

Salce's Lemma I

IVO HERZOG
The Ohio State University

PATHS 2024
May 13 - 17, 2024
Cetraro, ITALY

Outline

- 1 Salce's Lemma**
- 2 The Flat Cover Conjecture
- 3 Phantom Morphisms
- 4 Ideal Cotorsion Pairs
- 5 Convergence of the Phantom Filtration

Cotorsion Pairs

Definition

A pair $(\mathcal{F}, \mathcal{C})$ of subcategories of an exact category $(\mathcal{A}; \mathcal{E})$ is a **cotorsion pair** if

$$\mathcal{C} = \mathcal{F}^\perp = \{C \in \mathcal{A} : (\forall F \in \mathcal{F}) \text{Ext}(F, C) = 0\}$$

and

$$\mathcal{F} = {}^\perp\mathcal{C} = \{F \in \mathcal{A} : (\forall C \in \mathcal{C}) \text{Ext}(F, C) = 0\}.$$

For $A \in \mathcal{A}$, an **\mathcal{F} -precover** is a morphism $f: F \rightarrow A$ such that for every $F' \in \mathcal{F}$

$$\begin{array}{ccc} & F' & \\ & \downarrow f' & \\ F & \xrightarrow{f} & A \end{array}$$

A **\mathcal{C} -preenvelope** of A is defined dually.

Special Approximations

A **special \mathcal{F} -precover** is a morphism $f: F \rightarrow A$ that is part of a conflation (exact sequence)

$$C \longrightarrow F \xrightarrow{f} A$$

with $C \in \mathcal{C}$. For if $F' \in \mathcal{F}$, then

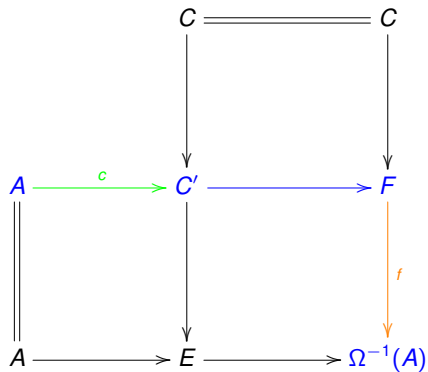
$$\begin{array}{ccccc}
 C & \longrightarrow & X & \longrightarrow & F' \\
 \parallel & & \downarrow & \lrcorner & \downarrow f' \\
 C & \longrightarrow & F & \xrightarrow{f} & A
 \end{array}$$

Special \mathcal{C} -preenvelopes are defined dually.

Theorem (Salce's Lemma)

If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in an exact category $(\mathcal{A}; \mathcal{E})$ with enough projective and injective objects, every object has a special \mathcal{F} -precover if and only if every object has a special \mathcal{C} -preenvelope. Such cotorsion pairs are called **complete**.

Proof of Salce's Lemma



Ext¹

Ext: $\mathbf{R}\text{-Mod} \times \overline{\mathbf{R}\text{-Mod}} \rightarrow \mathbf{Ab}$

Outline

- 1 Salce's Lemma
- 2 The Flat Cover Conjecture**
- 3 Phantom Morphisms
- 4 Ideal Cotorsion Pairs
- 5 Convergence of the Phantom Filtration

Flat Modules

Definition

A left R -module F is **flat** if it satisfies one of the following equivalent properties:

- 1 $\text{Tor}_1(-, F) = 0$;
- 2 every map from a finitely presented module M ,

$$\begin{array}{ccc}
 M & \xrightarrow{m} & F \\
 \vdots & & \parallel \\
 P & \dashrightarrow & F
 \end{array}$$

factors through a finitely generated projective module P ;

- 3 $F = \varinjlim P_i$ is a direct limit of finitely generated projective modules.

Purity

Definition (PM Cohn)

A short exact sequence Σ is **pure exact** if every morphism $m: M \rightarrow X$ from a finitely presented module M factors as in

$$\Sigma: 0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

The diagram illustrates the factoring property of a pure exact sequence. It shows a short exact sequence $\Sigma: 0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$. A finitely presented module M is positioned above the sequence. A solid arrow labeled m points from M down to X . A dotted arrow points from M down to Z , representing the factorization of m through the map $Z \rightarrow X$ in the sequence.

If X is flat, then every short exact sequence Σ is pure exact. If Y is **pure injective**, then $\text{Ext}(X, Y) = 0$.

