Diffusion geometry in deformable shape analysis

Alex Bronstein, Michael Bronstein, Umberto Castellani

May 14, 2012

Dimensions of media

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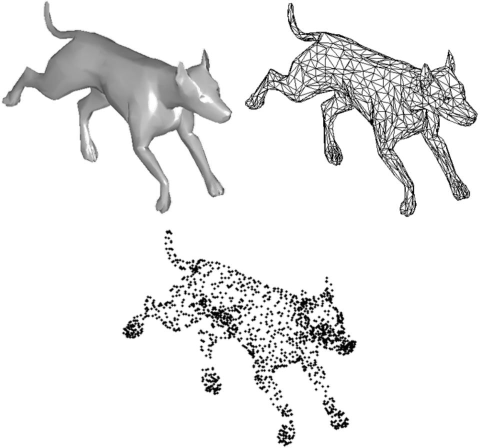
3D shapes vs. images

Array of pixels

3D shapes vs. images

Array of pixels



Splines, Mesh, Point cloud, etc

3D shapes vs. images

Affine, projective

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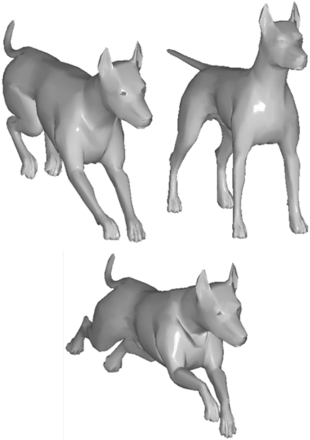
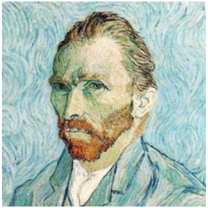


Wealth of nonrigid deformations

3D shapes vs. images

$\frac{1}{2}$  $+ \frac{1}{2}$  $=$ 

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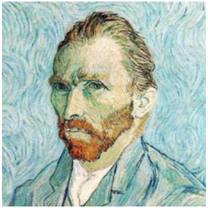
$\frac{1}{2}$  $+ \frac{1}{2}$  $=$ 

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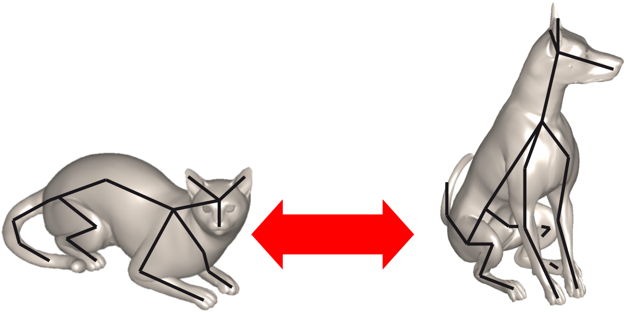
Setting



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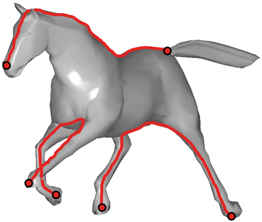
- Endow shapes with a structre
- Similarity, correspondence, retrieval, etc. = similarity and correspondence between structures
- *Invariance* under bending, scale, affine transformations, etc.

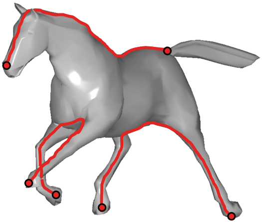
Structure



Global structure

Metric space

Structure



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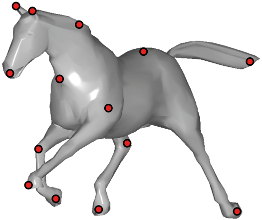

Local structure
Point descriptors

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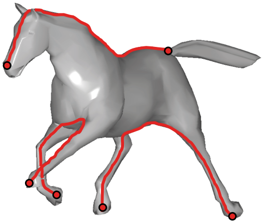

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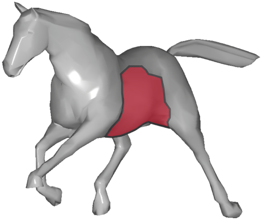
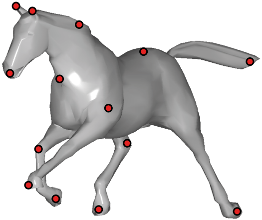


Glocal structure
Stable regions



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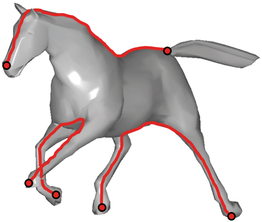

