Abstract representation theory via coherent Auslander-Reiten diagrams

Purity, Approximation Theory and Spectra in Cetraro

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Contents

Abstract representation theory

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  - ∗ obstructions: $\mathrm{D}(\mathcal{A}^Q) \not\simeq \mathrm{D}(\mathcal{A})^Q$ (*coherent vs. incoherent diagrams*)

    non-functoriality of the cone...

Representations over ?

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let's use stable ∞-categories!

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- any of your favorite stable homotopy theories...

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We write

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Q finite acyclic and R any ring

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Theorem (Abstract reflection functors)

For v a source of Q (finite), and $\mathcal{C}$ stable, there is an equivalence

$$S^- : \mathcal{C}^Q \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}} \mathcal{C}^{\sigma_v Q} : S^+ \qquad \text{(DJW'21, RŠ'18)}$$

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Coherent Auslander-Reiten diagrams

Auslander-Reiten quivers

- Auslander-Reiten quiver: one of the central objects in the representation theory of f.d. algebras

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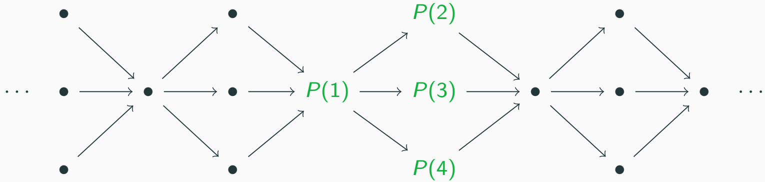

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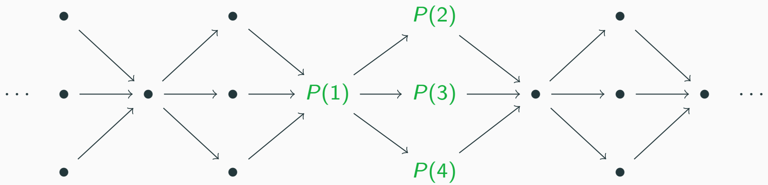

- Happel:

$$\Gamma_Q := \Gamma(\mathrm{D}^b(kQ)) \cong \mathbb{Z}Q \qquad \text{for } Q \text{ Dynkin}$$

$$\cong (\mathbb{Z} \times \mathbb{Z}Q) \sqcup \{\text{regular comps}\} \qquad \text{otherwise}$$

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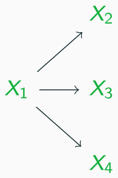

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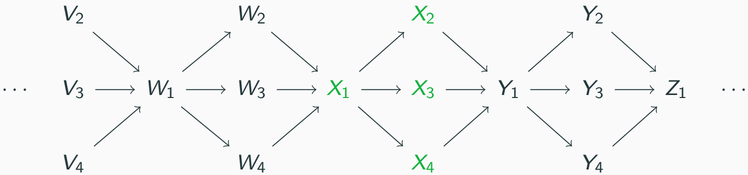

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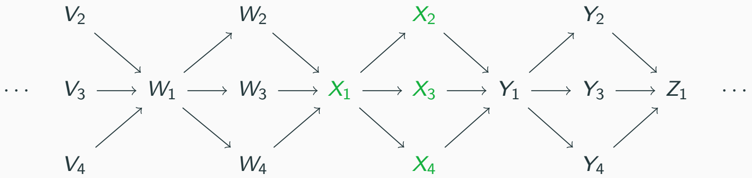

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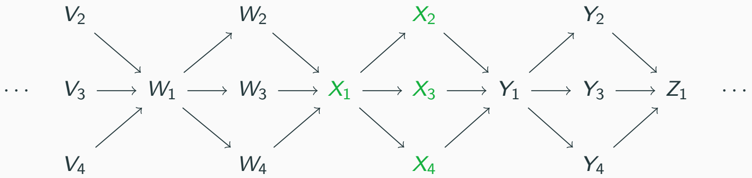

