Salce's Lemma I

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Outline

1 Salce's Lemma

2 The Flat Cover Conjecture

3 Phantom Morphisms

4 Ideal Cotorsion Pairs

5 Convergence of the Phantom Filtration

Cotorsion Pairs

Definition

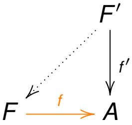
A pair $(\mathcal{F}, \mathcal{C})$ of subcategories of an exact category $(\mathcal{A}; \mathcal{E})$ is a cotorsion pair if

$$\mathcal{C} = \mathcal{F}^{\perp} = \{ C \in \mathcal{A} :\ (\forall F \in \mathcal{F})\ \mathrm{Ext}(F, C) = 0 \}$$

and

$$\mathcal{F} = {}^{\perp}\mathcal{C} = \{ F \in \mathcal{A} :\ (\forall C \in \mathcal{C})\ \mathrm{Ext}(F, C) = 0 \}.$$

For $A \in \mathcal{A}$, an $\mathcal{F}$-precover is a morphism $f\colon F \to A$ such that for every $F' \in \mathcal{F}$



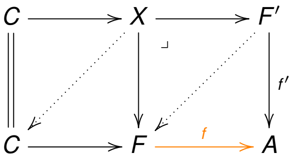
A $\mathcal{C}$-preenvelope of A is defined dually.

Special Approximations

A special $\mathcal{F}$-precover is a morphism $f\colon F \to A$ that is part of a conflation (exact sequence)

$$C \rightarrowtail F \overset{f}{\twoheadrightarrow} A$$

with $C \in \mathcal{C}$. For if $F' \in \mathcal{F}$, then

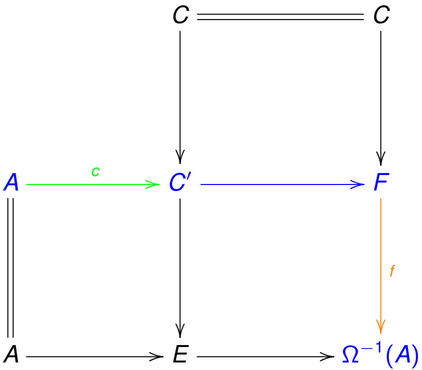


Special $\mathcal{C}$-preenvelopes are defined dually.

Theorem (Salce's Lemma)

If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in an exact category $(\mathcal{A}; \mathcal{E})$ with enough projective and injective objects, every object has a special $\mathcal{F}$-precover if and only if every object has a special $\mathcal{C}$-preenvelope. Such cotorsion pairs are called complete.

Proof of Salce's Lemma



Ext^1

$$\mathrm{Ext}\colon \text{R-}\underline{\mathrm{Mod}} \times \text{R-}\overline{\mathrm{Mod}} \to \mathrm{Ab}$$

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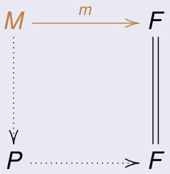
Flat Modules

Definition

A left R-module F is flat if it satisfies one of the following equivalent properties:

1. $\mathrm{Tor}_1(-, F) = 0$;
2. every map from a finitely presented module M,



   factors through a finitely generated projective module P;
3. $F = \varinjlim P_i$ is a direct limit of finitely generated projective modules.

Purity

Definition (PM Cohn)

A short exact sequence Σ is pure exact if every morphism $m\colon M \to X$ from a finitely presented module M factors as in

$$
\begin{array}{ccccccccc}
 & & & & & & M & & \\
 & & & & & \swarrow & \downarrow{\scriptstyle m} & & \\
\Sigma\colon 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0
\end{array}
$$

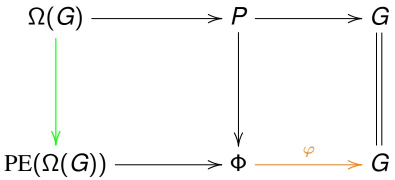
If X is flat, then every short exact sequence Σ is pure exact. If Y is pure injective, then $\mathrm{Ext}(X, Y) = 0$.

Pure Injective Modules

Proposition

${}^{\perp}(\text{R-PInj}) = \text{R-Flat}.$

Suppose that G is not flat.