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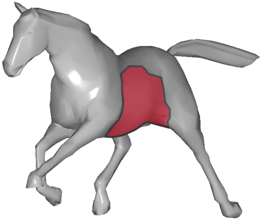

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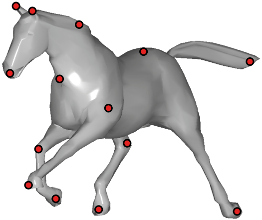

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Agenda

- Diffusion processes on surfaces
- Spectral point of view
- *Global structure*: diffusion geometry
- *Local structure*: diffusion kernel descriptors
- *Semi-local structure*: maximally stable components
- Extensions

Diffusion processes on surfaces

- *Heat equation*

$$\left(\Delta_X + \frac{\partial}{\partial t}\right) u = 0$$

governs heat propagation on manifold X

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- *Solution* $u(x,t)$: heat distribution at point x at time t
- *Initial condition* $u_0(x)$: heat distribution at time $t = 0$
- *Boundary condition* if manifold has a boundary

Laplace-Beltrami operator Δ_X

For two smooth functions $f, g : X \rightarrow \mathbb{R}$ and standard inner product on X

$$\langle f, g \rangle = \int_X f(x) g(x) \, da$$

the Laplacian satisfies the following properties:

- *Constant eigenfunction*: $\Delta_X f = 0$ if $f = \mathrm{const}$

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- *Maximum principle*: functions satisfying $\Delta_X f = 0$ (harmonic) have no minima/maxima in the interior of X

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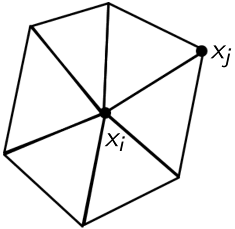
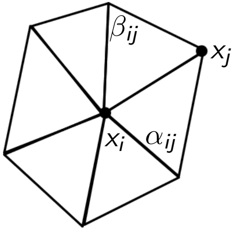
- In matrix notation

$$L_X f = A^{-1} L f$$

  where $A = \mathrm{diag}\{a_i\}$ and $(L)_{ij} = \mathrm{diag}\left\{\sum_{k \neq i} w_{ik}\right\} - w_{ij}$

Discretization of the Laplacian





Discrete Laplacian

$$w_{ij} = \begin{cases} 1 & : \; x_j \in \mathcal{N}_1(x_i) \\ 0 & : \; \text{else} \end{cases}$$

$a_i = 1$ (umbrella operator); or
$a_i = |\mathcal{N}_1(x_i)|$, valence (Tutte)

Discretized Laplacian

$$w_{ij} = \begin{cases} \cot \alpha_{ij} + \cot \beta_{ij} & : \; x_j \in \mathcal{N}_1(x_i) \\ 0 & : \; \text{else} \end{cases}$$

$a_i =$ sum of areas of triangles sharing vertex x_i

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Reason: Flat plate must have zero bending energy.

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  Indispensable for discretization of PDE solutions

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- Many attempts have been made to construct discrete Laplacians satisfying above desired properties
- **"No free lunch theorem" (Wardetzky et al., 2007)** There is no discrete Laplacian satisfying the above properties simultaneously!

Eigendecomposition of Laplacian

- On *compact* domains Laplacian admits *countable orthogonal eigendecomposition*

$$\Delta_X \phi_i = \lambda_i \phi_i$$

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- *Spectral decomposition theorem*

$$(\Delta_X f)(x) = \sum_{i \geq 0} \lambda_i \phi_i(x) \cdot \langle \phi_i, f \rangle$$

Discrete equivalent: $L_X f = \sum_{i \geq 0} \lambda_i \phi_i \phi_i^{\mathrm{T}} f$

Finite elements

- Discretize $\{\lambda_i, \phi_i\}$ directly!