Pure Injective Modules

Proposition

$${}^{\perp}(\mathbf{R}\text{-PInj}) = \mathbf{R}\text{-Flat}.$$

Suppose that G is not flat.

$$\begin{array}{ccccc}
 \Omega(G) & \longrightarrow & P & \longrightarrow & G \\
 \downarrow & & \downarrow & & \parallel \\
 \text{PE}(\Omega(G)) & \longrightarrow & \Phi & \xrightarrow{\varphi} & G
 \end{array}$$

Definition

A module M is **cotorsion** if $\text{Ext}(\mathbf{R}\text{-Flat}, M) = 0$. Thus

$$\mathbf{R}\text{-Cotor} = (\mathbf{R}\text{-Flat})^{\perp} = ({}^{\perp}(\mathbf{R}\text{-PInj}))^{\perp}.$$

The Eklof-Trlifaj Lemma (2001)

The Flat Cover Conjecture (Enochs, 1981)

Every R -module A has a flat precover $f: F \rightarrow A$, Equivalently, the cotorsion pair $(R\text{-Flat}, R\text{-Cotor})$ is complete.

The Eklof-Trlifaj Lemma (2001)

Let A and F_0 be R -modules. There exists a short exact sequences

$$0 \longrightarrow A \longrightarrow C \longrightarrow F \longrightarrow 0$$

such that $F \in \text{Filt}(F_0)$ and $\text{Ext}(F_0, C) = 0$

Let $F_0 = \coprod \{ F' \mid F' \in R\text{-Flat}, |F'| \leq \aleph_0 + |R| \}$.

Proof of the FCC (Bican, El Bachir, Enochs, 2001)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Omega(X) & \longrightarrow & P & \longrightarrow & X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & C & \longrightarrow & F_X & \longrightarrow & X \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & F & \xlongequal{\quad} & F & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The diagram illustrates a commutative diagram used in the proof of the Flat Cover Conjecture. It consists of two rows of exact sequences and several vertical maps connecting them.

- The top row is an exact sequence: $0 \rightarrow \Omega(X) \rightarrow P \rightarrow X \rightarrow 0$.
- The bottom row is an exact sequence: $0 \rightarrow C \rightarrow F_X \rightarrow X \rightarrow 0$.
- Vertical maps connect the rows:
 - A green arrow points from $\Omega(X)$ to C .
 - A black arrow points from P to F_X .
 - A black arrow points from X in the top row to X in the bottom row, with a double line indicating an isomorphism.
 - Black arrows point from C to F and from F_X to F .
 - A double line connects the two F objects, indicating an isomorphism.
 - Black arrows point from each F to a 0 at the bottom.
 - Black arrows point from 0 to $\Omega(X)$ and 0 to P in the top row.
 - Black arrows point from 0 to C and 0 to F_X in the bottom row.

Outline

- 1 Salce's Lemma
- 2 The Flat Cover Conjecture
- 3 Phantom Morphisms**
- 4 Ideal Cotorsion Pairs
- 5 Convergence of the Phantom Filtration

Phantom Morphisms

Definition (Benson and Gnacadja, Neeman, Adams)

A morphism $\varphi: A \rightarrow X$ of left R -modules F is **phantom** if it satisfies one of the following equivalent properties:

- $\text{Tor}_1(-, \varphi) = 0$;
- pre composing φ with a map from a finitely presented module M ,

$$\begin{array}{ccc}
 M & \xrightarrow{m} & A \\
 \vdots & & \downarrow \varphi \\
 P & \dashrightarrow & X
 \end{array}$$

factors through a finitely generated projective module P ;

- $\varphi: A = \varinjlim M_i \rightarrow B$ is a direct limit $\varphi = \varinjlim (f_i: M_i \rightarrow B)$ of morphism that factor through finitely generated projective modules.

Pulling Back Along a Phantom

If φ is a phantom, then the pullback of any short exact sequence is pure exact:

$$\begin{array}{ccccccc}
 & & & & M & & \\
 & & & & \downarrow m & & \\
 0 & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & A \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \varphi \\
 0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X \longrightarrow 0
 \end{array}$$

The diagram illustrates a pullback square. The top row is a short exact sequence $0 \rightarrow Y \rightarrow C \rightarrow A \rightarrow 0$. The bottom row is another short exact sequence $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$. The vertical map from C to Z is the pullback of $\varphi: A \rightarrow X$ along the map $Y \rightarrow C$. A dotted line represents the map $M \rightarrow Z$ that completes the pullback square $M \rightarrow C \rightarrow Z \rightarrow X$. The map $m: M \rightarrow A$ is the map from the pullback to A .

If Y is pure injective, then $\text{Ext}(\varphi, Y): \text{Ext}(X, Y) \rightarrow \text{Ext}(A, Y)$ is zero.

