Topological Quantum Computation
A very basic introduction

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PhD Course on Quantum Computing

Part II

Models of Anyons

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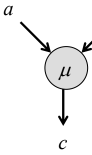
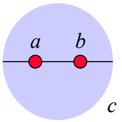
- The F-matrix (expressing associativity of fusion)
- The R-matrix (braiding rules)

Fusion

Fusion Rules express the interaction of two anyons:



$(\mu = 1, 2, 3, \ldots, N_{ab}{}^{c})$



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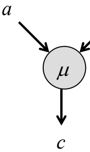

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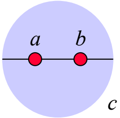
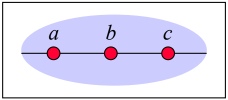


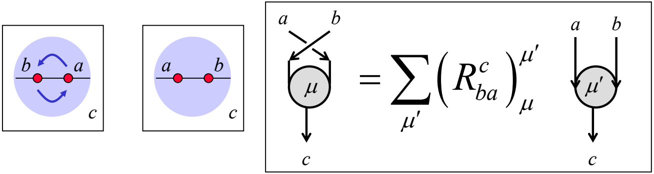
F-matrix encodes the associativity of the fusion rules:



$$(a \times b) \times c = a \times (b \times c)$$

Braiding Operator

Braiding Two neighboring anyons are exchanged counterclockwise:



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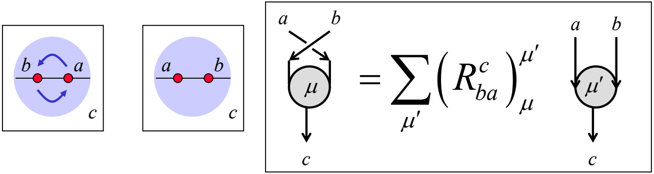

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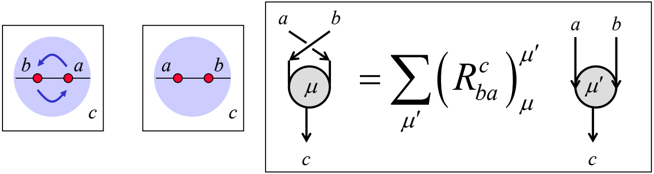

***R*-matrix** Unitary matrix representing the swapping

***B*-matrix** Since two particles may have no direct fusion channel, B is in general a composition of R and F matrices.

Fibonacci Theory

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Fusion Rules

- $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$,
- $\mathbf{1} \otimes \tau = \tau \otimes \mathbf{1} = \tau$,
- $\tau \otimes \tau = \mathbf{1} \oplus \tau$.

Thus for three charges:

- $\tau \otimes \tau \otimes \tau = 2 \cdot \tau \oplus 1 \cdot \mathbf{1}$.

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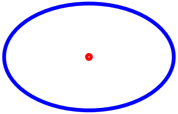
We need to specify the *F*-matrix and *R*-matrix.

Fibonacci Theory: Fusion Space



One particle in the vacuum:
The (enclosed) total charge is τ.

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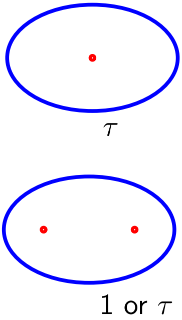

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For two particles, we apply the fusion rules. The resulting total charge can be either 1 or τ.
$\Rightarrow$ space dimension $dim = 2$

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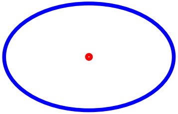

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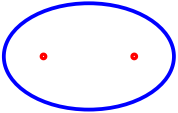

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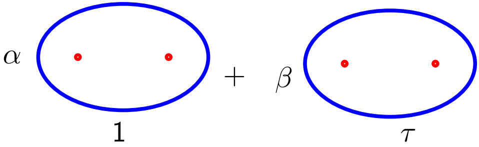

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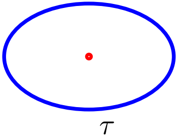

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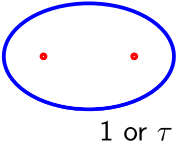

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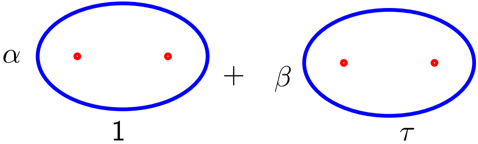

Note: The storing of information is **non-local**!

