

Topological Quantum Computation

A very basic introduction

Alessandra Di Piero

`alessandra.dipierro@univr.it`

Dipartimento di Informatica
Università di Verona

PhD Course on Quantum Computing

Part III

- 1 Recap
- 2 Universality of TQC
- 3 Conclusions

Fibonacci Anyons Basics

$$1 \times 1 = 0 + 1$$

This means that two Fibonacci anyons can have total q-spin 0 or 1, or be in **any quantum superposition of these two states**.

$$\alpha \text{ (two red dots)}_0 + \beta \text{ (two red dots)}_1$$

Two dimensional Hilbert space

Three Fibonacci anyons \rightarrow Three dimensional Hilbert space

Diagram illustrating the decomposition of a set into three parts: α , β , and γ . Each part contains nested ellipses and red dots representing elements. Part α has an outer ellipse labeled α and an inner ellipse labeled 0 , with two red dots inside. Part β has an outer ellipse labeled β and an inner ellipse labeled 1 , with two red dots inside. Part γ has an outer ellipse labeled γ and an inner ellipse labeled 1 , with two red dots inside. The parts are separated by plus signs.

For N Fibonacci anyons Hilbert space dimension is $\text{Fib}(N-1)$

The F Matrix

Changing fusion bases:

$$\sum_a F_{ab}^c \text{ (diagram with two nested ellipses, left inner ellipse has two red dots, right inner ellipse has one red dot, labeled 'a' and 'c') } = \text{ (diagram with two nested ellipses, left inner ellipse has one red dot, right inner ellipse has two red dots, labeled 'b' and 'c') }$$

$$\underbrace{\begin{pmatrix} -\tau & \sqrt{\tau} & 0 \\ \sqrt{\tau} & \tau & 0 \\ \hline 0 & 0 & 1 \end{pmatrix}}_{F_{ab}^c} \begin{pmatrix} \text{diagram with two nested ellipses, left inner ellipse has two red dots, right inner ellipse has one red dot, labeled '0' and '1'} \\ \text{diagram with two nested ellipses, left inner ellipse has one red dot, right inner ellipse has two red dots, labeled '1' and '1'} \\ \hline \text{diagram with two nested ellipses, left inner ellipse has one red dot, right inner ellipse has two red dots, labeled '1' and '0'} \end{pmatrix} = \begin{pmatrix} \text{diagram with two nested ellipses, left inner ellipse has two red dots, right inner ellipse has one red dot, labeled '0' and '1'} \\ \text{diagram with two nested ellipses, left inner ellipse has one red dot, right inner ellipse has two red dots, labeled '1' and '1'} \\ \hline \text{diagram with two nested ellipses, left inner ellipse has one red dot, right inner ellipse has two red dots, labeled '1' and '0'} \end{pmatrix}$$

$$\tau = \frac{\sqrt{5}-1}{2}$$

The R Matrix

Exchanging Particles:

$$\text{Diagram 1} = e^{-i4\pi/5} \text{Diagram 2}$$

$$\text{Diagram 3} = e^{i3\pi/5} \text{Diagram 4}$$

$$R = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$

F and R must satisfy certain consistency conditions (the “pentagon” and “hexagon” equations). For Fibonacci anyons these equations *uniquely determine* F and R .

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1. *Topological Quantum Compiling* , L. Hormozi, G. Zikos and N. Bonesteel, *Phy. Rev. Lett. B* 75, 165310 — 2007.
 2. *Topological Quantum Compiling* , S. Simon, N. Bonesteel, M. Freedman, N. Petrovic and L. Hormozi *Phy. Rev. Lett.* 96, 070503 — 2006.

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Recall:

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- How to map anyons to qubits
- How to implement braiding in the qubit space
- How to measure fusion outcomes in the qubit basis.

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For Fibonacci anyons

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- Thus we need $n-2$ qutrit or $m = \lceil 3(n-1)/2 \rceil$ qubit.

