Salce's Lemma II

IVO HERZOG
The Ohio State University

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Cetraro, ITALY

Outline

Exact Categories

Definition: Exact Category

An exact category $(\mathcal{A}; \mathcal{E})$ consists of an additive category $\mathcal{A}$ together with a distinguished class $\mathcal{E}$ of kernel-cokernel pairs, denoted

$$A \rightarrowtail B \twoheadrightarrow C,$$

closed under isomorphism and satisfying:

1. every identity morphism is an inflation and a deflation;
2. inflations, resp., deflations, are closed under composition; and
3. $\mathcal{E}$ is closed under pullbacks and pushouts.

Every additive category $\mathcal{A}$ may be equipped with the trivial exact structure $(\mathcal{A}; \mathcal{E}_0)$ whose conflations are the trivial (split) kernel-cokernel pairs.

The Exact Category of Arrows

If $(\mathcal{A}; \mathcal{E})$ is an exact category, then the category $\mathrm{Arr}(\mathcal{A})$ of arrows may be endowed with an exact structure $\mathrm{Arr}(\mathcal{E})$ so that the conflations are of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

Every such conflation of arrows admits a pushout-pullback factorization:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h_1} & & \Vert & & \\
0 & \longrightarrow & Y & \longrightarrow & W & \longrightarrow & A & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

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Mono-Epi Conflations

Definition

A conflation of arrows is mono-epi (ME) if it admits a factorization

$$\begin{CD}
0 @>>> B @>>> C @>>> A @>>> 0\\
@. @| @VV{h^1}V @VV{f}V @.\\
0 @>>> B @>>> V @>>> X @>>> 0\\
@. @VV{g}V @VV{h^2}V @| @.\\
0 @>>> Y @>>> Z @>>> X @>>> 0
\end{CD}$$

This is denoted by $h = h^2 h^1 = g \star f$.

Theorem (H, Fu)

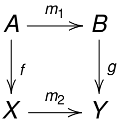
The functor $1 \colon (\mathcal{A}; \mathcal{E}) \to (\mathrm{Arr}(\mathcal{A}); \mathrm{Arr}(\mathcal{E}))$, $A \mapsto 1_A$, is exact and

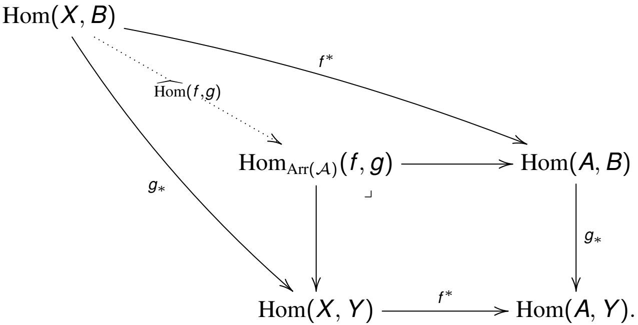
$$(\mathcal{A}; \mathcal{E}) \to (\mathrm{Arr}(\mathcal{A}); \mathrm{ME}) \subseteq (\mathrm{Arr}(\mathcal{A}); \mathrm{Arr}(\mathcal{E}))$$

is the smallest exact substructure containing $(\mathcal{A}; \mathcal{E})$.

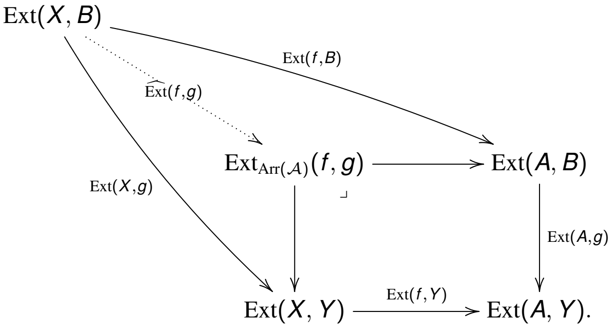
Leibniz Hom (Riehl and Verity)

A morphism $f \to g$ of arrows in $\mathrm{Arr}(\mathcal{A})$ is a pair (m_1, m_2) of horizontal maps





Leibniz Ext


Ext(X, B)
Ext(f,B)
$\widehat{\mathrm{Ext}}(f,g)$
$\mathrm{Ext}_{\mathrm{Arr}(\mathcal{A})}(f, g)$
Ext(A, B)
Ext(X,g)
Ext(A,g)
Ext(f,Y)
Ext(X, Y)
Ext(A, Y).

