Topological Quantum Computation
A very basic introduction

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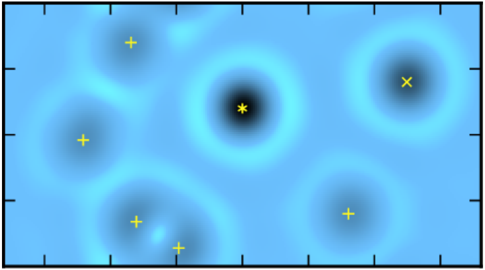
PhD Course on Quantum Computing

Part I

Quasiholes



Quasiholes in the Z_3 *Read-Rezayi state* (parafermion model), which may describe the filling $\mu = 12/5$ FQHE state, demonstrated to be Fibonacci anyons (when sufficiently separated)

Braiding Non-Abelian Quasiholes in Fractional Quantum Hall States, Yang-Le Wu, B. Estienne, N. Regnault, and B. Andrei Bernevig, *Phys. Rev. Lett. 113, 116801* — Published 8 September 2014.

Statistics for Quantum Computation

A fundamental symmetry in Physics: the wave function remains unchanged if we exchange two identical particles.

- About 30 years ago...
  - Physicists could only experience one consequence of this, i.e. the existence in three dimensions of bosons and fermions.
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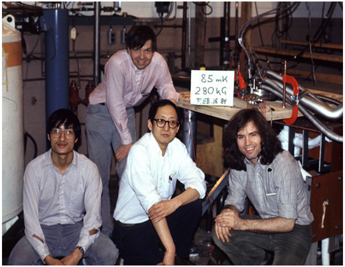
- However, in two dimensions a variety of statistical behaviours should be possible.
  Apart from $+1$ or a -1, arbitrary phase factors should be obtained when two particles are exchanged.
- Then comes the FQHE experiment (1982)
  - Starting with a theoretical construction of two dimensional models, physical systems were soon identified with effective two-dimensional behaviour.
  - Particles with such an exotic statistics were called anyons.

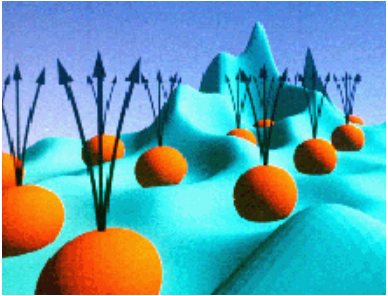
Fractional Quantum Hall Effect



1982 - The fractional quantum hall effect was measured with a dilution refrigerator in the hybrid magnet in the Francis Bitter National Magnet Laboratory at MIT, Cambridge, MA. From the left: Albert Cheng, Peter Berglund, **Dan Tsui** and **Horst Stormer**

The Physics Nobel Prize 1998 was given for the discovery and theory of the Fractional Quantum Hall Effect to Daniel Tsui, Horst Stormer and **Bob Laughlin**

Fractional Quantum Hall Effect



Confined gases of electrons in two dimensions in the presence of sufficiently strong magnetic field and low temperatures give rise to the Fractional Quantum Hall Effect:

> *When an electron is added to a FQH state it can be fractionalized — i.e., it can break apart into fractionally charged quasiparticles*

(Laughlin, *Phys. Rev.* **B23** 5632, 1981)

Order of Matter

- State of matter: gas, liquid, solid, plasma, ...

