A representability theorem for triangulated categories

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Representability problem

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- R is a commutative base ring.
- $\mathcal{D}$ is an additive R-linear category; it has finite biproducts.
- $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ is a contravariant functor.

Problem

Characterize H such that $H \cong \mathcal{D}(-, D)$ for some $D \in \mathcal{D}$. More generally, characterize H such that it is isomorphic to a direct summand of $\mathcal{D}(-, D)$, for some $D \in \mathcal{D}$.

We call $\mathcal{D}$ *karoubian*, if every idempotent endomorphism in $\mathcal{D}$ splits.

Finitely generated and finitely presented functors

The functor H is called

- *finitely generated* if there is a natural epimorphism $\mathcal{D}(-, D) \to H$, for some $D \in \mathcal{D}$;
- *finitely presented* if there is an exact sequence $\mathcal{D}(-, C) \to \mathcal{D}(-, D) \to H \to 0$.

Denote by $\mathrm{Hom}_{\mathcal{D}}(H, H')$ the class of all natural transformations between two functors $H, H' : \mathcal{D} \to \mathrm{Mod}\text{-}R$. Construct the category $\mathrm{mod}(\mathcal{D})$ with:

- objects: finitely presented functors $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$.
- morphisms: $\mathrm{Hom}_{\mathcal{D}}(H, H')$ for $H, H' \in \mathrm{mod}(\mathcal{D})$.

Weak kernels and properties of $\mathrm{mod}(\mathcal{D})$

From now we want the category $\mathrm{mod}(\mathcal{D})$ to be abelian.
Facts:

- [Freyd'66]: Equivalently, $\mathcal{D}$ has *weak kernels*, that is given $Y \to Z$ in $\mathcal{D}$, there is $X \to Y$ such that the sequence of functors
$$\mathcal{D}(-, X) \to \mathcal{D}(-, Y) \to \mathcal{D}(-, Z)$$
is exact; in this case the composite morphism $X \to Y \to Z$ vanishes, and we call it a *weak kernel sequence*.
- Triangles are examples of weak kernel sequences, therefore $\mathrm{mod}(\mathcal{D})$ is abelian, provided that $\mathcal{D}$ is triangulated.
- Projective objects in the abelian category $\mathrm{mod}(\mathcal{D})$ are direct summands of representable functors.

Universal property of $\mathrm{mod}(\mathcal{D})$

Proposition (Freyd '66, Krause '98)

If $\mathcal{A}$ is an abelian category, and $F : \mathcal{D} \to \mathcal{A}$ is a covariant functor, then there is a unique, up to a natural isomorphism, right exact (covariant) functor $\widehat{F} : \mathrm{mod}(\mathcal{D}) \to \mathcal{A}$ such that $\widehat{F}\mathcal{D}(-, X) = F(X)$ for all $X \in \mathcal{D}$.

Corollary (Krause 98)

The functor $\widehat{F}$ is exact if and only if F is weak exact, *that is it sends a (equivalently all) weak kernel sequence(s) $X \to Y \to Z$ to exact sequence(s) $F(X) \to F(Y) \to F(Z)$ in $\mathcal{A}$.*

An abstract representability result

Proposition

Let $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ a contravariant functor. Then there is a natural isomorphism $\widehat{H} \cong \mathrm{Hom}_{\mathcal{D}}(-, H)$. In particular, H is finitely presented if and only if $\widehat{H}$ is representable.

An abelian category is said to be *Frobenius*, provided that projective and injective objects coincide.

Corollary

Suppose $\mathrm{mod}(\mathcal{D})$ is Frobenius, and let the contravariant functor $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ be weak exact. Then H is isomorphic to direct summand of a representable functor if and only if H is finitely presented. If, in addition, $\mathcal{D}$ is karoubian, then H is representable if and only if H is finitely presented.

Coreflective subcategories of triangulated categories

Corollary (Neeman, '10, Casacuberta-Gutiererrez-Rosicky, '14)

Let $\mathcal{D}$ be a karoubian triangulated category and let $\mathcal{S} \subseteq \mathcal{D}$ be a traingulated subcategory which is closed under direct summands. Then $\mathcal{S}$ is precovering if and only if the inclusion functor $\mathcal{S} \to \mathcal{D}$ has a right adjoint (i. e. $\mathcal{S}$ is coreflective).

Recall that an $\mathcal{S}$-*precover* of $D \in \mathcal{D}$ is a morphism $S \to D$ such that the induced map $\mathcal{D}(X, S) \to \mathcal{D}(X, D)$ is surjective for all $X \in \mathcal{S}$. Further $\mathcal{S}$ is called *precovering*, provided that every $D \in \mathcal{D}$ has an $\mathcal{S}$-precover.