Special Phantom Precovers

In the situation above,

$$\begin{array}{ccccc}
 \Omega(G) & \longrightarrow & P & \longrightarrow & G \\
 \downarrow & & \downarrow & & \parallel \\
 \text{PE}(\Omega(G)) & \longrightarrow & \Phi & \xrightarrow{\varphi} & G,
 \end{array}$$

$\varphi: \Phi \rightarrow G$ is a phantom morphism, $\text{Ext}(\varphi, \text{R-PIinj}) = 0$.

If the ring R is **left perfect**, then the projective cover $P \rightarrow G$ is the flat cover.

Cotorsion vs Pure Injective Envelopes of Syzygies

Example (L. Gregory)

Let D be a DVR with primitive element π and field of fractions Q . Let Λ be a D -order for which $Q\Lambda$ is separable over Q .

A representation V of Λ is **torsion free reduced** if ${}_D V$ is torsion free with no nonzero divisible summands. Every syzygy is torsion free reduced.

The cotorsion envelope of V is the π -adic completion, $\text{CE}(V) = \overline{V}_\pi$.

The pure injective envelope $W = \text{PE}(V)$ is determined, up to isomorphism, among pure injective representations by the quotient $W/\pi^m W$, (where m is the **Maranda constant** of Λ) regarded as a (pure injective) representation over the artin algebra $\Lambda_m = \Lambda/\pi^m \Lambda$.

Outline

- 1 Salce's Lemma
- 2 The Flat Cover Conjecture
- 3 Phantom Morphisms
- 4 Ideal Cotorsion Pairs**
- 5 Convergence of the Phantom Filtration

Ideals

Definition

Let \mathcal{A} be an additive category, $\text{Hom}: \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{Ab}$.

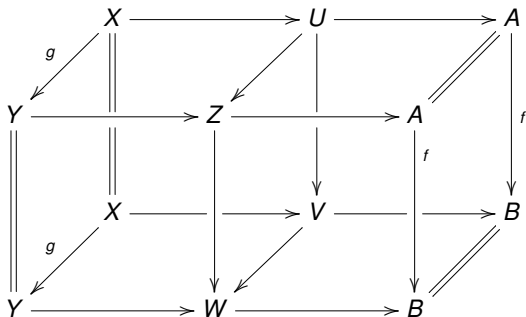
An **ideal** $\mathcal{I} \triangleleft \mathcal{A}$ is a subbifunctor $\mathcal{I}(-, -) \subseteq \text{Hom}_{\mathcal{A}}(-, -)$.

Examples

- 1 If $\mathcal{A} = \text{R-proj}$, then $\mathcal{I} \triangleleft \mathcal{A} \mapsto \mathcal{I}(R, R) \triangleleft R$ is a bijective correspondence.
 - 2 If $\mathcal{C} \subseteq \mathcal{A}$ is an additive subcategory, then $\langle \mathcal{C} \rangle$ is the ideal of morphisms that factor through an object in \mathcal{C} .
- We say $A \in \mathcal{I}$ is an **object** in \mathcal{I} if $1_A \in \mathcal{I}(A, A)$;
 - If \mathcal{I} and \mathcal{J} are ideals of \mathcal{A} , then $\mathcal{I}\mathcal{J} \triangleleft \mathcal{A}$;
 - \mathcal{I} is **idempotent** if $\mathcal{I}^2 = \mathcal{I}$.

Let $f: A \rightarrow B$ and $g: X \rightarrow Y$

$$\begin{array}{ccc}
 \text{Ext}(B, X) & \xrightarrow{\text{Ext}(f, X)} & \text{Ext}(A, X) \\
 \text{Ext}(B, g) \downarrow & \searrow \text{Ext}(f, g) & \downarrow \text{Ext}(A, g) \\
 \text{Ext}(B, Y) & \xrightarrow{\text{Ext}(f, Y)} & \text{Ext}(A, Y)
 \end{array}$$



Salce's Lemma

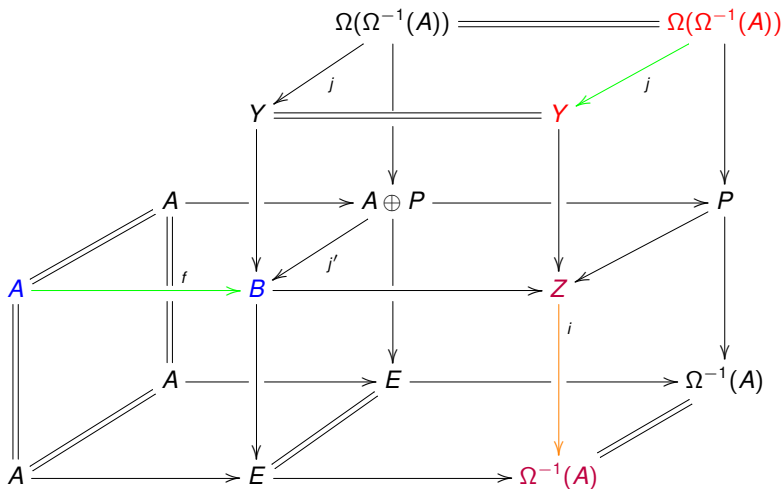
If $(\mathcal{A}; \mathcal{E})$ has enough projective objects, we may take a special \mathcal{I} -precover to be of the form