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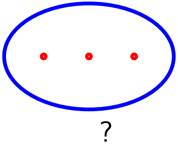

What is the total charge resulting from the fusion of three particles?

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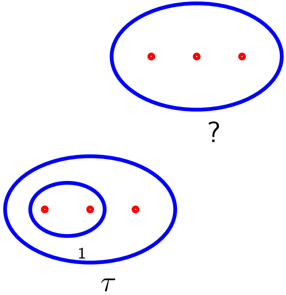

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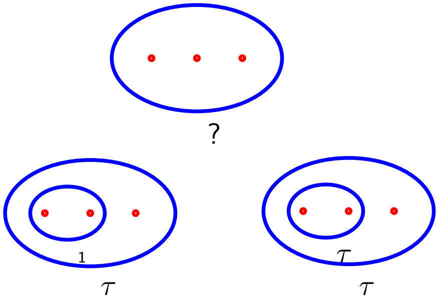

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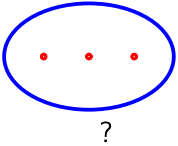

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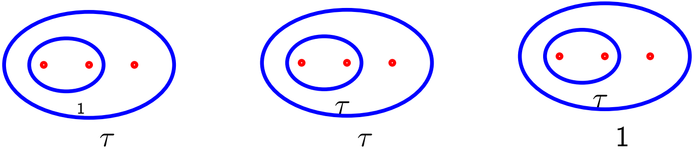

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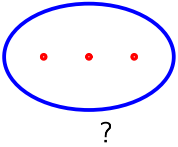

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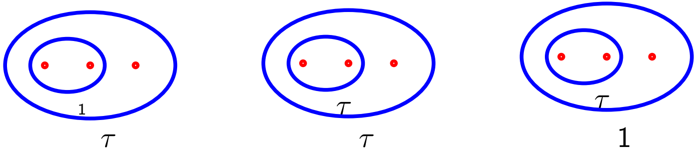
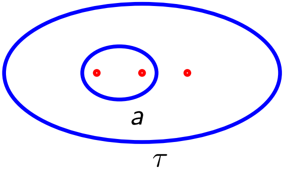


3-dimensional Hilbert space

The F-matrix



$a = 1, \tau$

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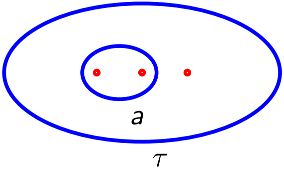

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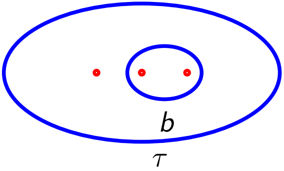

$b = 1, \tau$

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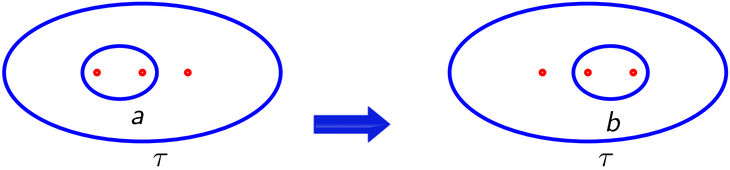

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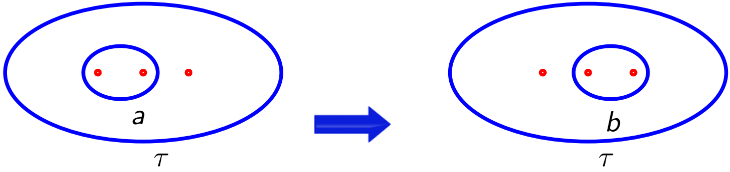

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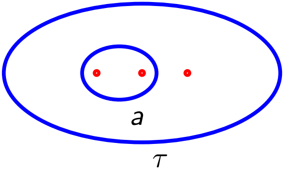
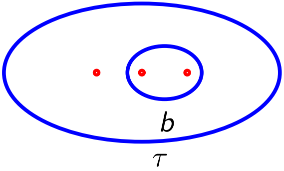
Base change : F-matrix

The F-matrix



$$= \sum_b \mathbf{F}_{ab}$$



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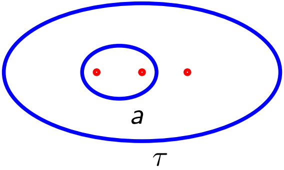