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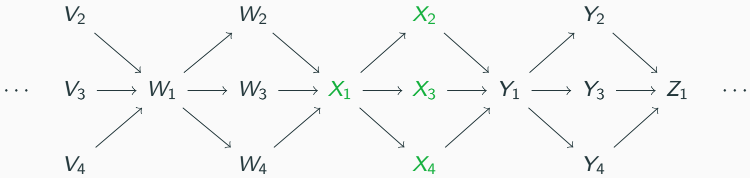

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$$(\tau X)_i = \mathbb{R}\underline{\mathrm{Hom}}(P(i), \tau X) = \mathbb{R}\underline{\mathrm{Hom}}(\tau^{-1} P(i), X) = \widetilde{X}(\tau^{-1} P(i))$$

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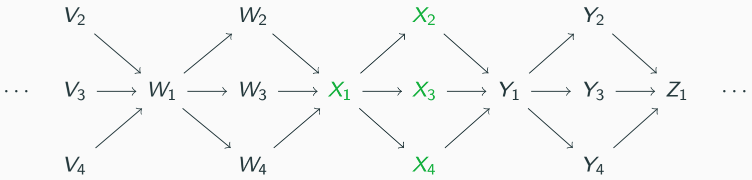

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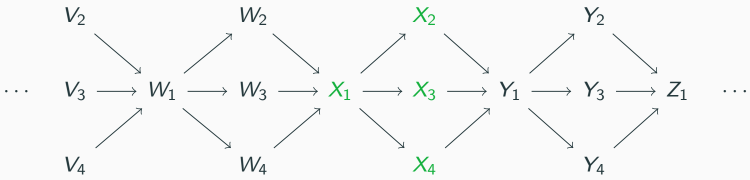

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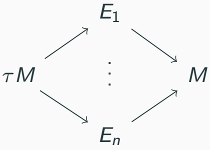

$$\longmapsto \qquad \widetilde{X}(M) \to \bigoplus \widetilde{X}(E_i) \to \widetilde{X}(\tau M)$$

Mesh representations of the repetitive quiver

- In the abstract setting ($\mathcal{C}$ stable), we are able to define a functor:

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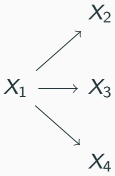

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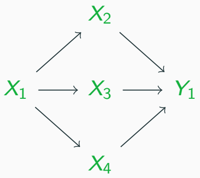

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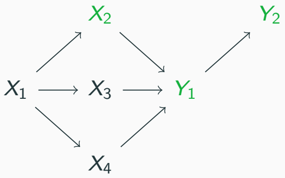

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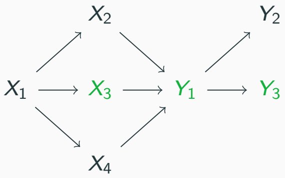

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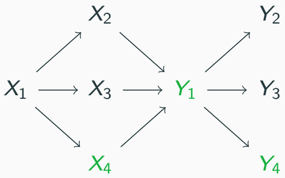

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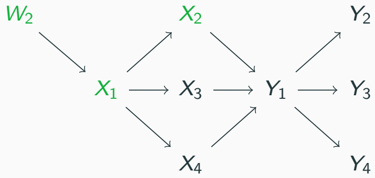

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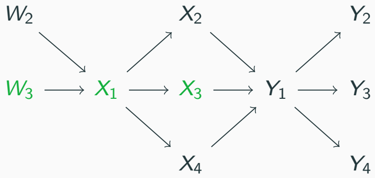

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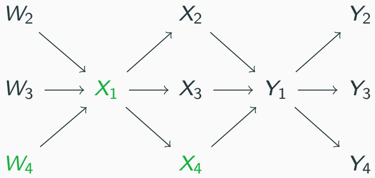

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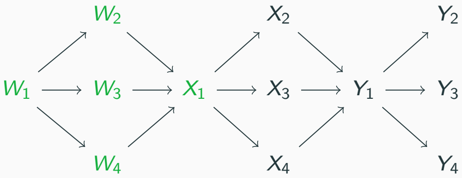

