Gabriel-Popescu Theorem revisited

Septimiu Crivei

"Babeş–Bolyai" University, Cluj–Napoca

(joint work with Constantin Năstăsescu)

Purity, Approximation Theory and Spectra
Grand Hotel San Michele, Cetraro, May 15-17, 2024
Dedicated to Manolo Saorín on his 65th birthday

Outline

Gabriel-Popescu Theorem

P. Gabriel, N. Popescu, *Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes*, C. R. Acad. Sci. Paris **258** (1964), 4188–4190.

Theorem (Gabriel-Popescu)

Let $\mathcal{A}$ be a Grothendieck category with a generator G, $R = \mathrm{End}_{\mathcal{A}}(G)$ and $\mathrm{Mod}(R)$ the category of unitary right R-modules. Then $T = \mathrm{Hom}_{\mathcal{A}}(G, -) : \mathcal{A} \to \mathrm{Mod}(R)$ is fully faithful and has an exact left adjoint S.

- The proof is inspired by a result by Giraud, which characterizes Grothendieck toposes.
- The original proof seemed rather complicated, especially the part on the exactness of the functor S.

A bit of history

Revisited by several authors:

- M. Takeuchi, *A simple proof of Gabriel and Popesco's theorem*, J. Algebra **18** (1971), 112–113.
  It keeps the first part of the original proof and simplifies the part on the exactness of the left adjoint S of T.

- F. Ulmer, *A flatness criterion in Grothendieck categories*, Invent. Math. **19** (1973), 331–336.
  It gives an exactness criterion for the left adjoint S of T.

- B. Mitchell, *A quick proof of the Gabriel-Popesco theorem*, J. Pure Appl. Algebra **20** (1981), 313–315.
  It uses some different ideas: an ingenious lemma and the result that every Grothendieck category has enough injective objects, already known from Grothendieck's Tôhoku paper from 1957.

Some other versions for functor categories:

- M. Prest, *Elementary torsion theories and locally finitely presented categories*, J. Pure Appl. Algebra **18** (1980), 205–212.

- G. Garkusha, *Grothendieck categories*, Algebra i Analiz **13** (2001), 1–68 (Russian). Engl. transl. in St. Petersburg Math. J. **13** (2002), 149–200.

- F. Castaño-Iglesias, P. Enache, C. Năstăsescu, B. Torrecillas, *Gabriel-Popescu type theorems and applications*, Bull. Sci. Math. **128** (2004), 323–332.

E. M. Vitale, *Localizations of algebraic categories II*, J. Pure Appl. Algebra **133** (1998), 317–326.

W. Lowen, *A generalization of the Gabriel-Popescu theorem*, J. Pure Appl. Algebra **190** (2004), 197–211.

M. Porta, *The Popescu-Gabriel theorem for triangulated categories*, Adv. Math. **225** (2010), 1669–1715.

Y. Imamura, *Grothendieck enriched categories*, Appl. Categ. Struct. **30** (2022), 1017–1041.

F. Genovese and J. Ramos González, *A derived Gabriel-Popescu theorem for t-structures via derived injectives*, Int. Math. Res. Not. IMRN **2023** (2023), 4695–4760.

S. Crivei, C. Năstăsescu, *The Gabriel-Popescu Theorem revisited*, preprint.
Main ingredients: generalized Mitchell Lemma and one-sided exact categories.

One-sided exact categories

- Rosenberg 2007, Rump 2010, 2011, Henrard-van Roosmalen 2019

Definition (Bazzoni-C. 2013)

An *inflation-exact category* is an additive category $\mathcal{C}$ endowed with a distinguished class of kernels, called *inflations* and denoted by $\rightarrowtail$, satisfying the axioms:

[R0] The identity morphism $1_0 : 0 \to 0$ is an inflation.
[R1] The composition of any two inflations is again an inflation.
[R2] The pushout of any inflation along an arbitrary morphism exists and is again an inflation.

The pushout of an inflation $i : A \rightarrowtail B$ along $A \to 0$ yields its cokernel $d : B \to C$, called *deflation* and denoted by $d : B \twoheadrightarrow C$. Then the kernel-cokernel pair $A \rightarrowtail B \twoheadrightarrow C$ is called *conflation*.

A *strongly inflation-exact category* is an inflation-exact category $\mathcal{C}$ satisfying:
[R3] If $i : A \to B$ and $p : B \to C$ are morphisms in $\mathcal{C}$ such that i *has a cokernel* and pi is an inflation, then i is an inflation.

Many homological lemmas still hold in strongly inflation-exact categories:

Short Five Lemma, 3×3 Lemma.

Coreflective subcategories revisited

A full subcategory of an additive category is *coreflective* if the inclusion functor has a right adjoint.

Theorem (Cortés-Izurdiaga-C.-Saorín 2023)

Let $\mathcal{C}$ be an additive category such that every morphism has a pseudokernel and a pseudocokernel. Let $\mathcal{B}$ be a subcategory of $\mathcal{C}$. Consider the following statements:

1. *$\mathcal{B}$ is a coreflective subcategory of $\mathcal{C}$.*
2. *$\mathcal{B}$ is precovering, closed under direct summands and every morphism in $\mathcal{B}$ has a pseudocokernel in $\mathcal{C}$ which belongs to $\mathcal{B}$.*

Then (1) $\Longrightarrow$ (2). If $\mathcal{C}$ has split idempotents, then (2) $\Longrightarrow$ (1).