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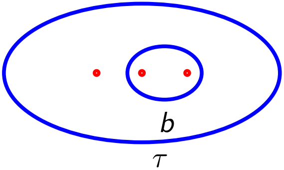

$a = 1, \tau$

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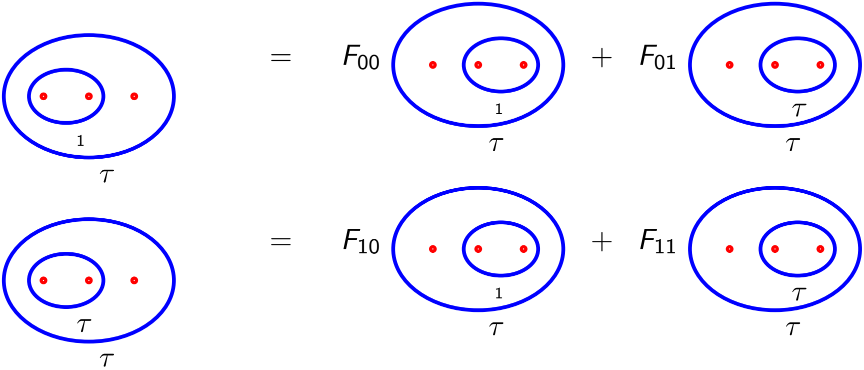
$$\mathbf{F} = \begin{bmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{bmatrix}$$

Base Change

The only two non trivial cases:



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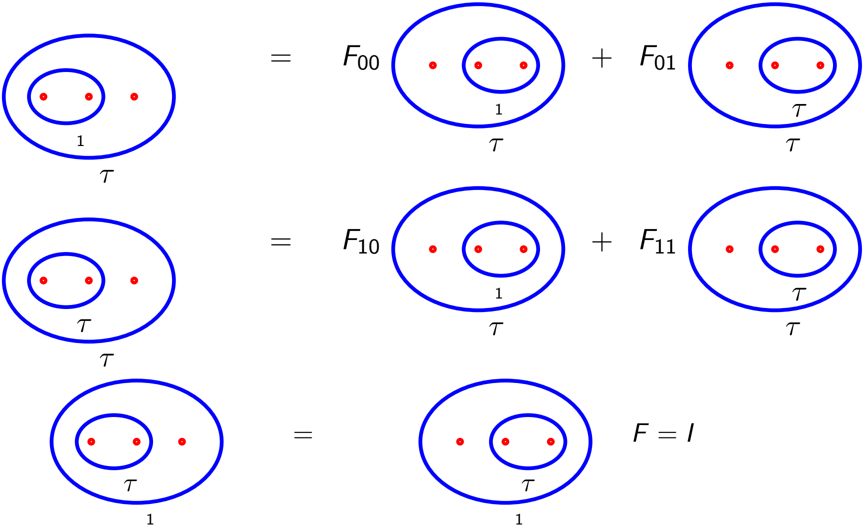

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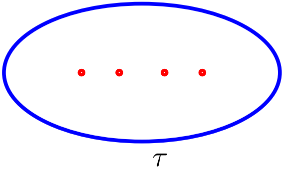

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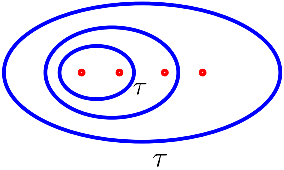

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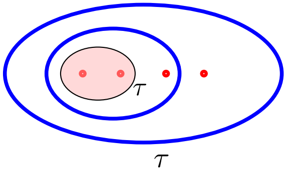

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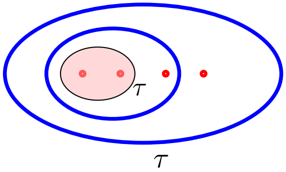

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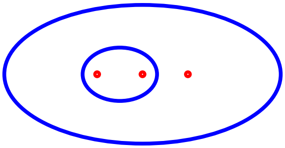

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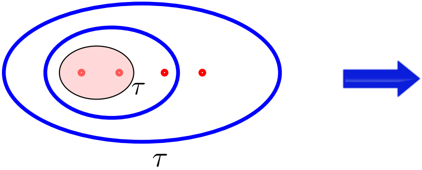