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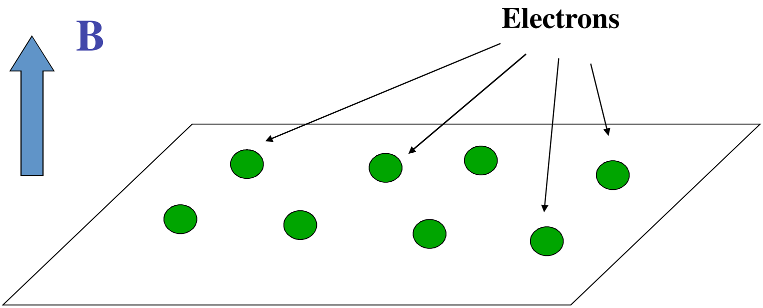
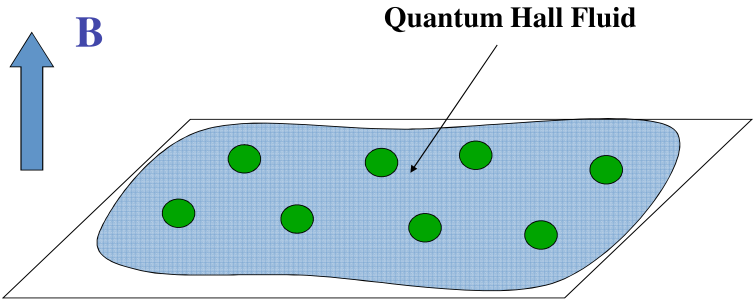
- FQH states contain a new type of order, called topological order, that is not associated with any symmetries and cannot be described by the Landau theory.

Charge Fractionalisation



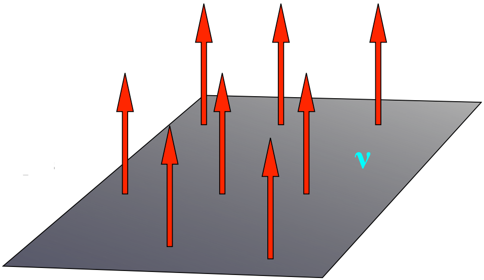
A two dimensional gas of electrons in a strong magnetic field **B**.

Charge Fractionalisation



An incompressible quantum liquid can form when the Landau level filling fraction $\nu = n_{elec}(hc/eB)$ is a rational fraction.

Charge Fractionalisation



Incompressible quantum liquid.

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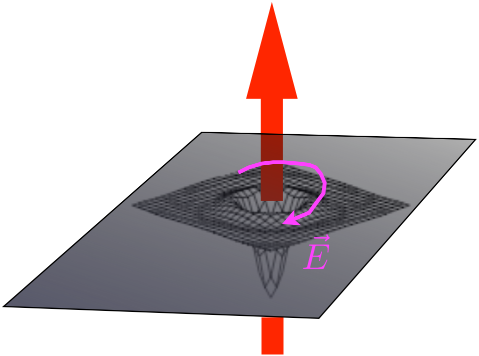
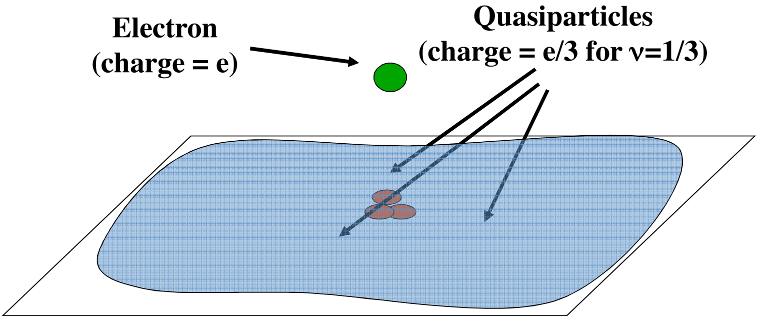


When a magnetic flux is turned on, the liquid is pushed out forming a hole.
If B is the magnetic field and the Landau level filling fraction $\nu = n_{elec}(hc/eB)$ is a rational fraction, we get the FQHE.

Charge Fractionalisation



When an electron is added to a FQH state it can break apart into fractionally charged quasiparticles.