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$$\sum_{i=1}^{m} u_i \underbrace{\langle \Delta_X \alpha_i, \alpha_j \rangle}_{a_{ij}} \quad = \quad \lambda \sum_{i=1}^{m} u_i \underbrace{\langle \alpha_i, \alpha_j \rangle}_{b_{ij}}$$

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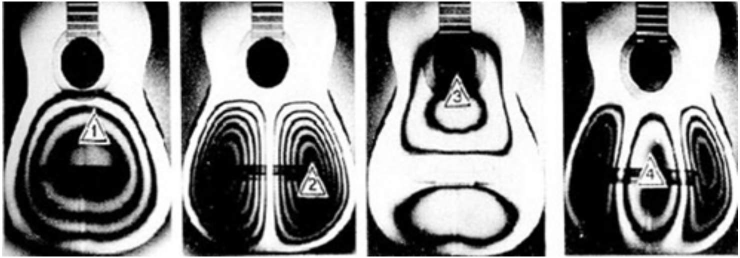
- In matrix notation: $Au = \lambda Bu$

To see the sound

Chladni plates



- Solutions to *stationary Helmholtz equation*

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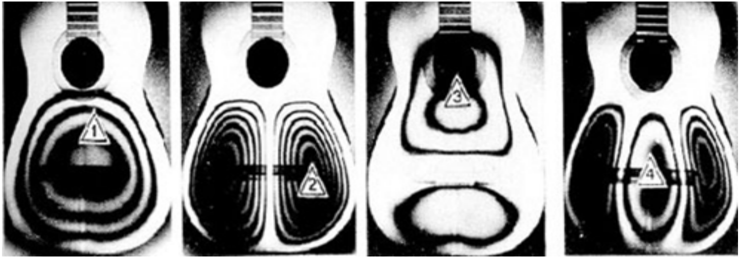

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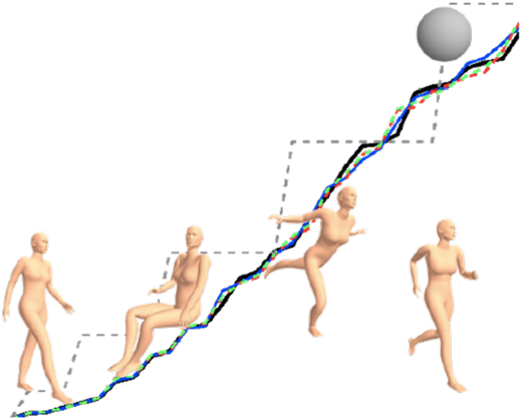
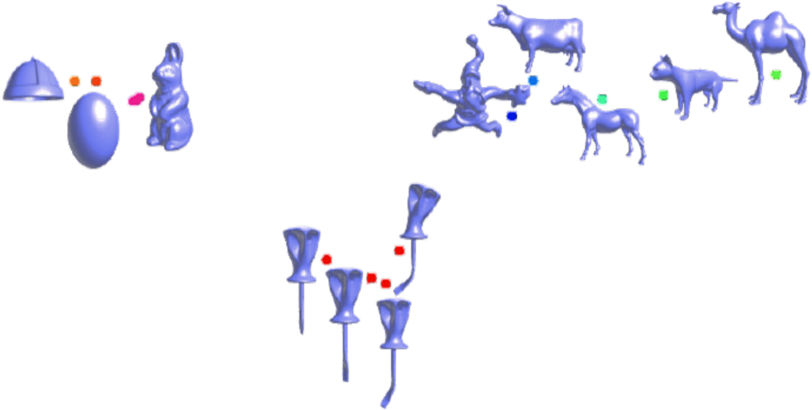
- Laplacian *eigenfunctions* = plate vibration modes

Shape DNA



(Reuter *et al.*, 2006) use Laplacian spectrum $\{\lambda_i\}$ as an isometry-invariant shape descriptor – *shape DNA*

Shape DNA



Shape similarity using Shape DNA

"Can we hear the shape of the drum?"

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- **Can one hear the shape of the drum?** (*Mark Kac*)

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The following shape properties can be recovered ("heard") from the spectrum of the Laplacian:

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The following shape properties can be recovered (“heard”) from the spectrum of the Laplacian:

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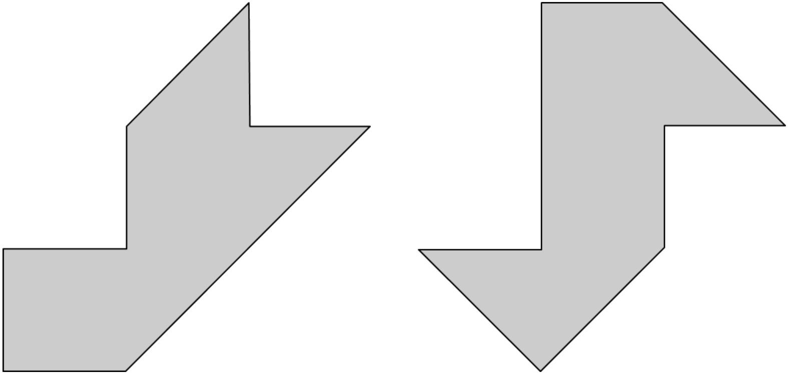
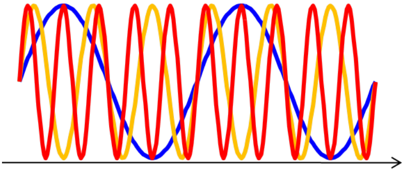
- **Can we hear the metric?**

One cannot hear the shape of the drum!