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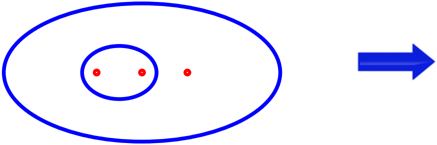

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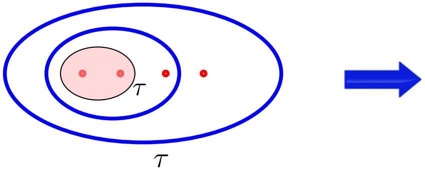

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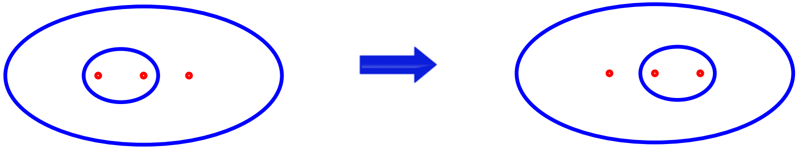

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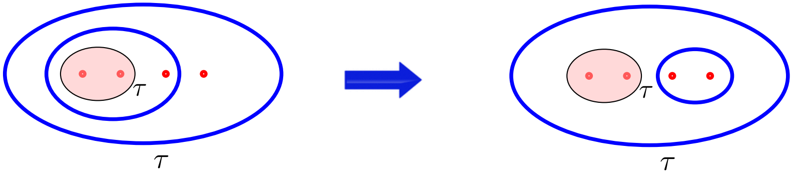

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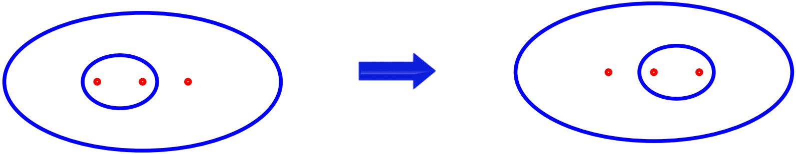
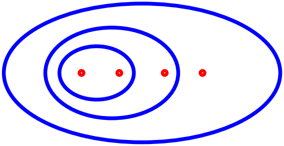


Consistency of F



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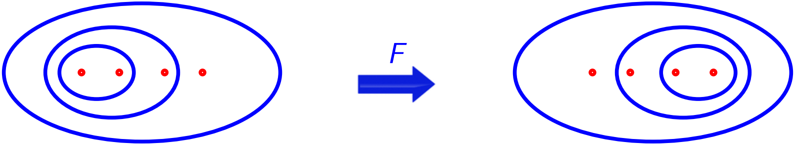

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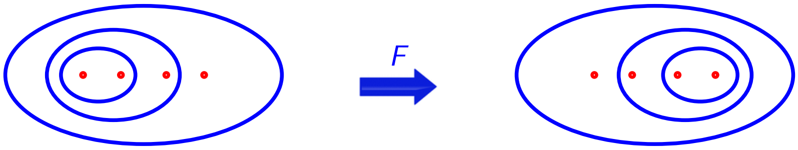
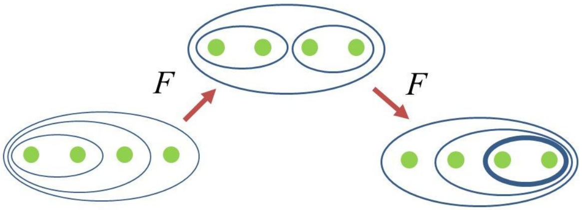


How many different ways to go from one base to the other?