Coreflective subcategories revisited

Corollary (Cortés-Izurdiaga-C.-Saorín 2023)

Let $\mathcal{A}$ be a preabelian category and $\mathcal{B}$ be an additive subcategory of $\mathcal{A}$. Then $\mathcal{B}$ is coreflective iff it is precovering and closed under cokernels.

If $\mathcal{U}$ is a set of objects of an abelian category, then $\mathrm{Gen}(\mathcal{U})$ is a coreflective subcategory.

Corollary (Cortés-Izurdiaga-C.-Saorín 2023)

1. *Let $\mathcal{U}$ be a set of objects of an AB3 abelian category $\mathcal{A}$. Then $\mathrm{Pres}(\mathcal{U})$ is a coreflective subcategory iff $\mathrm{Pres}(\mathcal{U})$ is closed under cokernels in $\mathcal{A}$.*
2. *Let $\mathcal{A}$ be a Grothendieck category. Then $\mathrm{fg}(\mathcal{A})$ is a skeletally small subcategory, and $\mathrm{Gen}(\mathrm{fg}(\mathcal{A})) = \mathrm{Pres}(\mathrm{fg}(\mathcal{A}))$ is a coreflective subcategory.*
3. *Let $\mathcal{A}$ be a Grothendieck category. Then $\mathrm{fp}(\mathcal{A})$ is a skeletally small subcategory, and $\mathrm{Pres}(\mathrm{fp}(\mathcal{A}))$ is a coreflective subcategory.*

Coreflective subcategories are right quasi-abelian categories

Question

What (one-sided) exact structure to consider on a coreflective subcategory of an AB5 category to be compatible with direct limits?

An additive category is *right quasi-abelian* if it it is preabelian, and the pushout of a kernel along an arbitrary morphism is again a kernel (Schneiders, Rump).

Coreflective subcategories are right quasi-abelian categories, and the class of all kernel-cokernel pairs define an inflation-exact structure.

Example (González-Férez-Marín 2011)

Let R be a non-unital ring and consider the category $\mathrm{Mod}(R)$ of right R-modules M such that $M \otimes_R R \cong M$ (*firm* modules). Then $\mathrm{Mod}(R)$ is a coreflective subcategory of the AB5 category $\mathrm{Mod}(R^*)$ of right R^*-modules over the unitary Dorroh extension R^* of R, but in general direct limits are not exact in $\mathrm{Mod}(R)$.

Proposition (Bazzoni-C. 2013)

Let $\mathcal{C}$ be a strongly inflation-exact category, $\mathcal{D}$ a right quasi-abelian category (in particular, a coreflective full subcategory), and $L : \mathcal{D} \to \mathcal{C}$ an additive functor which preserves cokernels (in particular, a left adjoint). Then there is an induced strongly inflation-exact structure on $\mathcal{D}$ defined by the property:

A kernel j in $\mathcal{D}$ is an inflation iff $L(j)$ is an inflation in $\mathcal{C}$.

A functor between inflation-exact categories is *conflation-exact* if it preserves conflations.

Lemma (C.-Năstăsescu)

Let $\mathcal{A}$ be an AB5 category endowed with the exact structure given by all short exact sequences, and $\mathcal{Q}$ a coreflective full subcategory of $\mathcal{A}$ endowed with the strongly inflation-exact structure induced from $\mathcal{A}$. Then direct limits are conflation-exact in $\mathcal{Q}$.

Notation

- $\mathcal{C}$: additive category
- $\mathcal{A}$: AB5 category
  (abelian with coproducts and exact direct limits)
- $\mathcal{U}$: set of objects of $\mathcal{A}$
- $\mathrm{Gen}(\mathcal{U})$: the full subcategory of $\mathcal{A}$ consisting of the $\mathcal{U}$-generated objects
- $\mathrm{Pres}(\mathcal{U})$ is the full subcategory of $\mathcal{A}$ consisting of the $\mathcal{U}$-presented objects
- $\mathcal{A}_{\mathcal{U}}$: the class of objects M of $\mathcal{A}$ such that for every morphism $f : \bigoplus_{U \in F} U \to M$ in $\mathcal{A}$ with F a finite subset of $\mathcal{U}$, $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$.
- $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$: the category of additive contravariant functors from $\mathcal{U}$ to Ab, which has the generating family of f.g. projective objects $(h_U)_{U \in \mathcal{U}}$
- $T : \mathcal{A} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is given by $T(X) = \mathrm{Hom}_{\mathcal{A}}(-, X)|_{\mathcal{U}}$ on objects X, $T(f) = \mathrm{Hom}_{\mathcal{A}}(-, f)|_{\mathcal{U}} : T(X) \to T(Y)$ on morphisms $f : X \to Y$
- T has a left adjoint $S : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$
  Denote by $\nu : ST \to 1_{\mathcal{A}}$ the counit of the adjunction (S, T)
- $\mathrm{Stat}(T)$ is the class of objects A of $\mathcal{A}$ such that ν_A is an isomorphism
- $\mathrm{Ker}(S)$ is the class of objects K in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ such that $S(K) = 0$