Iterated extensions

From now on $\mathcal{D}$ is triangulated.
Consider two subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{D}$ We define:

$$\mathcal{A}[\mathbb{Z}] = \{A[n] \mid A \in \mathcal{A}, n \in \mathbb{Z}\}$$

$$\mathcal{A} * \mathcal{B} = \{Y \in \mathcal{D} \mid \text{there is a triangle } X \to Y \to Z \rightsquigarrow \text{ with } X \in \mathcal{A}, Z \in B\}.$$

and inductively

- $\mathcal{A}^{*1} = \operatorname{smd} \mathcal{A}$ to be the closure of $\mathcal{A}$ under direct summands;
- $\mathcal{A}^{*(n+1)} = \operatorname{smd} (\mathcal{A}^{*n} * \mathcal{A}^{1})$, for $n \geq 1$.

Note that $0 \in \mathcal{A}^{*1}$, therefore $\mathcal{A}^{*1} \subseteq \mathcal{A}^{*2} \subseteq \dots$.

Relatively finitely generated functors

Keep $\mathcal{A} \subseteq \mathcal{D}$ and let $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ contravariant. We say that H is $\mathcal{A}$-*finitely generated* if there is $A \in \mathcal{A}$ and a natural transformation $\alpha : \mathcal{D}(-, A) \to H$ such that

$$\alpha^X : \mathcal{D}(X, A) \to H(X) \to 0$$

is exact for all $X \in \mathcal{A}$.

Remark

H is finitely generated exactly if it is $\mathcal{D}$-finitely generated and the two definitions are consistent, in the sense that H is $\mathcal{A}$-finitely generated if and only if its restriction $H|_{\mathcal{A}}$ is finitely generated.

The key lemma

Lemma

Let $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ be a cohomological functor, and let $\mathcal{A} \subseteq \mathcal{D}$ be a (full) subcategory such that $\mathcal{A}[1] = \mathcal{A}$. Suppose that H is $\mathcal{A}$-finitely generated and, for any $D \in \mathcal{D}$, the kernel of every natural transformation $\mathcal{D}(-, D) \to H$ is $\mathcal{A}$-finitely generated too. Then H is $\mathcal{A}^{*n}$-finitely generated for all $n \geq 1$.

Representability for triangulated categories

Theorem

Let $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ *be a cohomological functor, and let* $\mathcal{A} \subseteq \mathcal{D}$ *be a (full) subcategory such that* $\mathcal{A}[1] = \mathcal{A}$, $\mathcal{A}$ *is precovering and* $\mathcal{A}^{*n} = \mathcal{D}$ *for some* $n \geq 1$. *Suppose that* H *is* $\mathcal{A}$*-finitely generated and, for any* $D \in \mathcal{D}$, *the kernel of every natural transformation* $\mathcal{D}(-, D) \to H$ *is* $\mathcal{A}$*-finitely generated too. Then* H *is isomorphic to a direct summand of a representable functor. In particular, if* $\mathcal{D}$ *is karoubian, then* H *is representable.*

The case of "small" triangulated categories

The object G is called *strong generator* for $\mathcal{D}$ if $\mathcal{D} = G[\mathbb{Z}]^{*n}$ for some $n \geq 1$. The functor $H \to \mathrm{Mod}\text{-}R$ is called G-finite if the R-module $\bigoplus_{n\in\mathbb{Z}} H(G[n])$ is finitely generated (in particular, this implies $H(G[n]) = 0$ for $|n| \gg 0$). Taking $\mathcal{A} = \mathrm{add}\, G$ we get a new proof for:

Corollary (Rouquier, '08, Neeman, 18)

Assume R is noetherian and let G be a strong generator for $\mathcal{D}$, such that the functor $\mathcal{D}(-, D)$ is G-finite, for every $D \in \mathcal{D}$. Then a contravariant functor $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ is isomorphic to a direct summand of a representable functor if and only if it is cohomological and G-finite. If further $\mathcal{D}$ is karoubian, then H is representable if and only if it is cohomological and G-finite.

The case of "big" triangulated categories

Corollary (compare with Rouquier '08)

Assume $\mathcal{D}$ has coproducts and $H : \mathcal{D} \to \mathrm{Mod}\text{-}R$ is a contravariant functor. Suppose that there is a set $\mathcal{G}$ of objects in $\mathcal{D}$ such that $\mathcal{D} = \mathcal{A}^{*n}$, where $\mathcal{A} = \mathrm{Add}\mathcal{G}[\mathbb{Z}]$. Then H is representable if and only if H is cohomological and sends coproducts into products.

Remark

Classical Brown representability for compactly (well) generated triangulated categories follows. The main advantage of the prezent approach is that it can be directly dualized in order to find a criterion for Brown representability for the dual.