Outline

Products of Special Precovering Ideals

The Ideal Christensen's Lemma (Fu, H)

It $(\mathcal{I}_1, \mathcal{J}_1)$ and $(\mathcal{I}_2, \mathcal{J}_2)$ are complete ideal cotorsion pairs, then so is

$$(\mathcal{I}_1\mathcal{I}_2, \mathcal{J}_2 \diamond \mathcal{J}_1),$$

where $(\mathcal{I}_1\mathcal{I}_2)^\perp = \mathcal{J}_2 \diamond \mathcal{J}_1$, the ideal of arrows $j_2 \star j_1$ with $j_1 \in \mathcal{J}_1$ and $j_2 \in \mathcal{J}_2$.

Example

$(\Phi^2, \langle \mathrm{R\text{-}PInj} \star \mathrm{R\text{-}PInj} \star R\text{-}\mathrm{Inj}\rangle)$ is a complete ideal cotorsion pair.

Finite Powers of the Phantom Ideal

Recall $\Phi \supseteq \Phi^2 \supseteq \Phi^3 \supseteq \cdots \supseteq \langle R\text{-Flat}\rangle$. Then $(\Phi^n)^\perp$ is the object ideal generated by modules M with a filtration

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots M_n \subseteq M_{n+1},$$

where M_{n+1}/M_n is injective and the other factor M_{i+1}/M_i pure injective.

Definition

A ring R is *semiprimary* if $R/J(R)$ is semisimple and $(J(R))^n = 0$ for some $n > 0$. In that case, $\langle R\text{-Flat}\rangle = \langle R\text{-Proj}\rangle$.

Theorem (Fu, H)

If R is a semiprimary ring with $(J(R))^n = 0$, then $\Phi^n = \langle R\text{-Proj}\rangle$.

The Benson-Gnacadja Conjecture

Corollary

If kG is a finite group ring over a field k, then $\underline{\Phi}^{\ell} = 0$ in kG-$\underline{\mathrm{Mod}}$, where ℓ is the Loewy length of kG.

Proof (with Benson):

It suffices to show that a phantom $\varphi \colon A \to B$ vanishes on $\mathrm{soc}(A)$. We may assume that B has no projective-injective summands. Let $S \subseteq A$ be a simple submodule and consider the restriction

If $f \neq 0$, we can take $P = E(S)$, contradicting the assumption on B.

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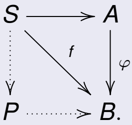

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Outline

Ideal Cotorsion Pair Generated by Pure Projectives

The cotorsion pair generated by pure projective modules is complete, given by $(\mathrm{FP\text{-}Proj}, \mathrm{FP\text{-}Inj})$, where $\mathrm{FP\text{-}Inj} = \mathrm{R\text{-}PProj}^{\perp}$.

Theorem (Enochs, Šťovíček)

$\mathrm{FP\text{-}Proj} = \mathrm{Filt}_{\omega}(\mathrm{PProj})$.

Let $\Psi = \langle \mathrm{R\text{-}mod}\rangle^{\perp} = \langle \mathrm{R\text{-}PProj}\rangle^{\perp}$ be the ideal of FP-ghost or cophantom morphisms. The ideal cotorsion pair $(\langle R\text{-}\mathrm{Proj} \star \mathrm{R\text{-}PProj}\rangle, \Psi)$ generated by the pure projective modules is complete.

The Cophantom Filtration

$$\Psi \supseteq \Psi^{(2)} \supseteq \cdots \supseteq \Psi^{(n)} \supseteq \cdots \supseteq \Psi^{(\alpha)} \supseteq \Psi^{(\alpha+1)} \supseteq \cdots \supseteq \langle \mathrm{FP\text{-}Proj}\rangle,$$

where $\Psi^{\alpha+1} = \Psi\Psi^{(\alpha)}$.

Convergence Theorem (Estrada, Fu, H, Odabaşi)

$$\Psi^{(\omega)} = \langle \mathrm{FP\text{-}Proj}\rangle.$$

Ghosts

Definition: Ghost Ideal

If $\mathcal{S} \subseteq \mathrm{R\text{-}Mod}$, $R \in \mathcal{S}$, let $\mathfrak{g} = \mathfrak{g}(\mathcal{S}) := \langle \mathcal{S} \rangle^{\perp}$.

The ideal cotorsion pair $({}^{\perp}\mathfrak{g}, \mathfrak{g})$ is complete.

Definition: Inductive Powers of an Ideal

Let $\mathcal{J}$ be an ideal and α be an ordinal. The ideal $\mathcal{J}^{(\alpha)}$ is generated by α-compositions of morphisms in $\mathcal{J}$, that is, structural morphism $a^0_{\alpha+1}$ from a continuous $(\alpha+1)$-system A $\in (\alpha+1, \mathcal{A})$ with successive structural morphisms $a^{\gamma}_{\gamma+1}$ in $\mathcal{J}$.