The Pentagon Equation



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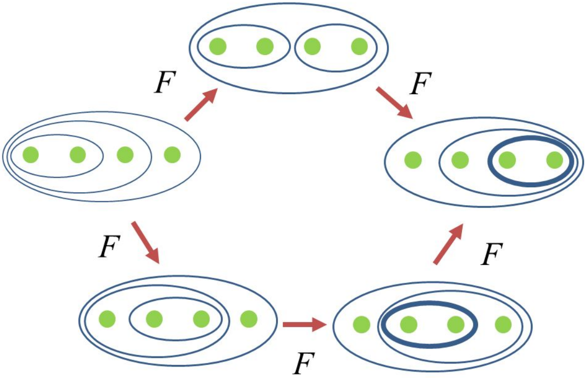
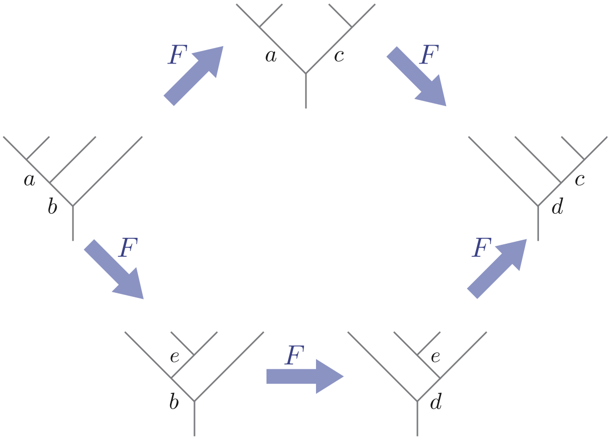


The pentagon equation is obtained by imposing the condition that the above diagram commutes.

The Pentagon Equation



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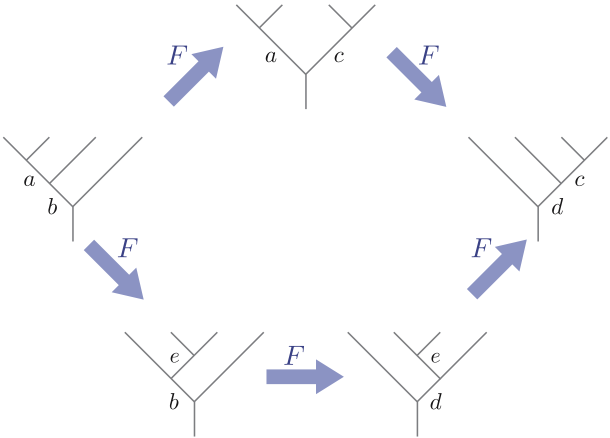
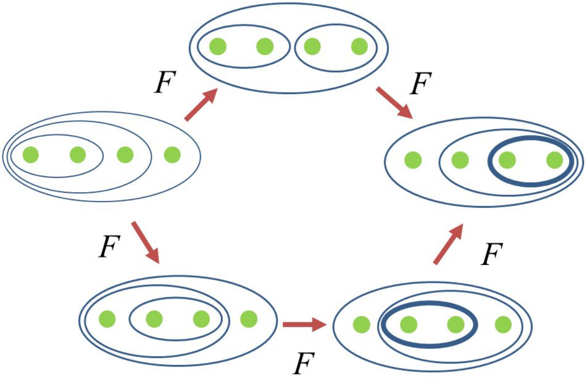


The only non-trivial case is the one with five τ's at the outer edges.
The explicit equation for Fibonacci anyons is:

$$(F_\tau^{\tau\tau c})_a^d (F_\tau^{a\tau\tau})_b^c = (F_d^{\tau\tau\tau})_e^c (F_\tau^{\tau e\tau})_b^d (F_b^{\tau\tau\tau})_a^e$$

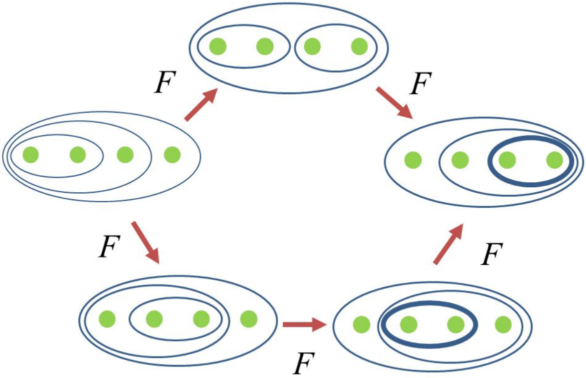
The Pentagon Equation



The unique unitary solution (up to irrelevant phase factor) is the unitary matrix (set $b = c = 1$ and use unitarity):

$$F = \begin{bmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{bmatrix} \text{ where } \varphi = \frac{1+\sqrt{5}}{2}$$

The Pentagon Equation



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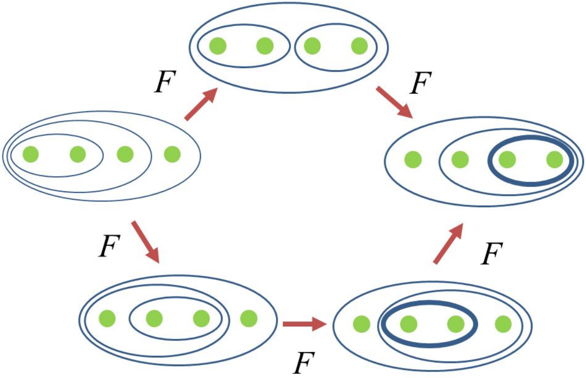

Note that for any number $n > 4$ of particles, any base change can always be reduced to paths on the diagram above and if the pentagon equation is satisfied, then the computation is correct.

