Topological Quantum Computation
A very basic introduction

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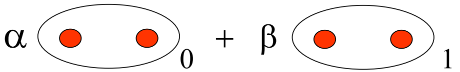
PhD Course on Quantum Computing

Part III

Fibonacci Anyons Basics

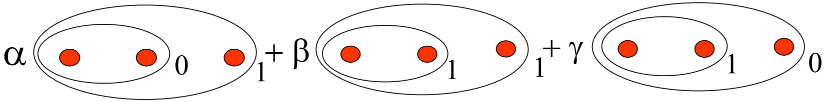
- The "fusion" rule for combining q-spin is: $\boxed{1 \times 1 = 0 + 1}$

This means that two Fibonacci anyons can have total q-spin 0 or 1, or be in any quantum superposition of these two states.



Two dimensional Hilbert space

Three Fibonacci anyons → Three dimensional Hilbert space

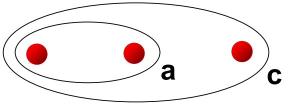


For **N** Fibonacci anyons Hilbert space dimension is **Fib(N-1)**

The F Matrix

Changing fusion bases:

$$\sum_a F_{ab}^c$$



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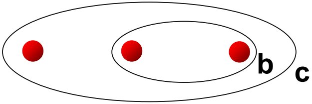
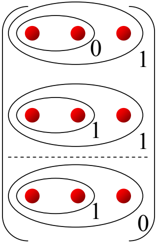


$$\underbrace{\left(\begin{array}{cc:c} -\tau & \sqrt{\tau} & 0 \\ \sqrt{\tau} & \tau & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right)}_{F_{ab}^c}$$



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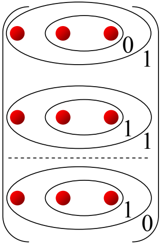
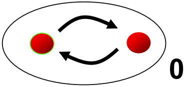


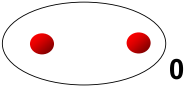
$$\tau = \frac{\sqrt{5}-1}{2}$$

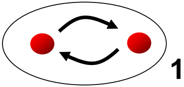
The R Matrix

Exchanging Particles:

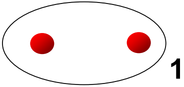


$$= e^{-i4\pi/5}$$





$$= e^{i3\pi/5}$$



$$R = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$

F and R must satisfy certain consistency conditions (the “pentagon” and “hexagon” equations). For Fibonacci anyons these equations *uniquely determine* F and R.

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1. *Topological Quantum Compiling* , L. Hormozi, G. Zikos and N. Bonesteel, *Phy. Rev. Lett. B 75, 165310* — 2007.

2. *Topological Quantum Compiling* , S. Simon, N. Bonesteel, M. Freedman, N. Petrovic and L. Hormozi *Phy. Rev. Lett. 96, 070503* — 2006.

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- How to map anyons to qubits
- How to implement braiding in the qubit space
- How to measure fusion outcomes in the qubit basis.

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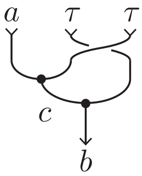
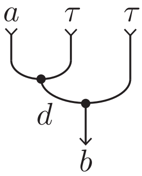


$$= \sum_{d} \left(B^{b}_{a\tau\tau}\right)^{d}_{c} \cdot$$



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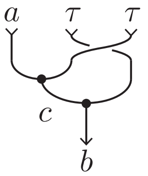

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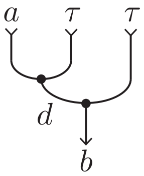

i.e., explicitly,

$$|a\tau; c\rangle \, |c\tau; b\rangle = \sum_{d} (B^{b}_{a\tau\tau})^{d}_{c} \, |a\tau; d\rangle \, |d\tau; b\rangle$$

with $|a\tau; c\rangle \, |c\tau; b\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$ and $|a\tau; d\rangle \, |d\tau; b\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$.

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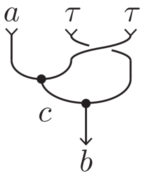

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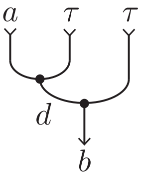

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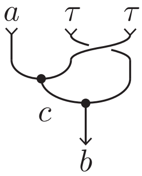

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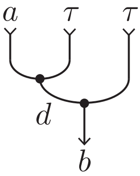

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$$B : \mathcal{H}_d \otimes \mathcal{H}_d = \mathbb{C}^{d^2} \to \mathbb{C}^{d^2}.$$

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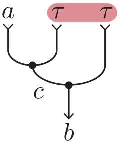

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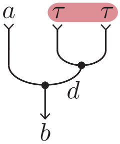

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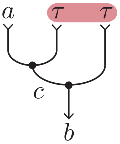

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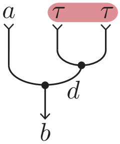

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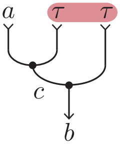

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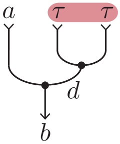

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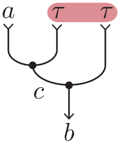

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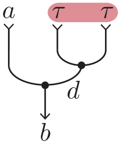

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Now we can perform a projective measurement on the second qubit in the $\{|\tau\tau;1\rangle\,, |\tau\tau;\tau\rangle\}$ basis and sample the probabilities of getting 1 and τ.

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- Topological Algorithms and Computational Architectures ?

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Hymn to the Anyon

Anyon, anyon, where do you roam?
Braid for a while before you go home.

Though you're condemned just to slide on a table,
A life in 2D also means that you're able
To be of a type neither Fermi nor Bose
And to know left from right — that's a kick, I suppose.

You and your buddy were made in a pair
Then wandered around, braiding here, braiding there.
You'll fuse back together when braiding is through
Well bid you adieu as you vanish from view.
. . .
Anyon, anyon, where do you roam?
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John Preskill