Closure properties of orthogonal classes associated to cosilting objects

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Notations

Let **D** be a triangulated category. For $\mathcal{C} \subseteq \mathbf{D}$ and $n \in \mathbb{Z}$, we denote

$$\mathcal{C}^{\perp_{>n}} = \{X \in \mathbf{D} \mid \mathrm{Hom}_{\mathbf{D}}(C, X[i]) = 0, \text{ for all } C \in \mathcal{C} \text{ and all } i > n\}$$

$$^{\perp_{>n}}\mathcal{C} = \{X \in \mathbf{D} \mid \mathrm{Hom}_{\mathbf{D}}(X, C[i]) = 0, \text{ for all } C \in \mathcal{C} \text{ and all } i > n\}.$$

Similar definitions are obtained by repacing $> n$ with $\geq n$, $\leq n$, $< n$, n etc.

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Torsion Pairs

A *torsion pair* in **D** is a pair $(\mathcal{U}, \mathcal{V})$ of subcategories **D** such that:

1. $\mathcal{U}^{\perp_0} = \mathcal{V}$ and $\mathcal{U} = {}^{\perp_0}\mathcal{V}$.
2. For every $X \in \mathbf{D}$ there exists a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

t-structures

A torsion pair $(\mathcal{U}, \mathcal{V})$ is called *t-structure* if in addition $\mathcal{U}$ is closed under positive: $\mathcal{U}[1] \subseteq \mathcal{U}$.

We say that $\mathcal{U}$ is the aisle, respectively $\mathcal{V}$ is the coaisle of the t-structure $(\mathcal{U}, \mathcal{V})$.

Remark that:

- $\mathcal{U}[1] \subseteq \mathcal{U} \Leftrightarrow \mathcal{V}[-1] \subseteq \mathcal{V}$;

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Silting and cosilting

An object $T \in \mathbf{D}$ is a *silting object* if the pair $(T^{\perp_{>0}}, T^{\perp_{\leq 0}})$ is a t-structure in $\mathbf{D}$.

The object $C \in \mathbf{D}$ is *cosilting* if $({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ is a t-structure.

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Outline

Neeman's Theorem

Suppose that **D is a well-generated triangulated category.**

Theorem (Neeman)

If $U \in \mathbf{D}$ and $\overline{\langle U\rangle}^{[-\infty,0]}$ is the smallest subcategory that contains U, is closed under positive shifts, coproducts and extensions then the pair $(\overline{\langle U\rangle}^{[-\infty,0]}, U^{\perp_{\leq 0}})$ is a t-structure.

Using this we deduce the following characterization

Theorem (Angeleri, Hrbek, Marks, Psaroudakis, Saórin, Vitória ...,)

An object $T \in \mathbf{D}$ is silting if and only if

- (S1) $T \in T^{\perp_{>0}}$.
- (S2) $T^{\perp_{>0}}$ is closed under coproducts.
- (S3) T generates $\mathbf{D}$, that is $T^{\perp_{\mathbb{Z}}} = \{0\}$.

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Some remarks

- The proof of Neeman's theorem uses the Brown Representability Theorem.
- It is not known if the well generated categories also satisfy the Brown Representability Theorem for the dual, hence we don't have a dual version for Nemman's Theorem.
- We cannot dualize the above corollary to obtain a similar result for cosilting objects.
- Such a characterization is known when ${}^{\perp_{>0}}U$ is already a coaisle of a t-structure, e.g. when U is pure-injective object (in compactly generated categories).
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A construction

Proposition

Let U be an object from $\mathbf{D}$. Assume that there exists a cocomplete pre-aisle $\mathcal{A}$ (i.e. it is closed under extensions, direct sums, direct summands, and positive shifts) in $\mathbf{D}$ such that

1) $U \in \mathcal{A}$,
2) $\mathrm{Hom}_{\mathbf{D}}(U, \mathcal{A}[n]) = 0$ for some positive integer n, and

(S1.5) $\mathrm{Add}(U) \subseteq U^{\perp_{>0}}$.

Then for every $X \in \mathbf{D}$ there exists a triangle $Y \to X \to Z \to Y[1]$ such that

a. $Z \in U^{\perp_{\leq 0}}$,
b. $Y \in U^{\perp_{>0}}$,
c. $\mathrm{Hom}_{\mathbf{D}}(Y, U^{\perp_{\leq 0}}) = 0$.

A construction

We construct inductively a sequence of morphisms $f_k : X_k \to X_{k+1}$ in the following way:

- $X_0 = X$;
- If X_k is constructed then we consider a triangle $U[k]^{(I_k)} \stackrel{\alpha_k}{\to} X_k \stackrel{f_k}{\to} X_{k+1}$, where α_k is an $\mathrm{Add}(U[k])$-preecover.

For every $i > k$, we consider the morphism $f_{ki} : X_k \to X_i$ that are obtained as the composition of the morphisms $f_k, \dots, f_{i-1}$. Moreover, $f_{ii} : X_i \to X_i$, $i \geq 0$, will be the indentity maps. We denote by S_{ki} the cone of f_{ki}.

From (S1.5) it follows that

- if $k \geq 0$ then $\mathrm{Hom}_{\mathbf{D}}(U[j], X_{k+1}) = 0$ for all $j = \overline{0, k}$;
- if $k < i$ then $S_{ki} \in \mathrm{Add}(U[k+1]) * \cdots * \mathrm{Add}(U[i]) \subseteq \mathcal{A}[k+1]$.

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Moving to homotopy colimits

Let $k \geq 0$.

There exists a commutative diagram such that all lines and columns are triangles:

$$\begin{array}{ccccc}
\oplus_{i\geq k} X_k & \xrightarrow{1-\text{shift}} & \oplus_{i\geq k} X_k & \longrightarrow & X_k \\
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\end{array}$$

Up to isomorphism, Z does not depends on k since it is the homotopy colimit of the sequence $(f_i)_{i\geq 0}$.

Using $k > n$ we obtain $Z \in U^{\perp_{\leq 0}}$. Using $k = 0$, we have $Y = C_0[-1]$ verifies (b) and (c). Since $X_0 = X$, the proof is complete.

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The characterization for (co)silting

Corollary

Assume the **D** is a triangulated category with coproducts. An object $U \in \mathbf{D}$ is silting if and only if:

(S1) $U \in U^{\perp_{>0}}$;

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Corollary

Assume the **D** is a triangulated category with products. An object $U \in \mathbf{D}$ is cosilting if and only if:

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(C2) ${}^{\perp_{>0}}U$ is closed under direct products,

(C3) ${}^{\perp_{\mathbb{Z}}}U = 0$.

Pure-injective cosilting

An object U in $\mathcal{C}$ is called (AMV) *partial cosilting* if the class ${}^{\perp_{>0}}U$ is a coaisle of a t-structure and $U \in {}^{\perp_{>0}}U$.

Proposition

The following are equivalent for an object U in a compactly generated category such that:

1. ${}^{\perp_{>0}}U$ is closed under products and "pure-subobjects";
2. ${}^{\perp_{>0}}U$ is closed under products, and T is pure-injective;
3. U is a pure injective (AMV) partial silting object.