An adjoint functor theorem

Theorem (C.-Năstăsescu)

Let $\mathcal{C}$ be an additive category with cokernels, $\mathcal{U}$ a set of objects of $\mathcal{C}$ and $T : \mathcal{C} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ an additive functor. The following are equivalent:

1. *T has a left adjoint $S : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{C}$ such that $S(h_U) = U$ for every $U \in \mathcal{U}$.*
2.
   1. *There exists a natural isomorphism $T \cong H$, where $H : \mathcal{C} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is the functor defined by $H(X) = \mathrm{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{U}}$ on objects X of $\mathcal{C}$, and correspondingly on morphisms.*
   2. *There exists the coproduct in $\mathcal{C}$ of any family of objects in $\mathcal{U}$.*

A result on coreflective subcategories

Corollary (C.-Năstăsescu)

Let $\mathcal{C}$ be cocomplete, and $\mathcal{U}$ a set of objects of $\mathcal{C}$. Let $T : \mathcal{C} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ be given by $T(X) = \mathrm{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{U}}$ on objects, and correspondingly on morphisms, and let $S : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{C}$ be its left adjoint.
Let $\mathcal{Q}$ be a coreflective full subcategory of $\mathcal{C}$ such that $\mathcal{U} \subseteq \mathcal{Q}$. Then $\mathrm{Im}(S) \subseteq \mathcal{Q}$, and there are 3 pairs of adjoint functors (S, T), (i, q) and (S', T') such that $T' = Ti$ and $iS' = S$:

$$\begin{array}{ccc}
\mathcal{C} & \underset{S}{\overset{T}{\rightleftarrows}} & (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \\
{\scriptstyle i}\uparrow\;\downarrow{\scriptstyle q} & & \| \\
\mathcal{Q} & \underset{S'}{\overset{T'}{\rightleftarrows}} & (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})
\end{array}$$

Generalized Mitchell Lemma

Lemma (C.-Năstasescu-L. Năstăsescu 2012)

Let $\mathcal{U} = (U_i)_{i \in I}$, A and B objects of $\mathcal{A}$ with $A \in \mathcal{A}_{\mathcal{U}}$, M a subobject of $T(A)$ and $G : M \to T(B)$ a morphism in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$. For every $\alpha \in \Lambda = \bigcup_{i \in I} M(U_i)$, denote by $i_\alpha : U_i \to \bigoplus_{\beta \in \lambda} U_\beta$ the canonical injection, where $U_\beta = U_i$ for $\beta \in M(U_i)$. Let $\psi : \bigoplus_{\beta \in \Lambda} U_\beta \to A$ be the unique morphism such that $\psi i_\beta = \alpha$ for every $\alpha \in \Lambda$, and $\phi : \bigoplus_{\beta \in \Lambda} U_\beta \to B$ the unique morphism such that $\phi i_\alpha = G_{U_i}(\alpha)$ for every $\alpha \in \Lambda$. Then ϕ factors through $\mathrm{Im}(\psi)$.

$$
\begin{array}{ccc}
M & \hookrightarrow & T(A) \\
{\scriptstyle G}\downarrow & & \\
T(B) & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\bigoplus_{\beta \in \Lambda} U_\beta & \xrightarrow{\psi} & A \\
{\scriptstyle \phi}\downarrow & & \\
B & &
\end{array}
$$

Generalized Mitchell Lemma

If $\mathcal{U} = \{U\}$ and $\mathrm{Gen}(\mathcal{U}) = \mathcal{A}$ one obtains the Mitchell Lemma.

Lemma (Mitchell 1981)

Let A and B be objects of a Grothendieck category $\mathcal{A}$ with generator U and $R = \mathrm{End}_{\mathcal{A}}(U)$, M a submodule of $T(A) = \mathrm{Hom}_{\mathcal{A}}(U, A)$ and $g : M \to T(B)$ an R-homomorphism. For every $m \in M$, denote by $i_m : U \to \bigoplus_{m \in M} U_m$ the canonical injection, where $U_m = U$ for $m \in M$. Let $\psi : \bigoplus_{m \in M} U \to A$ be the unique morphism such that $\psi i_m = m$ for every $m \in M$, and $\phi : \bigoplus_{m \in M} U \to B$ the unique morphism such that $\phi i_m = g(m)$ for every $m \in M$. Then ϕ factors through $\mathrm{Im}(\psi)$.

Corollary (C.-Năstăsescu)

$\mathrm{Gen}(\mathcal{U}) \cap \mathcal{A}_{\mathcal{U}} \subseteq \mathrm{Stat}(T)$.

Lemma (C.-Năstăsescu)

Let $A \in \mathrm{Gen}(\mathcal{U}) \cap \mathcal{A}_{\mathcal{U}}$*,* M *a subobject of* $T(A)$*, and* $i_A : M \to T(A)$ *the inclusion morphism. Then* $S(i_A) : S(M) \to ST(A)$ *is an inflation in* $\mathrm{Gen}(\mathcal{U})$*.*

If $\mathcal{U} = \{U\}$ and $\mathrm{Gen}(\mathcal{U}) = \mathcal{A}$ one obtains the Takeuchi Lemma.