Definition

A module M is cotorsion if $\mathrm{Ext}(R\text{-Flat}, M) = 0$. Thus

$$R\text{-Cotor} = (R\text{-Flat})^{\perp} = ({}^{\perp}(\text{R-PInj}))^{\perp}.$$

The Eklof-Trlifaj Lemma (2001)

The Flat Cover Conjecture (Enochs, 1981)

Every R-module A has a flat precover $f\colon F \to A$, Equivalently, the cotorsion pair $(R\text{-Flat}, R\text{-Cotor})$ is complete.

The Eklof-Trlifaj Lemma (2001)

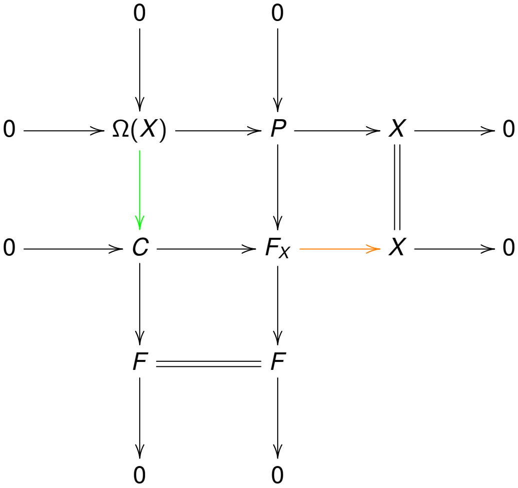
Let A and F_0 be R-modules. There exists a short exact sequences

$$0 \longrightarrow A \longrightarrow C \longrightarrow F \longrightarrow 0$$

such that $F \in \mathrm{Filt}(F_0)$ and $\mathrm{Ext}(F_0, C) = 0$

Let $F_0 = \coprod \{\, F' \mid F' \in R\text{-Flat},\ |F'| \leq \aleph_0 + |R| \,\}$.

Proof of the FCC (Bican, El Bachir, Enochs, 2001)



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Phantom Morphisms

Definition (Benson and Gnacadja, Neeman, Adams)

A morphism $\varphi\colon A \to X$ of left R-modules F is phantom if it satisfies one of the following equivalent properties:

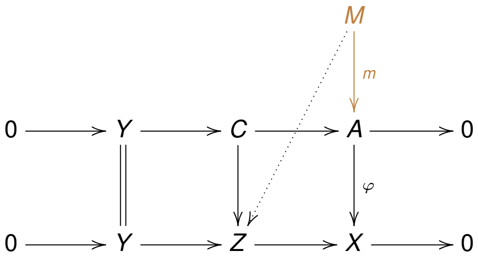
1. $\mathrm{Tor}_1(-, \varphi) = 0$;
2. pre compositing φ with a map from a finitely presented module M,

   $$\begin{array}{ccc} M & \xrightarrow{m} & A \\ \vdots & & \downarrow \varphi \\ P & \cdots\cdots\to & X \end{array}$$

   factors through a finitely generated projective module P;
3. $\varphi\colon A = \varinjlim M_i \to B$ is a direct limit $\varphi = \varinjlim (f_i\colon M_i \to B)$ of morphism that factor though finitely generated projective modules.

Pulling Back Along a Phantom

If φ is a phantom, then the pullback of any short exact sequence is pure exact:



If Y is pure injective, then $\mathrm{Ext}(\varphi, Y)\colon \mathrm{Ext}(X, Y) \to \mathrm{Ext}(A, Y)$ is zero.

Special Phantom Precovers

In the situation above,

$$\begin{array}{ccccc} \Omega(G) & \longrightarrow & P & \longrightarrow & G \\ \downarrow & & \downarrow & & \| \\ \mathrm{PE}(\Omega(G)) & \longrightarrow & \Phi & \xrightarrow{\varphi} & G, \end{array}$$

$\varphi\colon \Phi \to G$ is a phantom morphism, $\mathrm{Ext}(\varphi, \text{R-PInj}) = 0$.

If the ring R is left perfect, then the projective cover $P \to G$ is the flat cover.