Topological order

- FQH states are liquid and yet inside them electrons move around each other in a highly correlated manner.
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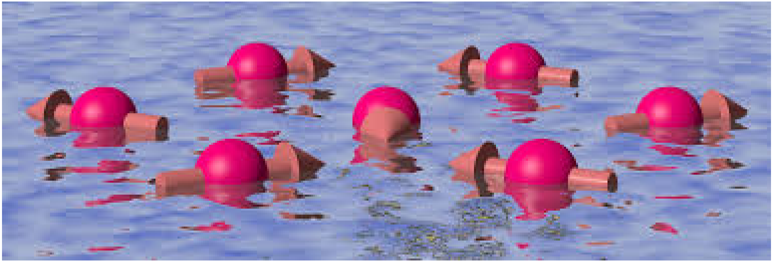
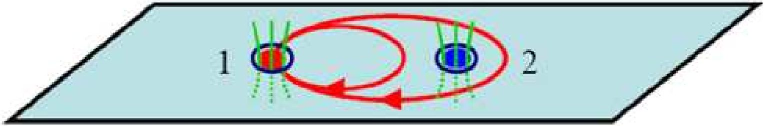


Quantum spin liquid

Anyons [Leinaas and Myrheim'77, Wilczek'82]



- To visualize the behavior of anyons one should think of them as being composite particles consisting of a magnetic flux Φ and a ring of electric charge q (see picture above).
- If particle 1 circulates particle 2, then its charge q goes around the flux Φ thereby acquiring a phase factor $U = e^{iq\Phi}$.

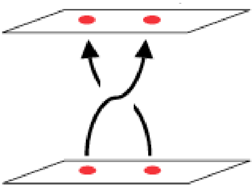
Anyonic Statistics

- The behaviour of a two-particle system in two-dimensions obeys statistics ranging continuously between Fermi-Dirac and Bose-Einstein statistics, as was first shown by Jon Magne Leinaas and Jan Myrheim of the University of Oslo in 1977.
- In Dirac notation:

$$|\psi_1\psi_2\rangle = e^{i\theta}\,|\psi_2\psi_1\rangle\,,$$

  where
  - $\vartheta = 0$ (phase $= +1$) for Bosons (Photons, He^4 atoms, Gluons ...)
  - $\vartheta = \pi$ (phase = -1) for Fermions (Electrons, Protons, Neutrons ...)
  - $\vartheta = 2\pi\nu - 1$ for Anyons

Particles Exchange



- The exchange generates a phase $e^{i\vartheta}$.
- If $\vartheta \in \{0, \pi\}$ $\Rightarrow$ permutation $\Rightarrow$ Bosons/Fermions.
- For general ϑ $\Rightarrow$ braids in $2+1$ dimensions $\Rightarrow$ Anyons.

Note that the resulting factor does not depend on the details of the path of particle 1 (local property).
It depends only on the number of times it circulates around it (global property, topology).

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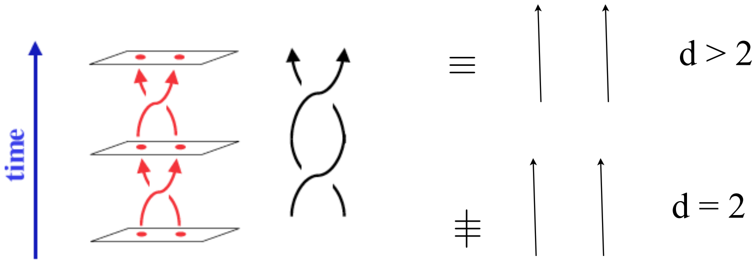

- There are no braids for $d > 2$, and we are back to just having the permutation symmetry, and thus only bosons and fermions.

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Non-Abelian Anyons [Moore and Read'91]

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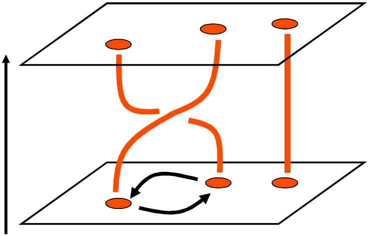
Unsolved problem in physics

What mechanism explains the existence of the $\nu = 5/2$ state in the fractional quantum Hall effect?

- There is no generally accepted experimental evidence for non-abelian statistics.