Lemma (Takeuchi 1971)

Let A *be an object of a Grothendieck category* $\mathcal{A}$*,* M *a subobject of* $T(A)$*, and* $i_A : M \to T(A)$ *the inclusion morphism. Then* $S(i_A) : S(M) \to ST(A)$ *is a monomorphism in* $\mathcal{A}$*.*

Theorem (C.-Năstăsescu)

Let $\mathcal{V}$ be a small preadditive category, $\mathcal{C}$ a strongly inflation-exact category having colimits and conflation-exact direct limits, and let $F : (\mathcal{V}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{C}$ be a functor which preserves colimits. Then the following are equivalent:

1. *F is conflation-exact.*
2. *For every object V of $\mathcal{V}$ and every f.g. subfunctor K of h_V, $F(I)$ is an inflation in $\mathcal{C}$, where $I : K \to h_V$ is the inclusion.*

Generalized Ulmer Theorem

Theorem (C.-Năstăsescu)

Assume that $\mathcal{U} \subseteq \mathcal{Q}$. Then the following are equivalent:

1. *The functor $S' : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{Q}$ is conflation-exact.*
2. *For every object $U \in \mathcal{U}$ and every f.g. subfunctor K of h_U, $S'(I)$ is an inflation in $\mathcal{Q}$, where $I : K \to h_U$ is the inclusion.*

If $\mathcal{Q} \subseteq \mathrm{Gen}(\mathcal{U})$, then they are further equivalent to:

3. *$\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$.*

If $\mathcal{Q} = \mathrm{Gen}(\mathcal{U}) = \mathcal{A}$ one obtains the Ulmer Theorem.

Theorem (Ulmer 1973)

The functor $S : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ is exact iff $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$.

Theorem (C.-Năstăsescu)

Let $\mathcal{U}$ be a set of objects of an AB5 category $\mathcal{A}$ such that $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$. Then:

1. *There exists a unique coreflective full subcategory $\mathcal{Q}$ of $\mathcal{A}$ such that $\mathcal{U} \subseteq \mathcal{Q} \subseteq \mathrm{Gen}(\mathcal{U}) \cap \mathcal{A}_{\mathcal{U}}$, namely*

$$\mathcal{Q} = \mathrm{Stat}(T) = \mathrm{Im}(S) = \mathrm{Pres}(\mathcal{U}) \cap \mathcal{A}_{\mathcal{U}} = \mathrm{Gen}(\mathcal{U}) \cap \mathcal{A}_{\mathcal{U}}.$$

2. *$T' : \mathcal{Q} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is fully faithful, and its left adjoint $S' : (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{Q}$ is conflation-exact.*

3. *T' induces an equivalence between $\mathcal{Q}$ and the full subcategory of $\mathrm{Ker}(S')$-closed objects of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$.*
   [An object M of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is $\mathrm{Ker}(S')$-closed if for every morphism $g : X \to X'$ in $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ with $\mathrm{Ker}(g), \mathrm{Coker}(g) \in \mathrm{Ker}(S')$, every morphism $X \to M$ extends uniquely to a morphism $X' \to M$.]

Corollary (Gabriel-Popescu 1964; Prest 1980; Kuhn 1994)

Let $\mathcal{A}$ be a Grothendieck category with a generating set of objects $\mathcal{U}$. Then the functor $T : \mathcal{A} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is fully faithful, and has an exact left adjoint S. Moreover, T induces an equivalence between $\mathcal{A}$ and the full subcategory of $\mathrm{Ker}(S)$-closed objects of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$.

In this case $\mathcal{U} \subseteq \mathrm{Gen}(\mathcal{U}) \cap \mathcal{A}_{\mathcal{U}} = \mathcal{A}$. The induced strongly inflation-exact structure in $\mathcal{A}$ coincides with the usual exact structure in $\mathcal{A}$ given by all short exact sequences, and so the left adjoint S of T is an exact functor in the usual sense.

Equivalence of categories

$\mathcal{U}$ is *self-small* if the canonical map $\bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{A}}(U, U_i) \cong \mathrm{Hom}_{\mathcal{A}}\left(U, \bigoplus_{i\in I} U_i\right)$ is an isomorphism for every object $U \in \mathcal{U}$ and every family $(U_i)_{i\in I}$ of objects of $\mathcal{U}$.

Corollary (C.-Năstăsescu)

Let $\mathcal{U}$ be a set of objects of an AB5 category $\mathcal{A}$ such that $\mathcal{U} \subseteq \mathcal{A}_{\mathcal{U}}$, and let $\mathcal{Q} = \mathrm{Gen}(\mathcal{U}) \cap \mathcal{A}_{\mathcal{U}}$. Then the functor $T' : \mathcal{Q} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is an equivalence of categories iff $\mathcal{U}$ is a self-small set of projective objects of $\mathcal{Q}$.

$\mathcal{U}$ is a *set of progenerators* of $\mathcal{A}$ if $\mathcal{U}$ is a self-small set of projective generators of $\mathcal{A}$.

