{Hom}_{\mathcal{G}}(G_\alpha, G_\beta) \neq \emptyset$, iff $\alpha \geq \beta$.

Vopěnka's Principle (VP)

There exist no large rigid systems in the category $\mathcal{G}$ of all graphs. That is, there exists no proper class of graphs $\{G_\alpha \mid \alpha \in \mathrm{Ord}\}$ such that $\mathrm{Hom}_{\mathcal{G}}(G_\alpha, G_\beta) = \emptyset$ for all ordinals $\alpha \neq \beta$ and $\mathrm{Hom}_{\mathcal{G}}(G_\alpha, G_\alpha) = \{\mathrm{id}_{G_\alpha}\}$ for each ordinal α.

Adámek-Rosický'1994, Wilson'2020

(i) VP implies WVP, and WVP implies existence of arbitrary large measurable cardinals.

(ii) WVP does not imply VP. Indeed, if supercompact cardinals exist, then there is a model of ZFC where WVP holds, but VP fails.

Preenveloping classes

Let R be a ring and $\mathcal{C}$ be a class of modules closed under direct summands.

Consider the following two conditions:

(i) $\mathcal{C}$ is preenveloping.

(ii) $\mathcal{C}$ is closed under direct products.

Then (i) implies (ii).

[Saorín-Šťovíček'2011] $\mathcal{C}$ is deconstructible, then (ii) is equivalent to (i).

[Adámek-Rosický'1994] If Weak Vopěnka's Principle holds, then the equivalence holds for all $\mathcal{C}$.

Precovering classes

Consider the following two conditions:

(i) $\mathcal{C}$ is precovering.

(ii) $\mathcal{C}$ is closed under direct sums.

Then (i) implies (ii).

[Saorín-Šťovíček'2011] If $\mathcal{C}$ is deconstructible, then (ii) is equivalent to (i).

However, there exist non-deconstructible classes of modules closed under direct sums that are not precovering.

Example

Let R be any non-right perfect ring. Then the class of all $\aleph_1$-projective (= flat Mittag-Leffler) modules is closed under transfinite extensions, but it is not precovering.

Adding more closure properties

Consider the following conditions for a class $\mathcal{C} \subseteq \mathrm{Mod}\text{–}R$.

(i) $\mathcal{C}$ is (pre-) enveloping and closed under submodules.

(ii) $\mathcal{C}$ is closed under submodules and direct products.

(iii) $\mathcal{C} = \mathrm{Cog}\,(M)$ for a module M.

Then (i) is equivalent to (ii), and it is implied by (iii).

However, there exist classes of modules that satisfy (ii), but not (iii).

Example

Let R be a Dedekind domain with a countable spectrum which is not a complete DVD. Then the class of all $\aleph_1$-projective (= flat Mittag-Leffler) modules is closed under submodules and direct products, but it is not of the form $\mathrm{Cog}\,(M)$ for any module M.
Key point of the proof: For each non-zero $\aleph_1$-projective module M there exists a continuous strictly increasing chain of $\aleph_1$-projective modules $(M_\alpha \mid \alpha \in \mathrm{Ord})$ such that $\mathrm{Hom}_R(M_\alpha, M_\beta) = 0$ for each $\beta < \alpha$.

The dual setting

Consider the following conditions for a class $\mathcal{C} \subseteq \mathrm{Mod}\text{–}R$.

(i) $\mathcal{C}$ is (pre-) covering and closed under homomorphic images.

(ii) $\mathcal{C}$ is closed under homomorphic images and direct sums.

(iii) $\mathcal{C} = \mathrm{Gen}\,(M)$ for a module M.

Then (i) is equivalent to (ii), and it is implied by (iii).

If Vopěnka's Principle holds, then (ii) implies (iii).

The necessity of large cardinals

Theorem

Assume that each class $\mathcal{C}$ of $\aleph_1$-projective groups which is closed under homomorphic images and direct sums is of the form $\mathcal{C} = \mathrm{Gen}\,(M)$ for a module M. Then Weak Vopěnka's Principle holds true.

To start the proof, we need to move from graphs to abelian groups:

Przeździecki'2014, Göbel-Przeździecki'2014

(i) There exists a functor G from the category $\mathcal{G}$ of all graphs to $\mathrm{Mod}\text{–}\mathbb{Z}$ which induces for all $X, Y \in \mathcal{G}$ a group isomorphism $\mathbb{Z}^{(\mathrm{Hom}_{\mathcal{G}}(X,Y))} \cong \mathrm{Hom}_{\mathbb{Z}}(G(X), G(Y))$ natural in both variables.