Non-Abelian Statistics



Fractional Abelian Statistics:

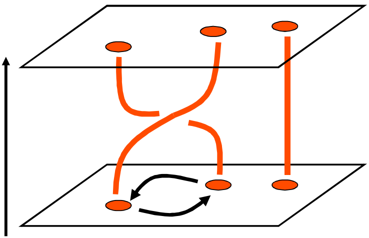
$|\psi_i\rangle$ initial state
$|\psi_f\rangle$ final state
$|\psi_f\rangle = e^{i\vartheta} \, |\psi_i\rangle$

Phase factors $\Rightarrow$ commute.

Fractional Non-Abelian Statistics

Degenerate Hilbert space: not a unique ground state for localised quasi-particles.

Non-Abelian Statistics



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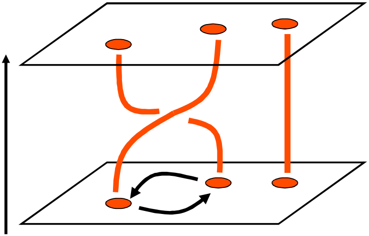
$$|\psi_i\rangle \quad = \quad \alpha\,|\psi_0\rangle + \beta\,|\psi_1\rangle$$

$$|\psi_f\rangle \quad = \quad \tilde{\alpha}\,|\psi_0\rangle + \tilde{\beta}\,|\psi_1\rangle$$

$|\psi_0\rangle$ and $|\psi_1\rangle$ are degenerate states.

Non-Abelian Statistics



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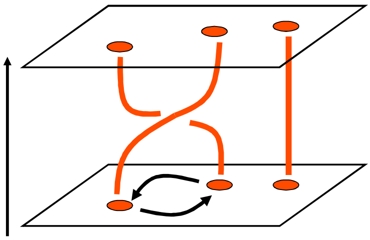

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$$|\psi_f\rangle = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Matrices $\Rightarrow$ do not commute.

On-going research

- Gregory Moore, Nicholas Read, and Xiao-Gang Wen pointed out that non-Abelian statistics can be realised in the fractional quantum Hall effect `http://arxiv.org/abs/hep-th/9202001`.
- Experimental evidence of non-Abelian anyons, although not yet conclusive, was presented in October, 2013 `http://arxiv.org/abs/1301.2639`.

Bob Willet (photo from Quanta Magazine)

- Last update: Yang-Le Wu, B. Estienne, N. Regnault, and B. Andrei Bernevig, *Phys. Rev. Lett.*, 113:116801, Sep 2014 (see initial picture)

Fault-tolerant quantum computing [Kitaev'97]

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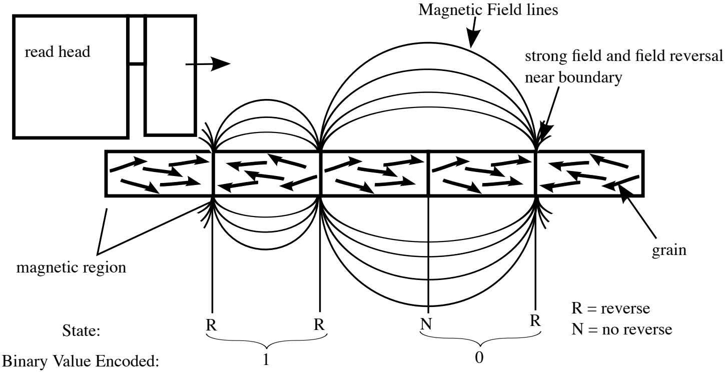
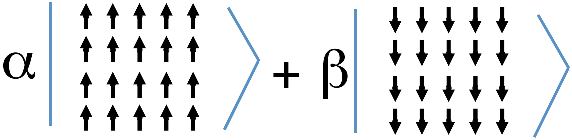
> *If a physical system were to have quantum topological (necessarily nonlocal) degrees of freedom, which were insensitive to local probes, then information contained in them would be automatically protected against errors caused by local interactions with the environment.*
>
> *This would be fault tolerance guaranteed by physics at the hardware level, with no further need for quantum error correction, i.e. topological protection.*

Alexei Kitaev

Storing a qubit

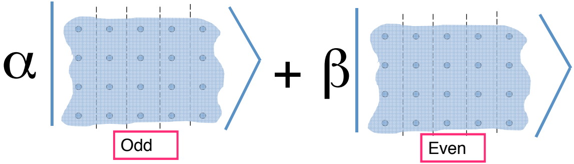


Storing a qubit



- Environment can measure the state of the qubit by a local measurement.
- Any quantum superposition will decohere almost instantly.