Corollary (Menini 1988)

Let $\mathcal{U}$ be a set of objects of a Grothendieck category $\mathcal{A}$. Then the functor $T : \mathcal{A} \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is an equivalence of categories iff $\mathcal{U}$ is a set of progenerators of $\mathcal{A}$.

Proposition (C.-Năstăsescu; Cortés-Izurdiaga-C.-Saorín 2023)

The following are equivalent:

1. $\mathrm{Gen}(\mathcal{U}) \subseteq \mathcal{A}_{\mathcal{U}}$.
2. *For every* $A \in \mathrm{Gen}(\mathcal{U})$ *and for every epimorphism* $f : \bigoplus_{i \in I} U_i \to A$ *with* $U_i \in \mathcal{U}$ *for every* $i \in I$, $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$.
3. *For every* $A \in \mathrm{Gen}(\mathcal{U})$ *and for every epimorphism* $f : B \to A$ *with* $B \in \mathrm{Gen}(\mathcal{U})$, $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$.
4. *Every cokernel in* $\mathrm{Gen}(\mathcal{U})$ *is a deflation.*
5. $\mathrm{Gen}(\mathcal{U})$ *is a left quasi-abelian category.*

If $\mathrm{Gen}(\mathcal{U})$ *is an abelian (exact) subcategory of* $\mathcal{A}$, *then* $\mathrm{Gen}(\mathcal{U}) \subseteq \mathcal{A}_{\mathcal{U}}$ *and* $\mathrm{Gen}(\mathcal{U})$ *is a Grothendieck category.*

Proposition (C.-Năstăsescu)

The following are equivalent:

1. $\mathrm{Pres}(\mathcal{U}) \subseteq \mathcal{A}_{\mathcal{U}}$.
2. $\mathrm{Pres}(\mathcal{U})$ *is a coreflective subcategory of* $\mathcal{A}$, *and for every* $A \in \mathrm{Pres}(\mathcal{U})$ *and for every epimorphism* $f : \bigoplus_{i \in I} U_i \to A$ *with* $U_i \in \mathcal{U}$ *for every* $i \in I$, $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$.
3. $\mathrm{Pres}(\mathcal{U})$ *is a coreflective subcategory of* $\mathcal{A}$, *and for every* $A \in \mathrm{Pres}(\mathcal{U})$ *and for every epimorphism* $f : B \to A$ *with* $B \in \mathrm{Gen}(\mathcal{U})$, $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$.
4. $\mathrm{Pres}(\mathcal{U})$ *is a coreflective subcategory of* $\mathcal{A}$, *and for every* $A \in \mathrm{Pres}(\mathcal{U})$ *and for every epimorphism* $f : B \to A$ *with* $B \in \mathrm{Pres}(\mathcal{U})$, $\mathrm{Ker}(f) \in \mathrm{Gen}(\mathcal{U})$.

If $\mathrm{Pres}(\mathcal{U})$ *is a coreflective abelian exact subcategory of* $\mathcal{A}$, *then* $\mathrm{Pres}(\mathcal{U}) \subseteq \mathcal{A}_{\mathcal{U}}$ *and* $\mathrm{Pres}(\mathcal{U})$ *is a Grothendieck category.*

Tilting and generalizations

The set $\mathcal{U}$ is:

1. *tilting* if $\mathrm{Gen}(\mathcal{U}) = \mathcal{U}^{\perp}$where $\mathcal{U}^{\perp}$ is the class of objects A of $\mathcal{A}$ such that $\mathrm{Ext}^1_{\mathcal{A}}(-, A)|_{\mathcal{U}} = 0$ (Yoneda Ext).
2. *self-tilting* if $\mathrm{Gen}(\mathcal{U}) = \mathrm{Pres}(\mathcal{U})$ and $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective.
3. *w-Σ-$\mathcal{U}$-projective* if every object of $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective, that is, it is projective with respect to short exact sequences $0 \to K \to \bigoplus_{i\in I} U_i \to C \to 0$ with $U_i \in \mathcal{U}$ for every $i \in I$ and $K \in \mathrm{Gen}(\mathcal{U})$.

tilting $\Rightarrow$ self-tilting $\Rightarrow$ w-Σ-$\mathcal{U}$-projective.

Corollary (C.-Năstăsescu)

Let $\mathcal{U}$ be a w-Σ-$\mathcal{U}$-projective set in $\mathcal{A}$. Then:

1. $\mathrm{Gen}(\mathcal{U}) \subseteq \mathcal{A}_{\mathcal{U}}$ *iff* $\mathrm{Gen}(\mathcal{U}) = \mathrm{Pres}(\mathcal{U})$.
2. $\mathrm{Pres}(\mathcal{U})$ *is a coreflective full subcategory of $\mathcal{A}$ and* $\mathrm{Pres}(\mathcal{U}) \subseteq \mathcal{A}_{\mathcal{U}}$.