Counter example of isospectral non-isometric shapes
(Gordon *et al.*, 1991)

Relation to harmonic analysis



1D signals

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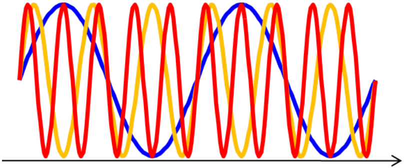
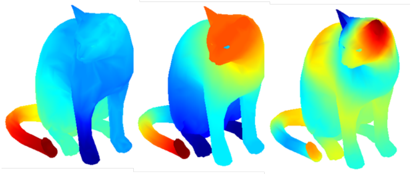
Relation to harmonic analysis





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3D shapes

$$\Delta_X \phi_i(x) = \lambda_i \phi_i(x)$$

Relation to harmonic analysis

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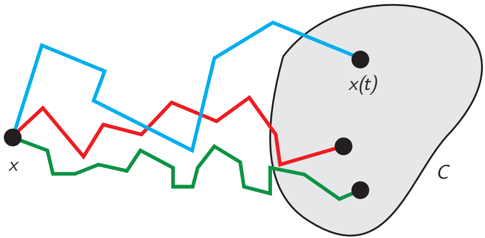
- Heat operator can be interpreted as a non shift-invariant version of convolution

Heat kernel

Probabilistic interpretation

- *Brownian motion* $x(t)$ starts at point x



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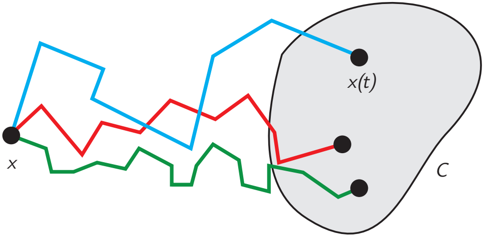

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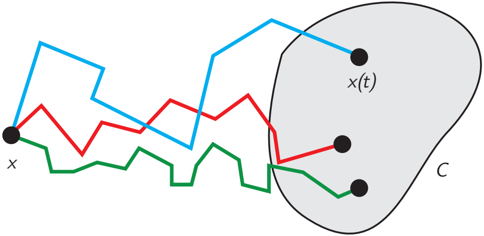

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Diffusion geometry

- Family of *diffusion metrics*

$$\begin{aligned} d^2(x,y) &= \|k(x,\cdot) - k(y,\cdot)\|^2_{L^2(X)} \\ &= \int_X (k(x,z) - k(y,z))^2 da(z) \end{aligned}$$

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- "Connectability" by random walks of any length
- **Scale invariant!**

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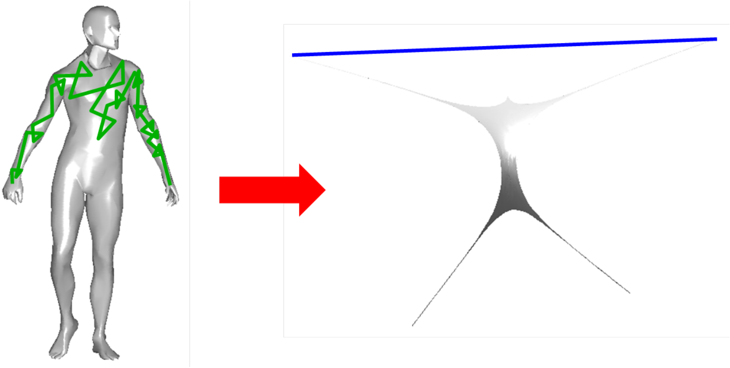
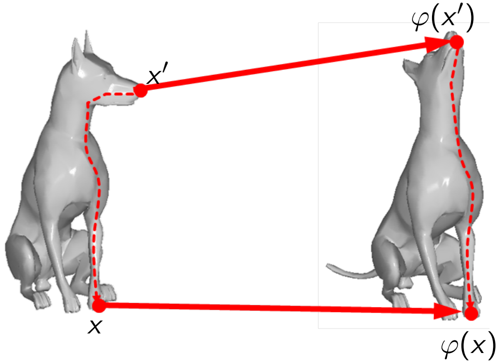
- Diffusion distance is represented by *Euclidean distance* in embedding space