(ii) The functor G can be constructed so that it takes values in the class of all $\aleph_1$-projective groups.

Proof

Assume WVP fails. Then there exists a proper class of graphs $(X_\alpha \mid \alpha \in \mathrm{Ord})$ such that for all ordinals α, β, $\mathrm{Hom}_{\mathcal{G}}(X_\alpha, X_\beta) \neq \emptyset$, iff $\alpha \geq \beta$.

Let $\mathcal{C}$ be the subclass of $\mathrm{Mod}\text{–}\mathbb{Z}$ generated by the groups $G(X_\alpha)$ ($\alpha \in \mathrm{Ord}$). W.l.o.g. $G(X_\alpha)$ is $\aleph_1$-projective for each $\alpha \in \mathrm{Ord}$. Since $\mathcal{C}$ is closed under direct sums and homomorphic images, $\mathcal{C}$ is a covering class. We will show that there is no abelian group $M \in \mathcal{C}$ such that $\mathcal{C} = \mathrm{Gen}\,(M)$.

If not, let α be the least ordinal such that M is generated by the groups $G(X_\beta)$ ($\beta < \alpha$). Then M is a homomorphic image of a direct sum of copies of these groups. Since $G(X_\alpha) \in \mathrm{Gen}\,(M)$, $G(X_\alpha)$ a homomorphic image of a direct sum of copies of M. Thus, there is a non-zero homomorphism from $G(X_\beta)$ to $G(X_\alpha)$ for some $\beta < \alpha$. Then $\mathrm{Hom}_{\mathcal{G}}(X_\beta, X_\alpha) \neq \emptyset$, a contradiction. □

Adding further closure properties

The case of envelopes

The following conditions are equivalent for a class $\mathcal{C} \subseteq \mathrm{Mod}\text{–}R$:

(i) $\mathcal{C}$ is (pre-) enveloping and closed under submodules and homomorphic images.

(iii) $\mathcal{C} = \mathrm{Mod}\text{–}(R/I)$ for a two-sided ideal I in R.

The case of covers

The following conditions are equivalent for a class $\mathcal{C} \subseteq \mathrm{Mod}\text{–}R$:

(i) $\mathcal{C}$ is (pre-) covering and closed under submodules and homomorphic images.

(ii) $\mathcal{C} = \sigma[M]$ for a module M.

Some of the asymmetries

(i) The conditions in the envelope case above imply those in the cover case (in ZFC), but not otherwise.

(ii) **WVP** implies the existence of a proper class of $\aleph_1$-projective groups $(A_\alpha \mid \alpha \in \mathrm{Ord})$ such that for all ordinals α, β, $\mathrm{Hom}_{\mathbb{Z}}(A_\alpha, A_\beta) \neq \emptyset$, iff $\alpha \geq \beta$.

However, the existence of a proper class of $\aleph_1$-projective groups $(A_\alpha \mid \alpha \in \mathrm{Ord})$ such that for all ordinals α, β, $\mathrm{Hom}_{\mathbb{Z}}(A_\alpha, A_\beta) \neq \emptyset$, iff $\alpha \leq \beta$ **is provable in ZFC**.

References I

M. Cortés-Izurdiaga, *Tor-pairs: Products and approximations*, Contemp. Math. 751(2020), 119 – 133.

L. Salce, *Cotorsion theories for abelian groups*, Symposia Math. 23(1979), 11 – 32.

M. Saorín, J. Šťovíček, *On exact categories and applications to triangulated adjoints and model structures*, Adv. Math. 228 (2011), 968 – 1007.

References II

J. Adámek, J. Rosický, *Locally Presentable and Accessible Categories*, LMSLNS 189, Cambridge Univ. Press, Cambridge 1994.

A. Ben Yassine, J. Trlifaj, Dualizations of approximations, $\aleph_1$-projectivity, and Vopěnka's Principles, arXiv:2401.11979v1.

R. Göbel, A. J. Przeździecki, *An axiomatic construction of an almost full embedding of the category of graphs into the category of R-objects*, J. Pure Appl. Algebra 218(2014), 208 – 217.

A. J. Przeździecki, *An almost full embedding of the category of graphs into the category of abelian groups*, Adv. Math. 257(2014), 527 – 545.

T.M. Wilson, *Weak Vopěnka's Principle does not imply Vopěnka's Principle*, Adv. Math. 363(2020), 106986.