Corollary (C.-Năstăsescu)

1. *Assume that $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective. Then $T' : \mathrm{Pres}(\mathcal{U}) \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is fully faithful conflation-exact, and has a conflation-exact left adjoint. Moreover, T' induces an equivalence between $\mathrm{Pres}(\mathcal{U})$ and the full subcategory of $\mathrm{Ker}(S')$-closed objects of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$.*
2. *Assume that $\mathcal{U}$ is self-tilting. Then $T' : \mathrm{Gen}(\mathcal{U}) \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is fully faithful conflation-exact, and has a conflation-exact left adjoint. Moreover, T' induces an equivalence between $\mathrm{Gen}(\mathcal{U})$ and the full subcategory of $\mathrm{Ker}(S')$-closed objects of $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$.*

Corollary (C.-Năstăsescu)

1. *T induces an equivalence with inverse S' between* $\mathrm{Pres}(\mathcal{U})$ *and the full subcategory of* $\mathrm{Ker}(S')$*-closed objects of* $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ *iff* $\mathcal{U}$ *is w-*Σ*-*$\mathcal{U}$*-projective.*
2. *T induces an equivalence with inverse S' between* $\mathrm{Gen}(\mathcal{U})$ *and the full subcategory of* $\mathrm{Ker}(S')$*-closed objects of* $(\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ *iff* $\mathcal{U}$ *is self-tilting.*

Equivalences of categories

Corollary (C.-Năstăsescu)

Assume that $\mathcal{A}$ has a family of cogenerators $\mathcal{V}$. Then:

1. *T induces an equivalence between $\mathrm{Pres}(\mathcal{U})$ and $\mathrm{Copres}(T(\mathcal{V}))$ iff $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective.*
2. *T induces an equivalence between $\mathrm{Gen}(\mathcal{U})$ and $\mathrm{Copres}(T(\mathcal{V}))$ iff $\mathcal{U}$ is self-tilting.*
3. *T induces an equivalence between $\mathrm{Pres}(\mathcal{U})$ and $\mathrm{Cogen}(T(\mathcal{V}))$ iff $\mathcal{U}$ is self-small w-Σ-$\mathcal{U}$-projective.*
4. *T induces an equivalence between $\mathrm{Gen}(\mathcal{U})$ and $\mathrm{Cogen}(T(\mathcal{V}))$ iff $\mathcal{U}$ is self-small self-tilting.*

Equivalences induced by the set of f.p. (f.g.) objects

Corollary (C.-Năstăsescu)

Let $\mathcal{G}$ be a Grothendieck category, let $\mathcal{U} = \mathrm{fp}(\mathcal{G})$ and let $\mathcal{V}$ be a family of cogenerators of $\mathcal{G}$.

1. *If $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective, then $T' : \mathrm{Pres}(\mathcal{U}) \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is fully faithful conflation-exact, and has a conflation-exact left adjoint.*
2. *T induces an equivalence between $\mathrm{Pres}(\mathcal{U})$ and $\mathrm{Cogen}(T(\mathcal{V}))$ iff $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective.*

Corollary (C.-Năstăsescu)

Let $\mathcal{G}$ be a Grothendieck category, let $\mathcal{U} = \mathrm{fg}(\mathcal{G})$ and let $\mathcal{V}$ be a family of cogenerators of $\mathcal{G}$.

1. *If $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective, then $T' : \mathrm{Gen}(\mathcal{U}) \to (\mathcal{U}^{\mathrm{op}}, \mathrm{Ab})$ is fully faithful conflation-exact, and has a conflation-exact left adjoint.*
2. *T induces an equivalence between $\mathrm{Gen}(\mathcal{U})$ and $\mathrm{Cogen}(T(\mathcal{V}))$ iff $\mathcal{U}$ is w-Σ-$\mathcal{U}$-projective.*

Corollary (Colpi 1990)

Let U be a right R-module with $A = \mathrm{End}_R(U)$. Consider $T = \mathrm{Hom}_R(U, -) : \mathrm{Mod}(R) \to \mathrm{Mod}(A)$ and its restriction T'.

1. *If U is w-Σ-quasi-projective, then $T' : \mathrm{Pres}(U) \to \mathrm{Mod}(A)$ is fully faithful conflation-exact, and has a conflation-exact left adjoint $S' = - \otimes_A U : \mathrm{Mod}(A) \to \mathrm{Pres}(U)$.*
2. *If U is self-tilting, then $T' : \mathrm{Gen}(U) \to \mathrm{Mod}(A)$ is fully faithful conflation-exact, and has a conflation-exact left adjoint $S' = - \otimes_A U : \mathrm{Mod}(A) \to \mathrm{Gen}(U)$.*

Module categories

Corollary (Bazzoni 2010; Colpi 1990; Castaño-Iglesias-Gómez-Torrecillas-Wisbauer 2003)

Let U be a right R-module with $A = \mathrm{End}_R(U)$, and let V be a cogenerator of $\mathrm{Mod}(R)$. Let $T = \mathrm{Hom}_R(U, -) : \mathrm{Mod}(R) \to \mathrm{Mod}(A)$ and $V^* = \mathrm{Hom}_R(U, V)$. Then:

1. *T induces an equivalence between $\mathrm{Pres}(U)$ and $\mathrm{Copres}(V^*)$ iff U is w-Σ-quasi-projective.*
2. *T induces an equivalence between $\mathrm{Gen}(U)$ and $\mathrm{Copres}(V^*)$ iff U is self-tilting.*
3. *T induces an equivalence between $\mathrm{Pres}(U)$ and $\mathrm{Cogen}(V^*)$ iff U is self-small w-Σ-quasi-projective.*
4. *T induces an equivalence between $\mathrm{Gen}(U)$ and $\mathrm{Cogen}(V^*)$ iff U is self-small self-tilting.*

Let R is an algebra over a commutative ring k, and let $\mathcal{C}$ be an R-coring (in particular, a coalgebra over k).