Diffusion maps



Correspondence



- Embed one shape into the other

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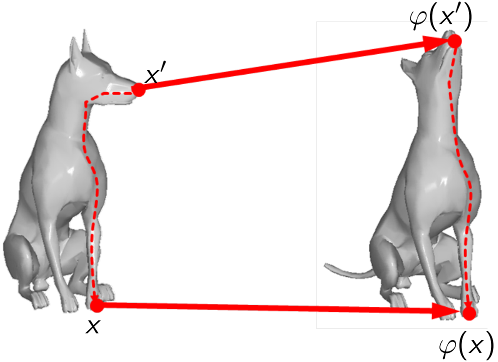

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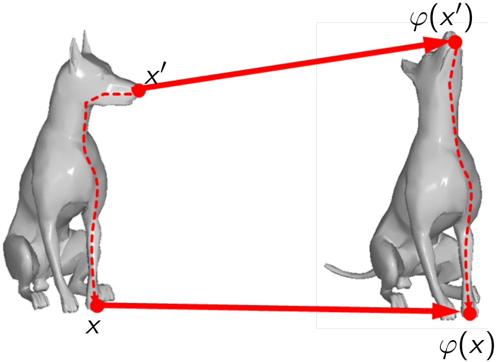
- Find *minimum distortion* correspondence

$$\min_{\varphi: X \to Y} \| d_X - d_Y \circ (\varphi \times \varphi) \|$$

Correspondence



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$$\min_{\varphi:X\to Y} \int\int (d_X(x,x') - d_Y(\varphi(x),\varphi(x')))^2 da(x) da(x')$$

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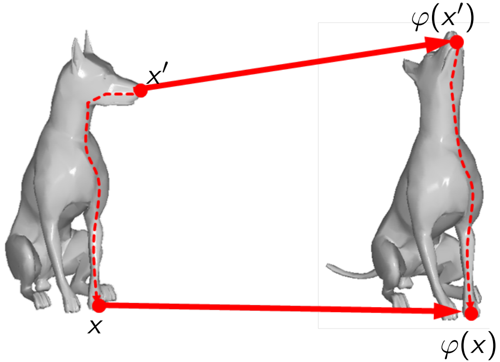

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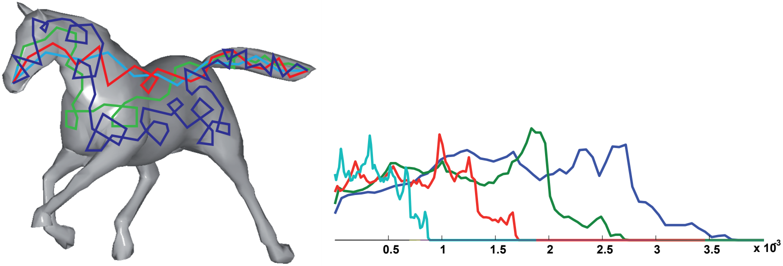
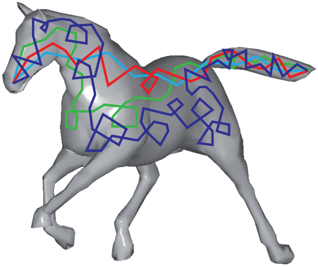
- Solved using *generalized multidimensional scaling*

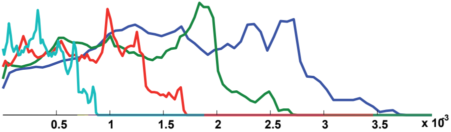
Diffusion distance distributions



- Represent shape as *distribution* of diffusion distances

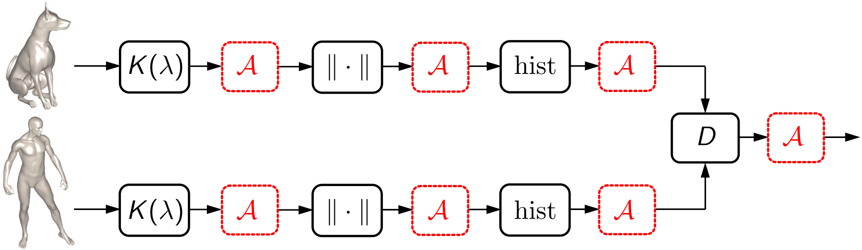
Diffusion distance distributions



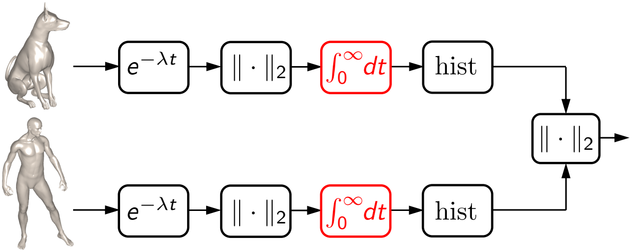


- Represent shape as *distribution* of diffusion distances
- Compare shapes using *divergence* of distributions

