A definable approach to tensor triangular geometry

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Joint work with Jordan Williamson

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Define the $\otimes$-Ziegler topology on T to have closed sets the $\otimes$-closed definable subcategories; the closure operation corresponds to

$$\mathrm{Def}^{\otimes}(\mathsf{X}) = \mathrm{Def}(c \otimes x : x \in \mathsf{X}, c \in \mathsf{T}^{\mathrm{c}})$$

We let $\mathrm{Zg}^{\otimes}(\mathsf{T})$ be the Ziegler spectrum equipped with the $\otimes$-Ziegler topology.

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Definition. The *homological spectrum* of T^{c}, denoted $\mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}})$, is a topological space whose points are homological primes, topologised by a basis of closed sets given by

$$\mathsf{supp}^{\mathsf{h}}(A) = \{\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}}) : \mathsf{y}A \notin \mathcal{B}\}$$

as A runs over T^{c}.

If $\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}})$, there is a unique pure injective object $E_{\mathcal{B}} \in \mathsf{T}$. The localisation adjunction

$$\mathsf{Mod}(\mathsf{T}^{\mathrm{c}}) \underset{R}{\overset{Q}{\rightleftarrows}} \mathsf{Mod}(\mathsf{T}^{\mathrm{c}})/\varinjlim \mathcal{B}$$

gives an injective object $R(\mathbb{E}Q\mathsf{y}\mathbb{1}) \in \mathsf{Mod}(\mathsf{T}^{\mathrm{c}})$, which is isomorphic to $\mathsf{y}E_{\mathcal{B}}$.

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However, $E_{\mathcal{B}}$ need not be indecomposable, so sending $\mathcal{B}$ to $E_{\mathcal{B}}$ does not give a map $\mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}}) \to \mathsf{pinj}(\mathsf{T})$.

Lemma. Let U be any compactly generated triangulated category and $\mathcal{S}$ a Serre subcategory of $\mathrm{mod}(\mathrm{U}^{\mathrm{c}})$. Then there is an equivalence of categories

$$\mathrm{y}^{-1}R\colon \mathrm{Inj}(\mathrm{Mod}(\mathrm{U}^{\mathrm{c}})/\varinjlim \mathcal{S}) \xrightarrow{\simeq} \mathrm{Pinj}(\mathrm{U}) \cap \mathcal{D}(\mathcal{S}).$$

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Corollary. Let $\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathsf{T}^{\mathrm{c}})$, then $\mathcal{D}(\mathcal{B}) = \mathrm{Def}^{\otimes}(E_{\mathcal{B}})$, and this is a simple $\otimes$-closed definable subcategory of T.

We get a well defined map

$$\Phi\colon \mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}}) \to \mathsf{KZg}^{\otimes}(\mathsf{T})$$

given by sending $\mathcal{B}$ to $[X]$, where $X \in \mathsf{pinj}(\mathsf{T}) \cap \mathsf{Def}^{\otimes}(E_{\mathcal{B}})$.

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Theorem. The map Φ gives a bijection

$$\mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}}) \to \mathsf{Cl}(\mathsf{KZg}^{\otimes}(\mathsf{T})).$$

The inverse is $[X] \mapsto \mathsf{mod}(\mathsf{T}^{\mathrm{c}}) \cap \mathsf{Ker}(\mathsf{y}X \otimes -)$.

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$$(A)_{\otimes} = \{X \in \mathsf{Cl}(\mathsf{KZg}^{\otimes}(\mathsf{T})) : \underline{\mathrm{Hom}}(A, X) = 0\}$$

for $A \in \mathsf{T}^{\mathrm{c}}$, and define the GZ-topology to be that having a basis of open sets given by $\{(A)_{\otimes} : A \in \mathsf{T}^{\mathrm{c}}\}$.

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Proposition. Φ induces a homeomorphism

$$\mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}}) \simeq \mathsf{Cl}(\mathsf{KZg}^{\otimes}(\mathsf{T}))^{\mathsf{GZ}}.$$

Balmer showed there is is a canonical surjective map $\mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}}) \to \mathsf{Spc}(\mathsf{T}^{\mathrm{c}})$ given by $\mathcal{B} \mapsto \mathsf{y}^{-1}\mathcal{B}$.

Proposition. (Barthel-Heard-Sanders, B.-Williamson) The canonical map is a surjection if and only if $\mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}})$ - equivalently $\mathsf{Cl}(\mathsf{KZg}^{\otimes}(\mathsf{T}))^{\mathsf{GZ}}$ - is T_0.

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Consider the following two conditions

(1) Λ preserves cohomological functors;

(2) $\overline{F}$ is lax monoidal and we have the projection formula

$$\overline{F}X \otimes Y \simeq \overline{F}(X \otimes \Lambda Y).$$

Theorem. Let $F\colon \mathsf{T} \to \mathsf{U}$ be a definable functor satisfying the above conditions. Then F preserves simple $\otimes$-closed definable subcategories. Thus, if $\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}})$,

$$\mathsf{pure}(F\mathsf{Def}^{\otimes}(E_{\mathcal{B}})) = \mathsf{Def}^{\otimes}(FE_{\mathcal{B}})$$

is a simple $\otimes$-closed definable subcategory of U.
In particular, the assignment

$$\mathcal{B} \mapsto \mathsf{Ker}(- \otimes \mathsf{y}FE_{\mathcal{B}}) \cap \mathsf{mod}(\mathsf{U}^{\mathrm{c}})$$

defines a map

$$\mathsf{Spc}^{\mathsf{h}}(F)\colon \mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathrm{c}}) \to \mathsf{Spc}^{\mathsf{h}}(\mathsf{U}^{\mathrm{c}}).$$