Cotorsion vs Pure Injective Envelopes of Syzygies

Example (L. Gregory)

Let D be a DVR with primitive element π and field of fractions Q. Let Λ be a D-order for which $Q\Lambda$ is separable over Q.

A representation V of Λ is torsion free reduced if ${}_D V$ is torsion free with no nonzero divisible summands. Every syzygy is torsion free reduced.

The cotorsion envelope of V is the π-adic completion, $\mathrm{CE}(V) = \overline{V}_\pi$.

The pure injective envelope $W = \mathrm{PE}(V)$ is determined, up to isomorphism, among pure injective representations by the quotient $W/\pi^m W$, (where m is the Maranda constant of Λ) regarded as a (pure injective) representation over the artin algebra $\Lambda_m = \Lambda/\pi^m\Lambda$.

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Ideals

Definition

Let $\mathcal{A}$ be an additive category, $\mathrm{Hom}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$.

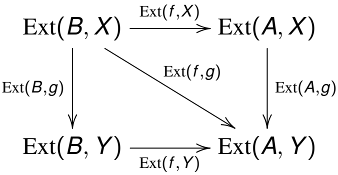
An ideal $\mathcal{I} \lhd \mathcal{A}$ is a subbifunctor $\mathcal{I}(-,-) \subseteq \mathrm{Hom}_{\mathcal{A}}(-,-)$.

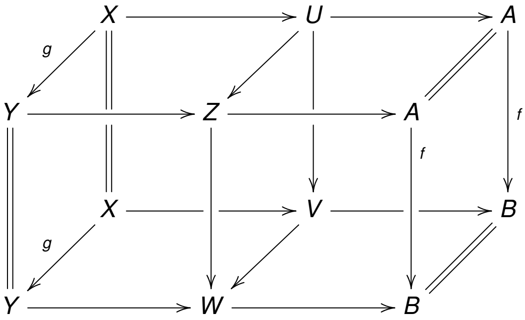
Examples

1. If $\mathcal{A} = \mathrm{R\text{-}proj}$, then $\mathcal{I} \lhd \mathcal{A} \mapsto \mathcal{I}(R,R) \lhd R$ is a bijective correspondence.
2. If $\mathcal{C} \subseteq \mathcal{A}$ is an additive subcategory, then $\langle \mathcal{C} \rangle$ is the ideal of morphisms that factor through an object in $\mathcal{C}$.

- We say $A \in \mathcal{I}$ is an object in $\mathcal{I}$ if $1_A \in \mathcal{I}(A,A)$;
- If $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathcal{A}$, then $\mathcal{I}\mathcal{J} \lhd \mathcal{A}$;
- $\mathcal{I}$ is idempotent if $\mathcal{I}^2 = \mathcal{I}$.

Let $f\colon A \to B$ and $g\colon X \to Y$

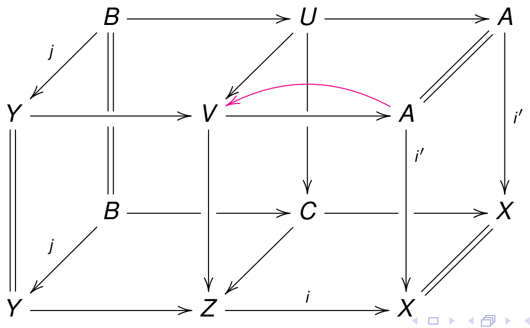




Special Ideal Precovers

Let $(\mathcal{I}, \mathcal{J})$ be an ideal cotorsion pair in an exact category $(\mathcal{A}; \mathcal{E})$. A special $\mathcal{I}$-precover of an object X is a morphism of exact sequences

$$\begin{array}{ccccc} B & \longrightarrow & C & \longrightarrow & X \\ \downarrow{\scriptstyle j} & & \downarrow & & \| \\ Y & \longrightarrow & Z & \xrightarrow{i} & X, \end{array}$$