Hilbert Space Dimensions

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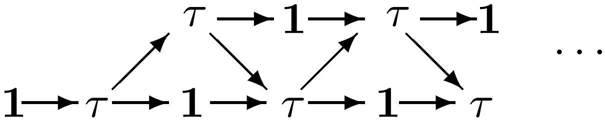

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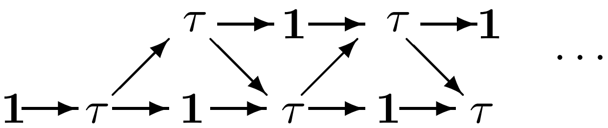

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$$\begin{tikzcd}
 & & \tau \arrow[r] \arrow[dr] & \mathbf{1} \arrow[r] & \tau \arrow[r] \arrow[dr] & \mathbf{1} & \\
\mathbf{1} \arrow[r] & \tau \arrow[r] \arrow[ur] & \mathbf{1} \arrow[r] & \tau \arrow[r] \arrow[ur] & \mathbf{1} \arrow[r] & \tau & \cdots
\end{tikzcd}$$

- Consider n τ-anyons in the plane with total charge τ
- Denote by $\mathbf{F}_n$ the ground state degeneracy
- Then $\mathbf{F}_{n+1} = \mathbf{F}_n + \mathbf{F}_{n-1}$ (Fibonacci sequence).

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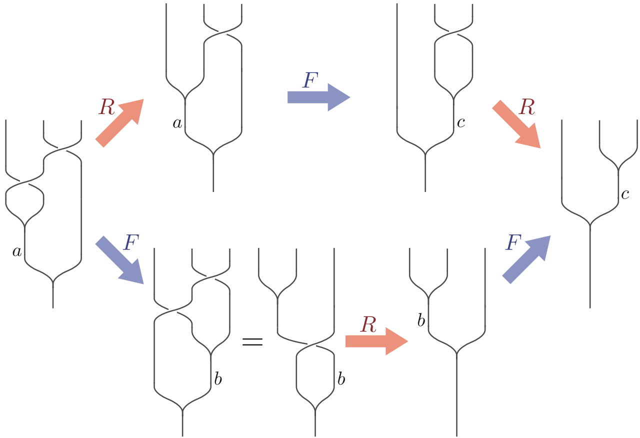
- $R_c^{b,a}$ is not the inverse of $R_c^{a,b}$.

The Hexagon Equation



The Hexagon equations enforce the condition that braiding is compatible with fusion. The equations are obtained by imposing the condition that the above diagram commutes.

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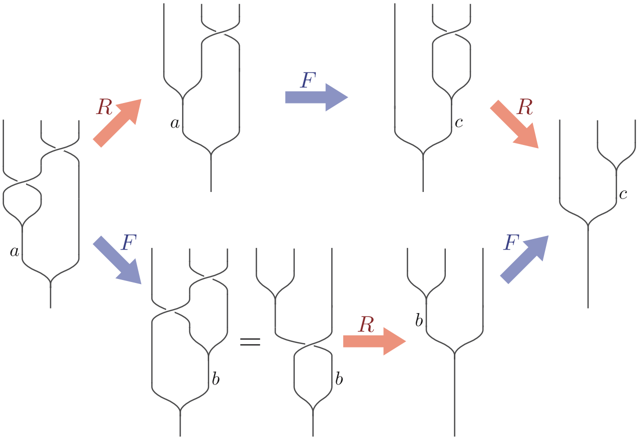

Explicit hexagon equation for Fibonacci anyons:

$$R_c^{\tau,\tau}(F_\tau^{\tau\tau\tau})_a^c R_a^{\tau,\tau} = \sum_b (F_\tau^{\tau\tau\tau})_b^c R_\tau^{\tau,b} (F_\tau^{\tau\tau\tau})_a^b$$