Diffusion distance distributions



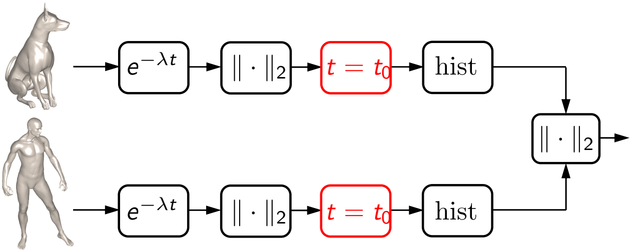
Particular case I: Rustamov's GPS embedding



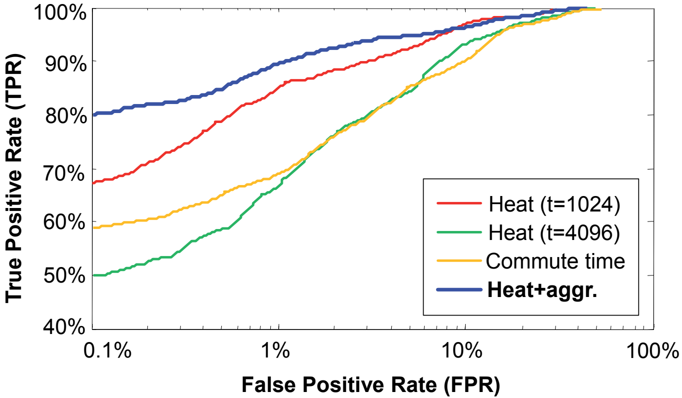
$$d^2_{\mathrm{CT}}(x,y) = \int_0^\infty \underbrace{\sum_{i \geq 0} e^{-2\lambda_i t}(\phi_i(x) - \phi_i(y))^2}_{\text{Diffusion distance } d_t^2(x,y)} dt$$

Diffusion distance distributions

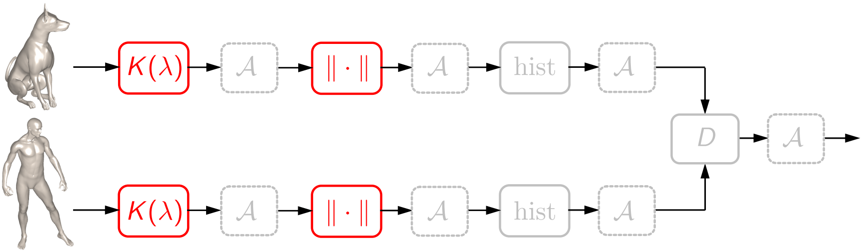
Particular case II: Mahmoudi & Sapiro



Diffusion distance distributions



Generalizations



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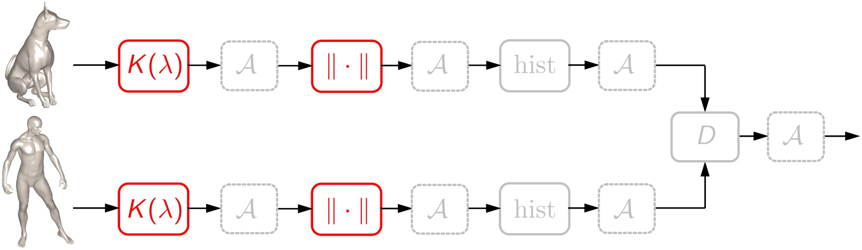
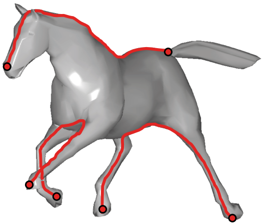
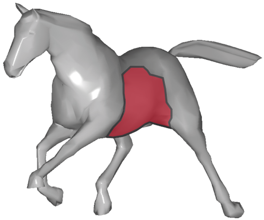


- Diffusion distances generated by other norms, e.g. $\|\cdot\|_{L^1(X)}$
- Construct (or learn) *optimal* task-specific diffusion kernels