Corollary (C.-Năstăsescu)

Let U be a right $\mathcal{C}$-comodule with $A = \mathrm{End}^{\mathcal{C}}(U)$. Consider $T = \mathrm{Hom}^{\mathcal{C}}(U, -) : \mathcal{M}^{\mathcal{C}} \to \mathrm{Mod}(A)$ and its restriction T'.

1. *Assume U is w-Σ-quasi-projective. Then $T' : \mathrm{Pres}(U) \to \mathrm{Mod}(A)$ is fully faithful conflation-exact, and has a conflation-exact left adjoint $S' = - \otimes_A U : \mathrm{Mod}(A) \to \mathrm{Pres}(U)$.*
2. *Assume U is self-tilting. Then $T' : \mathrm{Gen}(U) \to \mathrm{Mod}(A)$ is fully faithful conflation-exact, and has a conflation-exact left adjoint $S' = - \otimes_A U : \mathrm{Mod}(A) \to \mathrm{Gen}(U)$.*

Corollary (C.-Năstăsescu; Brzeziński-Wisbauer 2003; Castaño-Iglesias-Gómez-Torrecillas-Wisbauer 2003)

Let U be a right $\mathcal{C}$-comodule with $A = \mathrm{End}^{\mathcal{C}}(U)$, and let V be a cogenerator of $\mathcal{M}^{\mathcal{C}}$. Let $T = \mathrm{Hom}^{\mathcal{C}}(U, -) : \mathcal{M}^{\mathcal{C}} \to \mathrm{Mod}(A)$ and $V^* = \mathrm{Hom}^{\mathcal{C}}(U, V)$. Then:

1. *T induces an equivalence between $\mathrm{Pres}(U)$ and $\mathrm{Copres}(V^*)$ iff U is w-Σ-quasi-projective.*
2. *T induces an equivalence between $\mathrm{Gen}(U)$ and $\mathrm{Copres}(V^*)$ iff U is self-tilting.*
3. *T induces an equivalence between $\mathrm{Pres}(U)$ and $\mathrm{Cogen}(V^*)$ iff U is self-small w-Σ-quasi-projective.*
4. *T induces an equivalence between $\mathrm{Gen}(U)$ and $\mathrm{Cogen}(V^*)$ iff U is self-small self-tilting.*

A non-Grothendieck AB5 category

Example (Rickard 2020)

Rickard constructed a cocomplete abelian category that is not complete. Start with a fixed chain of fields $\{k_\alpha \mid \alpha \in \mathbf{On}\}$ indexed by the ordinals such that k_β / k_α is an infinite field extension for every $\alpha < \beta$.

- First, consider a category $\mathbf{C}$ as follows. An object of $\mathbf{C}$ consists of a k_α-vector space V_α for each ordinal α together a k_α-linear map $v_{\alpha,\beta} : V_\alpha \to V_\beta$ for each pair $\alpha < \beta$ of ordinals such that $v_{\alpha,\gamma} = v_{\beta,\gamma} v_{\alpha,\beta}$ whenever $\alpha < \beta < \gamma$. A morphism of $\mathbf{C}$ consists of a k_α-linear map $\varphi_\alpha : V_\alpha \to W_\alpha$ for each ordinal α such that $\varphi_\beta v_{\alpha,\beta} = w_{\alpha,\beta} \varphi_\alpha$ whenever $\alpha < \beta$, where $w_{\alpha,\beta} : W_\alpha \to W_\beta$ is a k_α-linear map corresponding to W_α whenever $\alpha < \beta$. If $\psi : U \to V$ and $\varphi : V \to W$ are morphisms in $\mathbf{C}$, their composition $\varphi\psi : U \to W$ is defined by $(\varphi\psi)_\alpha = \varphi_\alpha \psi_\alpha$ for every ordinal α. Then $\mathbf{C}$ is a (not locally small) abelian category with (small) products and coproducts in which (small) filtered colimits are exact. Kernels, cokernels, products and coproducts in $\mathbf{C}$ are "pointwise" constructions, which are obtained from the corresponding ones in the category of k_α-vector spaces for each ordinal α.

A non-Grothendieck AB5 category

Example (Rickard 2020)

- Next, for every ordinal α, consider the full subcategory α-**G** of **C** of the α-grounded objects in the following sense. An object V of **C** is called *α-grounded* if, for every $\beta > \alpha$, V_β is generated as a k_β-vector space by the image of the corresponding k_α-linear map $v_{\alpha,\beta} : V_\alpha \to V_\beta$. Then the full subcategory α-**G** of α-grounded objects of **C** is a Grothendieck category with generator $\bigoplus_{\beta \leq \alpha} M^\beta$, where $M^\beta_\gamma = k_\gamma$ for $\gamma \geq \beta$ and zero otherwise, and the associated linear maps $m^\beta_{\gamma,\delta} : k_\gamma \to k_\delta$ are the inclusions for $\beta \leq \gamma \leq \delta$.