$$\begin{array}{ccccc}
 \Omega(X) & \longrightarrow & P & \longrightarrow & X \\
 \downarrow j & & \downarrow & & \parallel \\
 Y & \longrightarrow & Z & \xrightarrow{i} & X.
 \end{array}$$

The Ideal Salce Lemma (Fu, Guil Asensio, H, Torrecillas)

Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough projective objects and enough injective objects. If every object has a special \mathcal{I} -precover, then every object has a special \mathcal{J} -preenvelope.

Proof of the Ideal Salce Lemma



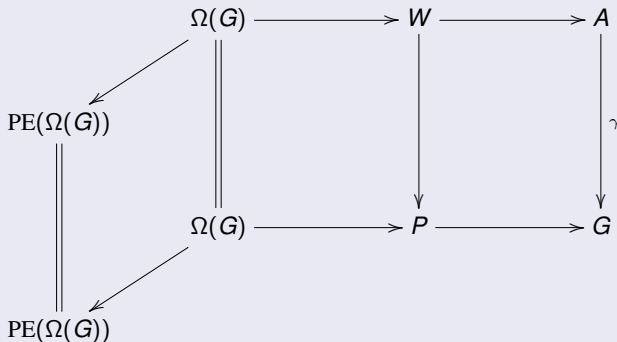
Outline

- 1 Salce's Lemma
- 2 The Flat Cover Conjecture
- 3 Phantom Morphisms
- 4 Ideal Cotorsion Pairs
- 5 Convergence of the Phantom Filtration**

Ideal Cotorsion Pair Generated by Pure Injective Modules

$${}^{\perp}\langle \mathbf{R}\text{-PIinj} \rangle = \Phi$$

We saw how $\text{Ext}(\Phi, \mathbf{R}\text{-PIinj}) = 0$. Suppose $\gamma: A \rightarrow G$ is not phantom.



Then $\text{Ext}(\gamma, \text{PE}(\Omega(G))) \neq 0$.

It follows from the proof of the Ideal Salce Lemma that $\Phi^{\perp} = \langle \mathbf{R}\text{-PIinj} \star \mathbf{R}\text{-Inj} \rangle$.

Xu's Theorem (1996)

$$\langle R\text{-Flat} \rangle \subseteq \cdots \subseteq \cdots \subseteq \Phi^2 \subseteq \Phi \subseteq \Phi^0 = \text{Hom}$$

Xu's Theorem. TFAE:

- 1 R-PInj is closed under extensions;
- 2 R-PInj = R-Cotor;
- 3 $\Phi = \langle R\text{-Flat} \rangle$; and
- 4 $\Phi^2 = \Phi$.

Proof:

- 1) \Rightarrow 2). Use Wakamatsu's Lemma.
- 2) \Rightarrow 3). $\Phi = {}^\perp \langle R\text{-PInj} \rangle = {}^\perp \langle R\text{-Cotor} \rangle = \langle R\text{-Flat} \rangle$.
- 4) \Rightarrow 3). An idempotent covering ideal is an object ideal. Apply existence of phantom **covers**.

REFERENCES:

- 1 Bican, El Bachir, and Enochs, All modules have flat covers, Bulletin of the LMS **33** (2001), 385-390.
- 2 Eklof, P.C., and Trlifaj, J., How to Make Ext Vanish, Bulletin of the LMS **33** (1) (2001), 41-51.
- 3 Enochs, E.E., Injective and flat covers, envelopes and resolvents, Israel Journal of Mathematics **39** (1981), 189-209.
- 4 Fu, X.H., Guil Asensio, P.A., Herzog, I., Torrecillas, B., Ideal Approximation Theory, Advances in Math **244** (2013), 750-790.
- 5 Gregory, L., Maranda's theorem for pure-injective modules and duality, Canadian J of Math **75**(2) (2023), 581-607.
- 6 Xu, J., *Flat Covers of Modules*, Lecture Notes in Mathematics **1634**, Springer, 1996