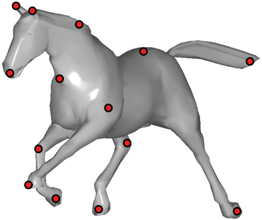
Local structure



Global structure
Metric space



Glocal structure
Stable regions



Local structure
Point descriptors

Diffusion kernel descriptors

- Associate each point x with a vector $(k_{t_1}(x,x), \ldots, k_{t_n}(x,x))$

(Sun *et al.*, SGP'09)

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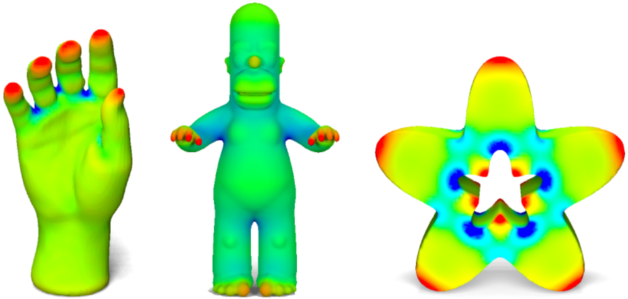
Diffusion kernel descriptors

- Associate each point x with a vector $(k_{t_1}(x,x), \ldots, k_{t_n}(x,x))$
- Multi-scale *point-wise descriptor*
- *Heat kernel signature* (HKS): $x \mapsto (h_{t_1}(x,x), \ldots, h_{t_n}(x,x))$

(Sun *et al.*, SGP'09)

Heat kernel signature



$$h_t(x,x) = \frac{1}{4\pi t}\left(1 + \frac{1}{3}\kappa(x)t + \mathcal{O}(t^2)\right)$$

$\kappa(x)$ = Gaussian curvature at point x

(Sun *et al.*, SGP'09)

Scale invariance

- **Original shape**
- Eigenvalues λ_i
- Eigenfunctions $\phi_i(x)$
- Heat kernel $h_t(x,x)$

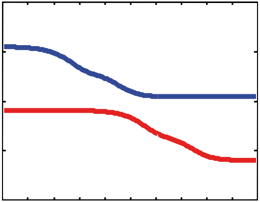
- **Scaled by $\frac{1}{\alpha}$**
- Eigenvalues $\alpha^2\lambda_i$
- Eigenfunctions $\alpha\phi_i(x)$
- Heat kernel $\alpha^2 h_{\alpha^2 t}(x,x)$
- **Not scale invariant!**

Scale invariance

Log scale space

$h_{e^\tau} \to \alpha^2 h_{e^{\tau+\beta}}$

$\beta = \log \alpha^2$



Scale $\to$ shift + factor

Scale invariance

Log scale space

$h_{e^\tau} \to \alpha^2 h_{e^{\tau+\beta}}$

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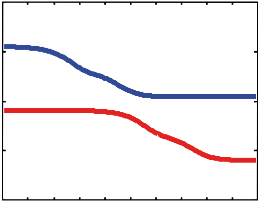
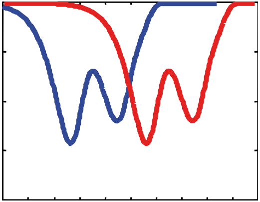
Log + derivative

$$\frac{d}{d\tau}\left(\log \alpha^2 + \log h_{e^{\tau+\beta}}\right)$$
$$= \frac{d}{d\tau} \log h_{e^{\tau+\beta}}$$





Scale $\to$ shift + factor

Undo factor

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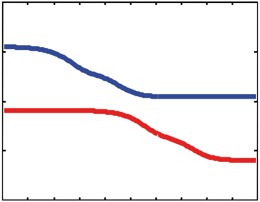
Log + derivative

$$\frac{d}{d\tau}\left(\log \alpha^2 + \log h_{e^{\tau+\beta}}\right) = \frac{d}{d\tau} \log h_{e^{\tau+\beta}}$$

Fourier magnitude

$$\mathcal{F}\left\{\frac{d}{d\tau} \log h_{e^{\tau+\beta}}\right\} = e^{\beta i \omega \pi} \mathcal{F}\left\{\frac{d}{d\tau} \log h_{e^\tau}\right\}$$



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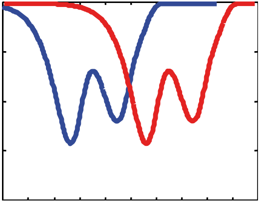
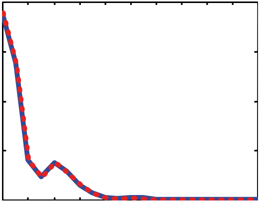