Storing a qubit

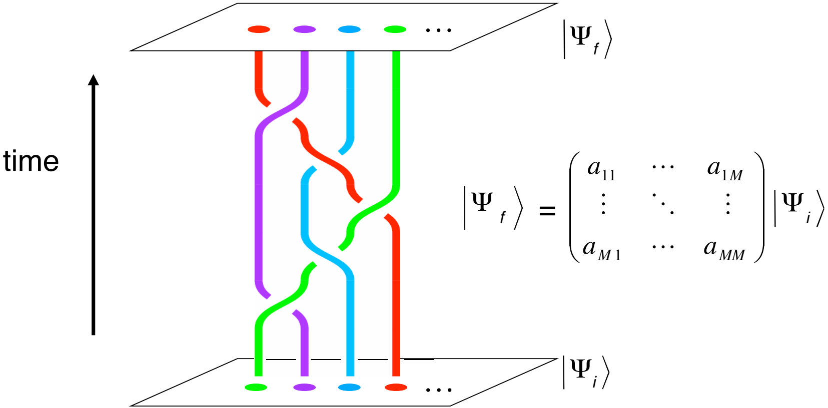


- Environment can only measure the state of the qubit by a global measurement.
- Quantum superposition should have long coherence time.

Computing with Anyons

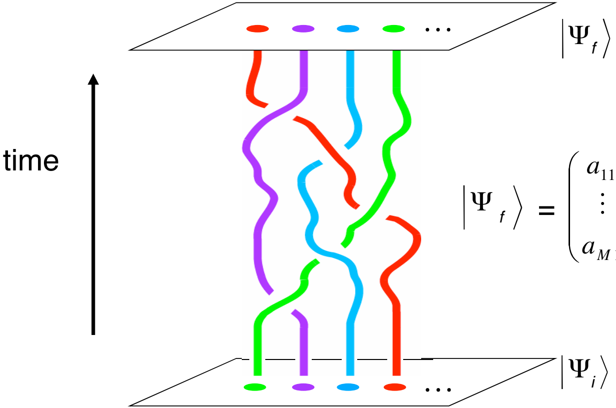
From: *Nature* **452**, 803-805 (2008)

Computing with Anyons



Initial and final state occupy the same position.

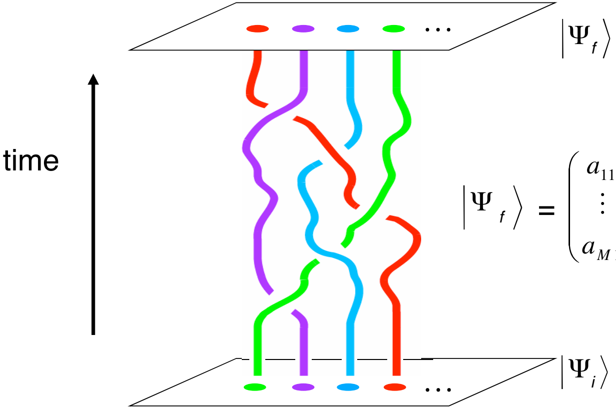
Computing with Anyons



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Matrix depends only on the topology of the braid swept out by anyon world lines!

Computing with Anyons



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Robust quantum computation (Kitaev '97, Freedman, Larsen and Wang '01)

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Fibonacci anyons

Charge $e/5$ quasiparticles studied in Read& Rezayi (1999). Slingerland& Bais (2001)

More theoretical, need quantum liquids with $\nu = 12/5$

Braiding is universal for quantum computation Freedman, Larsen& Wang (2002)