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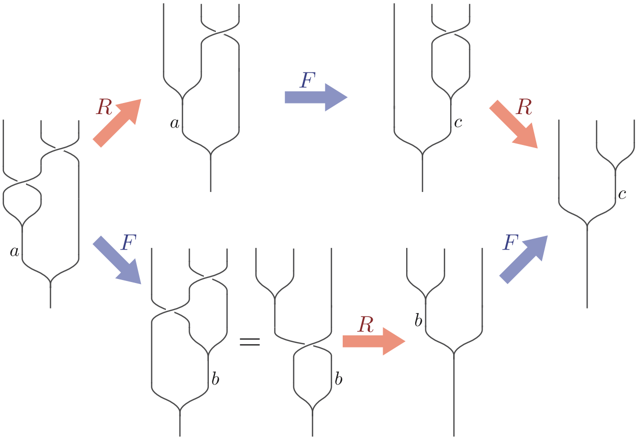

Only two solutions:

$$R = \begin{bmatrix} e^{i4\pi/5} & 0 \\ 0 & e^{-i3\pi/5} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{bmatrix}$$

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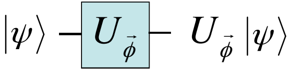
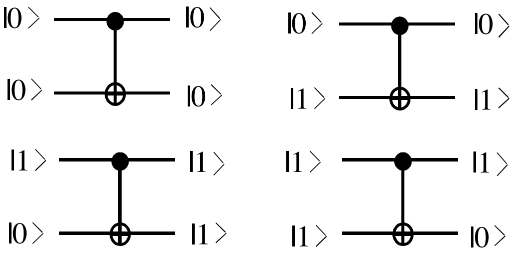
1. *Simulation of topological field theories by quantum computers*, M. Freedman, A. Kitaev and Z. Wang, *Comm. in Math. Phy. 227(3): 587-603* — 2002.
2. *A modular functor which is universal for quantum computation*, M. Freedman, M. Larsen and Z. Wang, *Comm. in Math. Phy. 227(3): 605-622* — 2002.

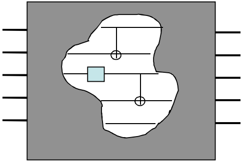
Universal Quantum Computation

Single Qubit Rotation



Controlled Not





Any N qubit operation can be carried out using these two gates.

$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

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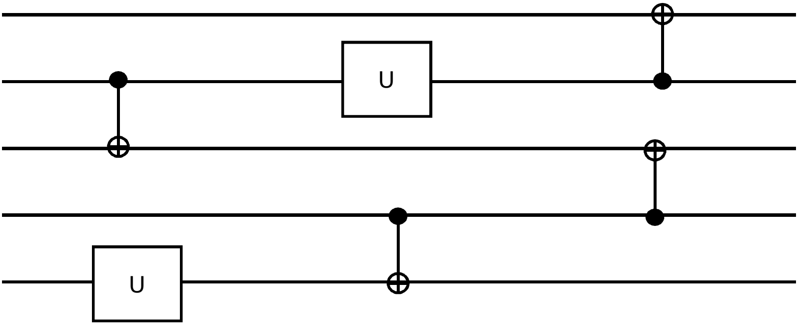
Theorem (Solovay-Kitaev)

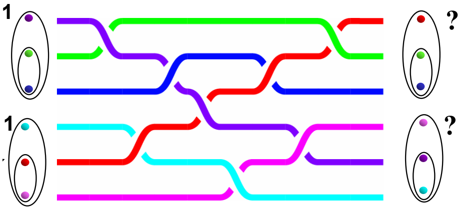
Let G be a finite set of elements in $\mathbf{SU}(d)$ containing its own inverses, and let $\varepsilon > 0$ be the desired accuracy. If G is dense in $\mathbf{SU}(d)$, then there exists a constant c s.t. for any $U \in \mathbf{SU}(d)$ there exists a finite sequence S of gates in G of length $\mathcal{O}(\log^c(\frac{1}{\varepsilon}))$ and with $d(U,S) < \varepsilon$.

Simulating Circuits with Fibonacci Anyons



What braid corresponds to this circuit?

Simulating Braids with Quantum Circuits



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