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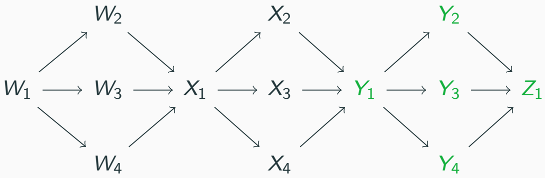

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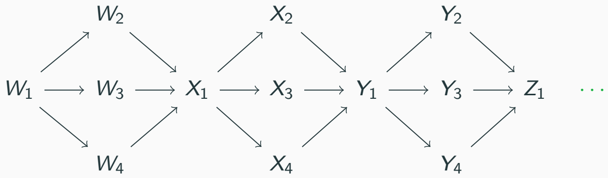

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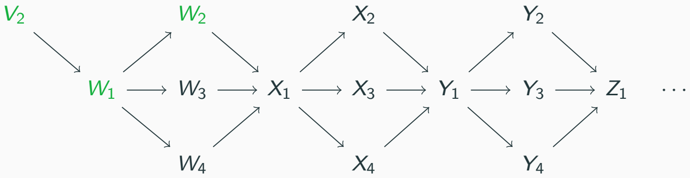

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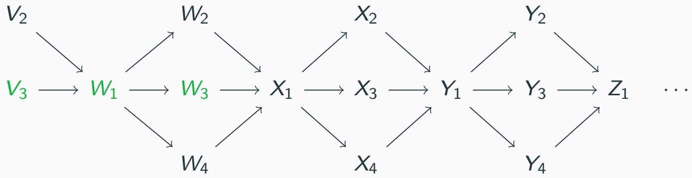

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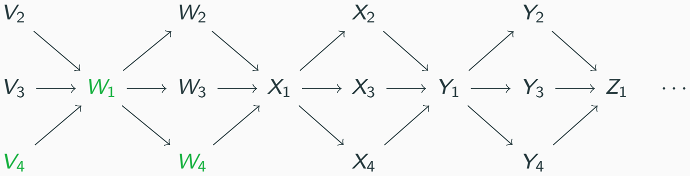

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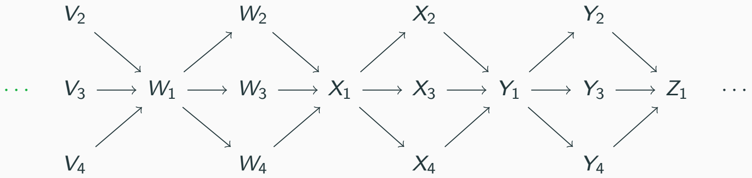

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Theorem (S.)

Restriction along $Q \subset \mathbb{Z}Q$ induces an equivalence of ∞-categories

$$\mathcal{C}^{\mathbb{Z}Q,\,\mathrm{mesh}} \simeq \mathcal{C}^{Q}$$

Applications

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- a functor $\mathrm{BAut}(\mathbb{Z}Q) \to \mathrm{CAT}_{\infty}$ sending $* \mapsto \mathcal{C}^{Q}$ and $\sigma \mapsto \sigma^{-*}$

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* $\Gamma_Q^{\mathrm{irr}} \cong \mathbb{Z}Q$ for Q Dynkin and $\Gamma_Q^{\mathrm{irr}} \cong \mathbb{Z} \times \mathbb{Z}Q$ otherwise
* the second case uses a variation of the main Thm:

$$\begin{array}{ccccc} \mathcal{C}^{Q} & \longrightarrow & \mathcal{C}^{\mathbb{Z}Q} & \longrightarrow & \mathcal{C}^{\mathbb{Z}\times\mathbb{Z}Q} \\ X & \longmapsto & \widetilde{X} & \longmapsto & (\Sigma^n \widetilde{X})_n \end{array}$$

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In general, we have produced a group homomorphism

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- for important examples of $\mathcal{C} = \mathcal{D}^b(\mathbb{Z}), \mathrm{Sp}^{\mathrm{fin}}, \ldots$ we expect to get many interesting elements of the integral and spectral Picard group, i.e. meaninful versions of τ, nakayama, suspension, etc.

Thank you for your attention!