- Finally, consider the full subcategory **G** of **C** consisting of the *grounded* objects in the sense that they are α-grounded for some ordinal α. Note that **G** is an abelian exact full subcategory of **C**. It follows that **G** is a (locally small) AB5 category that is not complete. Since every Grothendieck category must be complete, **G** also offers an example of an AB5 category which is not Grothendieck.

References I

- S. Bazzoni, *Equivalences induced by infinitely generated tilting modules*, Proc. Amer. Math. Soc. **138** (2010), 533–544.
- S. Bazzoni and S. Crivei, *One-sided exact categories*, J. Pure Appl. Algebra **217** (2013), 377–391.
- T. Brzeziński and R. Wisbauer, *Corings and comodules*, Cambridge University Press, New York, 2003.
- F. Castaño-Iglesias, P. Enache, C. Năstăsescu and B. Torrecillas, *Gabriel-Popescu type theorems and applications*, Bull. Sci. Math. **128** (2004), 323–332.
- F. Castaño-Iglesias, J. Gómez-Torrecillas and R. Wisbauer, *Adjoint functors and equivalences of subcategories*, Bull. Sci. Math. **127** (2003), 379–395.
- R. Colpi, *Some remarks on equivalences between categories of modules*, Comm. Algebra **18** (1990), 1935–1951.
- R. Colpi, *Tilting in Grothendieck categories*, Forum Math. **11** (1999), 735–759.
- R. Colpi and K. R. Fuller, *Tilting objects in abelian categories and quasitilted rings*, Trans. Amer. Math. Soc. **359** (2007), 741–765.
- M. Cortés-Izurdiaga, S. Crivei and M. Saorín, *Reflective and coreflective subcategories*, J. Pure Appl. Algebra **227** (2023), 107267.

- S. Crivei and C. Năstăsescu, *The Gabriel-Popescu Theorem revisited*, preprint.
- S. Crivei, C. Năstăsescu and L. Năstăsescu, *A generalization of the Mitchell Lemma: The Ulmer Theorem and the Gabriel-Popescu Theorem revisited*, J. Pure Appl. Algebra **216** (2012), 2126–2129.
- P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.
- P. Gabriel and N. Popescu, *Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes*, C. R. Acad. Sci. Paris **258** (1964), 4188–4190.
- G. Garkusha, *Grothendieck categories*, St. Petersburg Math. J. **13** (2002), 1–52.
- F. Genovese and J. Ramos González, *A derived Gabriel-Popescu theorem for t-structures via derived injectives*, Int. Math. Res. Not. IMRN **2023** (2023), 4695–4760.
- J. González Férez and L. Marín, *Exactness of direct limits in the category of firm modules*, J. Algebra **330** (2011), 298–313.
- R. Henrard and A.-C. van Roosmalen, *Localizations of (one-sided) exact categories*. arXiv:1903.10861

References III

Y. Imamura, *Grothendieck enriched categories*, Appl. Categ. Struct. **30** (2022), 1017–1041.

B. Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), 379–417.

N. J. Kuhn, *Generic representations of the finite general linear groups and the Steenrod algebra: I*, Amer. J. Math. **116** (1994), 327–360.

W. Lowen, *A generalization of the Gabriel-Popescu theorem*, J. Pure Appl. Algebra **190** (2004), 197–211.

C. Menini, *Gabriel-Popescu type theorems and graded modules.* In: Perspectives in Ring Theory (Antwerp, 1987), Kluwer, Dordrecht, 1988, pp. 239–251.

B. Mitchell, *A quick proof of the Gabriel-Popesco theorem*, J. Pure Appl. Algebra **20** (1981), 313–315.

C. Năstăsescu and F. Van Oystaeyen, *Methods of graded rings*, Lect. Notes Math. **1836**, Springer, Berlin, 2004.

M. Porta, *The Popescu-Gabriel theorem for triangulated categories*, Adv. Math. **225** (2010), 1669–1715.

M. Prest, *Elementary torsion theories and locally finitely presented categories*, J. Pure Appl. Algebra **18** (1980), 205–212.

References IV

- D. Quillen, *Higher algebraic K-theory. I.* In: Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math. **341**, 1973, Springer, Berlin, pp. 85–147.
- J. Rickard, *A cocomplete, but not complete abelian category*, Bull. Aust. Math. Soc. **101** (2020), 442–445.
- A. Rosenberg, *Homological algebra of non-commutative spaces, I*, preprint Max Planck, Bonn: MPIM2008-91, http://www.mpim-bonn.mpg.de/preprints.
- W. Rump, *On the maximal exact structure of an additive category*, Fund. Math. **214** (2011), 77–87.
- J. P. Schneiders, *Quasi-abelian categories and sheaves*, Mém. Soc. Math. Fr. (N.S.) **76** (1999), pp. 1–140.
- M. Takeuchi, *A simple proof of Gabriel and Popesco's theorem*, J. Algebra **18** (1971), 112–113.
- F. Ulmer, *A flatness criterion in Grothendieck categories*, Invent. Math. **19** (1973), 331–336.
- E. M. Vitale, *Localizations of algebraic categories II*, J. Pure Appl. Algebra **133** (1998), 317–326.