Salce's Lemma

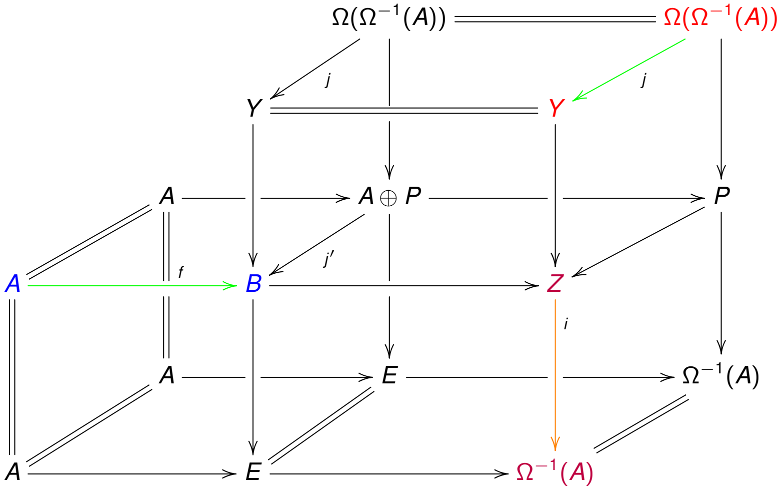
If $(\mathcal{A}; \mathcal{E})$ has enough projective objects, we may take a special $\mathcal{I}$-precover to be of the form

$$\begin{array}{ccccc} \Omega(X) & \longrightarrow & P & \longrightarrow & X \\ \downarrow{\scriptstyle j} & & \downarrow & & \| \\ Y & \longrightarrow & Z & \xrightarrow{i} & X. \end{array}$$

The Ideal Salce Lemma (Fu, Guil Asensio, H, Torrecillas)

Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough projective objects and enough injective objects. If every object has a special $\mathcal{I}$-precover, then every object has a special $\mathcal{J}$-preenvelope.

Proof of the Ideal Salce Lemma



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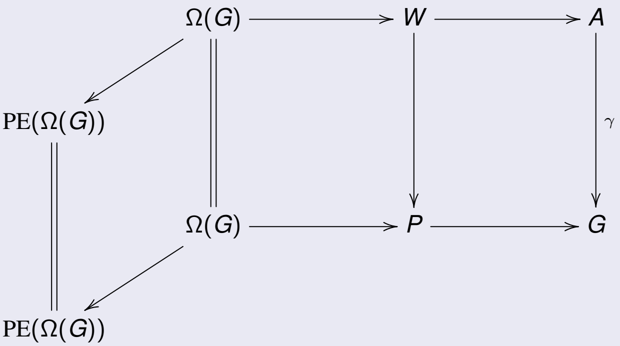
Ideal Cotorsion Pair Generated by Pure Injective Modules

${}^{\perp}\langle \text{R-PInj} \rangle = \Phi$

We saw how $\text{Ext}(\Phi, \text{R-PInj}) = 0$. Suppose $\gamma\colon A \to G$ is not phantom.



Then $\text{Ext}(\gamma, \text{PE}(\Omega(G))) \neq 0$.

It follows from the proof of the Ideal Salce Lemma that $\Phi^{\perp} = \langle \text{R-PInj} \star \text{R-Inj} \rangle$.

Xu's Theorem (1996)

$$\langle R\text{-Flat}\rangle \subseteq \cdots \subseteq \cdots \subseteq \Phi^2 \subseteq \Phi \subseteq \Phi^0 = \mathrm{Hom}$$

Xu's Theorem. TFAE:

1. $\mathrm{R\text{-}PInj}$ is closed under extensions;
2. $\mathrm{R\text{-}PInj} = \mathrm{R\text{-}Cotor}$;
3. $\Phi = \langle R\text{-Flat}\rangle$; and
4. $\Phi^2 = \Phi$.

Proof:

- 1) $\Rightarrow$ 2). Use Wakamatsu's Lemma.
- 2) $\Rightarrow$ 3). $\Phi = {}^{\perp}\langle \mathrm{R\text{-}PInj}\rangle = {}^{\perp}\langle \mathrm{R\text{-}Cotor}\rangle = \langle \mathrm{R\text{-}Flat}\rangle$.
- 4) $\Rightarrow$ 3). An idempotent covering ideal is an object ideal. Apply existence of phantom covers.

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