Braiding

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The diagram shows an equation between two braiding configurations. On the left, a line labeled a enters from the top left, goes down to a dot, then loops around to the right, passing over a line labeled b that enters from the bottom and goes up to a dot. From this dot, a line labeled c exits to the left. Above the crossing, two lines labeled τ enter from the top and cross each other. On the right, the same setup is shown but the line labeled c is replaced by a line labeled d that exits to the right. The equation is set equal to a summation over d of the matrix element $(B_{a\tau\tau}^b)_c^d$ multiplied by the right-hand diagram.

$$\text{Diagram 1} = \sum_d (B_{a\tau\tau}^b)_c^d \cdot \text{Diagram 2}$$

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$$\begin{array}{c} a \\ \tau \quad \tau \\ \text{braid} \\ c \downarrow b \end{array} = \sum_d (B_{a\tau\tau}^b)_c^d \cdot \begin{array}{c} a \\ \tau \quad \tau \\ \text{braid} \\ d \downarrow b \end{array}$$

i.e., explicitly,

$$|a\tau; c\rangle |c\tau; b\rangle = \sum_d (B_{a\tau\tau}^b)_c^d |a\tau; d\rangle |d\tau; b\rangle$$

with $|a\tau; c\rangle |c\tau; b\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$ and $|a\tau; d\rangle |d\tau; b\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d$.

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$$B : \mathcal{H}_d \otimes \mathcal{H}_d = \mathbb{C}^{d^2} \rightarrow \mathbb{C}^{d^2}.$$

Fusion

We can calculate the outcome of fusing two anyons in the standard basis by applying the F -matrix:

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The diagram shows an equality between two braiding configurations. On the left, three strands enter from the top: the leftmost is labeled a , the middle and rightmost are grouped by a red oval labeled τ and τ respectively. The middle and right strands cross each other, then both cross the left strand. A dot is placed on the middle strand between the two crossings, labeled c below it. The strands then merge into a single strand exiting at the bottom, labeled b . On the right, the same three strands enter, but the middle and right strands cross each other first, then the left strand crosses them. A dot is placed on the right strand between the two crossings, labeled d below it. The strands then merge into a single strand exiting at the bottom, labeled b . Between the two diagrams is an equals sign followed by a summation over d of the matrix element $(F_{a\tau\tau}^b)^d$.

$$\begin{array}{c} a \\ \downarrow \end{array} \begin{array}{c} \tau \quad \tau \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ c \\ \downarrow \\ b \end{array} = \sum_d (F_{a\tau\tau}^b)^d \cdot \begin{array}{c} a \\ \downarrow \end{array} \begin{array}{c} \tau \quad \tau \\ \downarrow \end{array} \begin{array}{c} \downarrow \\ d \\ \downarrow \\ b \end{array}$$

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We can calculate the outcome of fusing two anyons in the standard basis by applying the F -matrix:

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$$\begin{aligned}
 |a\tau; c\rangle |c\tau; b\rangle &= \sum_d (F_{a\tau\tau}^b)_c^d |ad; b\rangle |\tau\tau; d\rangle \\
 &= (F_{a\tau\tau}^b)_c^1 |a1; b\rangle |\tau\tau; 1\rangle + (F_{a\tau\tau}^b)_c^\tau |a\tau; b\rangle |\tau\tau; \tau\rangle
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 &= (F^b_{a\tau\tau})^1_c |a1; b\rangle |\tau\tau; 1\rangle + (F^b_{a\tau\tau})^\tau_c |a\tau; b\rangle |\tau\tau; \tau\rangle
 \end{aligned}$$

Now we can perform a **projective measurement** on the second qubit in the $\{|\tau\tau; 1\rangle, |\tau\tau; \tau\rangle\}$ basis and sample the probabilities of getting 1 and τ .

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- Topological Algorithms and Computational Architectures ?

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Hymn to the Anyon

*Anyon, anyon, where do you roam?
Braid for a while before you go home.*

*Though you're condemned just to slide on a table,
A life in 2D also means that you're able
To be of a type neither Fermi nor Bose
And to know left from right — that's a kick, I suppose.*

*You and your buddy were made in a pair
Then wandered around, braiding here, braiding there.
You'll fuse back together when braiding is through
Well bid you adieu as you vanish from view.*

...

*Anyon, anyon, where do you roam?
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John Preskill