The Ideal Eklof Lemma (Estrada, Fu, H, Odabaşi)

For every ordinal α, the ideal $\mathfrak{g}^{(\alpha)}$ is special preenveloping with

$$^{\perp}(\mathfrak{g}^{(\alpha)}) = \langle R\text{-Proj} \star \mathrm{Filt}_{\alpha}(\mathrm{Add}(\mathcal{S})) \rangle$$

The Dual Xu Theorem

Theorem (EFHO)

If R-PProj is closed under extension, then $\text{R-PProj} = \text{FP-Proj}$.

Proof:

By the Ideal Christensen Lemma, $\Psi^2 = \Psi$, so the filtration looks like

$$\Psi = \Psi^2 = \cdots = \Psi^n = \cdots \supseteq \Psi^{(\omega)} = \langle \text{FP-Inj} \rangle,$$

by the Enochs-Šťovíček bound and the Ideal Eklof Lemma. To get that $\Psi = \Psi^{(\omega)}$, apply the property of Ψ that if $A \in (\omega + 1, \text{R-Mod})$ is continuous, then

$$\Psi(\lim_{\rightarrow} A, Y) \cong \Psi(A, c(Y)).$$

Outline

The Trivial Exact Structure

The distinguished kernel-cokernel pairs in the trivial exact structure $(\mathcal{A};\mathcal{E}_0)$ are the split ones. A splitting is a self homotopy,

$$\begin{array}{ccccc} X_1 & \xrightarrow{i} & X_0 & \xrightarrow{p} & X_{-1} \\ \| & \swarrow^{r} & \| & \swarrow^{s} & \| \\ X_1 & \xrightarrow{i} & X_0 & \xrightarrow{p} & X_{-1}, \end{array}$$

where $ri = 1_{X_1}$, $ps = 1_{X_1}$, and $1_{X_0} = ir + sp$.

$$\begin{array}{ccccc} X_{-1} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & X_{-1}\oplus X_1 & \xrightarrow{[0,1]} & X_1 \\ \| & & \downarrow{\scriptstyle [s,i]} & & \| \\ X_{-1} & \xrightarrow{s} & X_0 & \xrightarrow{r} & X_1, \end{array}$$

The Connecting Map

Consider a morphism of trivial conflations

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_X} & X_0 & \xrightarrow{p_X} & X_{-1} \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} \\ Y_1 & \xrightarrow{i_Y} & Y_0 & \xrightarrow{p_Y} & Y_{-1}, \end{array}$$

equipped with respective splittings (r_X, s_X) and (r_Y, s_Y). There is a unique $\Delta = \Delta(f)$,

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_X} & X_0 & \xrightarrow{p_X} & X_{-1} \\ & \swarrow^{\Delta} & & & \\ Y_1 & \xrightarrow[i_Y]{} & Y_0 & \xrightarrow{p_Y} & Y_{-1}, \end{array}$$

such that $D_2(\Delta) = (-\Delta p_X, i_Y \Delta) = (r_Y f_0 - f_1 r_X, s_Y f_{-1} - f_0 s_X)$.

Covariant Adjustment

Consider a morphism of trivial conflations with an ME factorization

$$\begin{array}{ccccc}
X_0 & \xrightarrow{i_0} & Y_0 & \xrightarrow{p_0} & Z_0 \\
\| & & \downarrow{\scriptstyle y^1} & & \downarrow{\scriptstyle z} \\
X_0 & \xrightarrow{i} & Y & \xrightarrow{p} & Z_1 \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y^2} & & \| \\
X_1 & \xrightarrow{i_1} & Y_1 & \xrightarrow{p_1} & Z_1
\end{array}$$

equipped with respective splittings $(r_0.s_0)$, (r, s), and (r_1, s_1). We have

- $\Delta_0 : Z_0 \to X_0$ satisfying $-\Delta_0 p_0 = ry' - r_0$ and $i\Delta_0 = sz - y's_0$; and
- $\Delta_1 : Z_1 \to X_1$ satisfying $-\Delta_1 p = r_1 y^2 - xr$ and $i_1\Delta_1 = s_1 - y^2 s$.

If $\mathrm{Ext}(\text{Coker } z, x) = 0$, then $x\Delta_0 = \Delta z$, and

$$(r_0, s_0) \text{ and } (r_1 + (\Delta_1 + \Delta)p_1), s_1 - i_1(\Delta_1 + \Delta))$$

yield a splitting for $x \to y \to z$.

The Limit Induction Step

Inductive Step at a Limit Ordinal

Let $\lambda = \omega\alpha$ be a limit ordinal and

$$\Sigma\colon \mathsf{A} \longrightarrow \mathsf{B} \longrightarrow \mathsf{C}$$

and conflation if $(\lambda + 1)$-systems, $\Sigma \in (\lambda + 1, \mathcal{E})$. If A and C are continuous, and $\Sigma|_\lambda$ is trivial, then Σ is trivial and continuous.

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