Twist equivalence arising from a (partially) minimal projective resolution.

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Introduction

Given an object X in a Frobenius exact category satisfying mild conditions, this work constructs a derived autoequivalence of the endomorphism algebra of X. The idea is that properties of the stable endomorphism algebra of X are sufficient to specify properties of X and of its endomorphism algebra. As an application, it is possible to obtain a new non-trivial derived autoequivalence for very singular varieties.

Setup

Setup 1. Let $\mathcal{E}$ be an $\mathcal{R}$-linear Frobenius exact category with a an object P such that $\operatorname{add} P = \operatorname{proj} \mathcal{E}$. The stable category $\mathcal{D}$ is triangulated and write its suspension functor as Σ.

Assume in addition that $\mathcal{R}$ be a Cohen-Macaulay noetherian local ring of dimension d with coefficient field k and canonical module ω_R.

Setup 2. With the assumptions as in 1, let $X \in \mathcal{E}$ be such that

1. The object P is a summand of X,
2. X has infinite projective dimension in $\mathcal{E}$,
3. $\operatorname{add}_{\mathcal{D}}(X) \subset \mathcal{D}$ is Krull-Schmidt,
4. X is basic in $\mathcal{D}$.

Notation 3. With the X and $\mathcal{E}$ as in Setup 2, let

1. $\Lambda := \operatorname{End}_{\mathcal{E}}(X)$ and $\Lambda_{\text{con}} = \underline{\operatorname{End}}_{\mathcal{E}}(X)$.
2. $[\operatorname{proj} \mathcal{E}]$ is the ideal in Λ of maps $X \to X$ which factor through a projective object in $\mathcal{E}$.

Under set up 2, Λ has the dualizing complex

$$\omega_\Lambda := \mathbf{R}\operatorname{Hom}_{\mathcal{R}}(\Lambda, \omega_{\mathcal{R}}).$$

Partialy minimal projective resolution

Notation 4. Let $\mathcal{E}$ and X be as in setup 2. let

$$\mathcal{S} = \bigoplus_i S_i$$

where $\{S_i\}_i$ is representative set of simple Λ_{con}-modules.

Definition 5 (Donovan, Wemyss [DW19]). Λ_{con} is *d-relatively spherical* if $\operatorname{Ext}^i_\Lambda(\Lambda_{\text{con}}, \mathcal{S}) = 0$ for all $i \neq 0, d$.

Definition 6. Let $X \in \mathcal{D}$ and $n \in \mathbb{Z}$ with $n < 0$. We say that X is *n-rigid* if $\operatorname{Ext}^i_{\mathcal{D}}(X, X) = 0$ for all $n \leqslant i \leqslant -1$.

Theorem 7. *Let $\mathcal{E}$ and X be as in setup 2 with $d \geqslant 3$. If Λ_{con} is self-injective and $\Lambda_{\text{con}} \otimes^{\mathbf{L}} \omega_\Lambda \cong \Lambda_{\text{con}}$, then the following are equivalent*

1. *Λ_{con} is perfect over Λ and d-relatively spherical.*
2. *$\Sigma^{-d+1} X \cong X$ in $\mathcal{D}$ and X is $(-d+2)$-rigid in $\mathcal{D}$.*

The key point in the proof of theorem 7 is that any $X \in \mathcal{E}$ which satisfies the assumptions of theorem 7 is such that Λ_{con} admits the following projective resolution

$$0 \to Q_d \oplus \operatorname{Hom}_{\mathcal{E}}(X, X) \xrightarrow{f_d} Q_{d-1} \xrightarrow{f_{d-1}} Q_{d-2} \to \cdots \to Q_1 \xrightarrow{f_1} Q_0 \oplus \operatorname{Hom}_{\mathcal{E}}(X, X) \xrightarrow{f_0} \Lambda_{\text{con}} \to 0$$

where $Q_i \in \operatorname{add} \operatorname{Hom}_{\mathcal{E}}(X, P)$ and $\operatorname{Hom}_\Lambda(f_i, \mathcal{S}) = 0$ for all $i > 0$.

Autoequivalence

Notation 8. Fix $k \in \mathbb{Z}$ with $-d+2 \leqslant k \leqslant 0$. We denote

1. the algebra $\Lambda_{\Sigma^k} := \operatorname{End}_{\mathcal{E}}(P \oplus \Sigma^k X)$,
2. the $\Lambda_{\Sigma^{k-1}}$-Λ_{Σ^k} bimodules

$$T_k := \operatorname{Hom}_{\mathcal{E}}(P \oplus \Sigma^k X, P \oplus \Sigma^{k-1} X).$$

Corollary 9. *For $k \in \mathbb{Z}$ with $-d+2 \leqslant k \leqslant 0$, the $\Lambda_{\Sigma^{k-1}}$-Λ_{Σ^k} bimodule T_k is tilting.*

Therefore, the functors

$$\Phi_k \colon \mathbf{R}\operatorname{Hom}_{\Lambda_{\Sigma^k}}(T_k, -) \colon \ \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\Sigma^k}) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\Sigma^{k-1}})$$

for $k \in \mathbb{Z}$ with $-d+2 \leqslant k \leqslant 0$ are equivalences and, thus, so is their composition

$$\mathbf{R}\operatorname{Hom}_\Lambda(T_{-d+2} \otimes^{\mathbf{L}}_{\Lambda_{\Sigma-d+2}} T_{-d+3} \otimes^{\mathbf{L}}_{\Lambda_{\Sigma-d+3}} \cdots \otimes^{\mathbf{L}}_{\Lambda_{\Sigma^{-1}}} T_0, -) \colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_{\Sigma^{-d+1}}).$$

Now since $\Sigma^{-d+1} X \cong X$, this composition is a derived autoequivalence of Λ which turns out to be functoially isomorphic to the noncommutative twist functor

$$\mathcal{T} := \mathbf{R}\operatorname{Hom}_\Lambda([\operatorname{proj} \mathcal{E}], -) \colon \ \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda).$$

defined by Donovan and Wemyss [DW16].

References

[DW16] W. Donovan and M. Wemyss. Noncommutative deformations and flops. *Duke Math. J.* 165.8 (2016), pp. 1397–1474.

[DW19] W. Donovan and M. Wemyss. Noncommutative enhancements of contractions. *Advances in Mathematics* 344 (2019), pp. 99–136.