Undo factor



Undo shift

Scale invariant heat kernel signature

(B&Kokkinos, CVPR'09)

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Affine invariance

- Let $C(p)$ be a curve on manifold X

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- Demand $\det(x_1, x_2, C_{pp}) = 1$ and obtain *equi-affine invariant* arclength

$$\begin{aligned} dp^2 &= \det(x_1, x_2, x_{11}du_1^2 + 2x_{12}du_1du_2 + x_{22}du_2^2) \\ &= \tilde{g}_{11}du_1^2 + 2\tilde{g}_{12}du_1du_2 + \tilde{g}_{22}du_2^2 \end{aligned}$$

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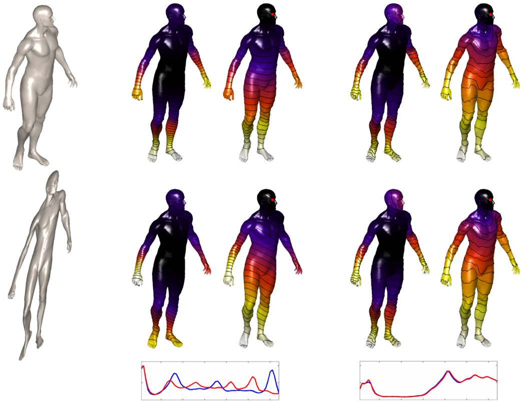
- Positive definite only on *convex* surfaces
- Construct g from $\hat{g}$ enforcing positivity of eigenvalues
- g is a *valid metric* on manifolds with non-vanishing curvature

Affine invariance

- Define *equi-affine Laplacian* Δ_g
- **Equi-affine Laplacian + scale-invariance = affine-invariance**



Fusing geometric and photometric information

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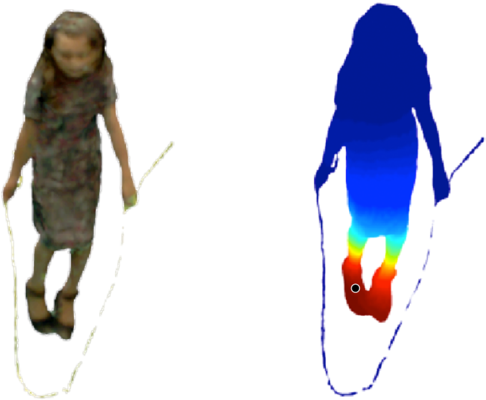
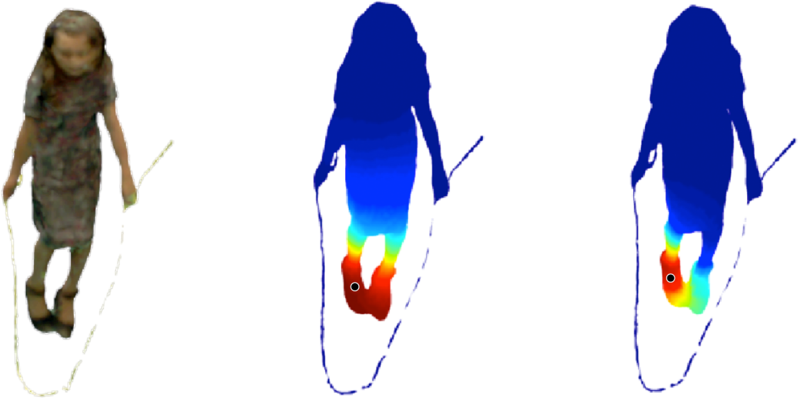


Geometric

Fusing geometric and photometric information



Geometric

Geometric+
photometric

Color = embedding

Embedding

$\xi : X \to \xi(X) \subset \mathcal{E}$

Pullback metric

$(\xi^* h)_{\mu\nu} = h_{ij} \partial_\mu \xi^i \partial_\nu \xi^j$

Manifold X

Manifold $\mathcal{E}$

metric h_{ij}

Embedding

$$\xi = (\xi_g, \xi_p)$$

Pullback metric

$$(\xi^* h)_{\mu\nu} = \langle \partial_\mu \xi_g, \partial_\nu \xi_g \rangle_{\mathbb{R}^3} + \eta^2 \langle \partial_\mu \xi_p, \partial_\nu \xi_p \rangle_{\mathbb{R}^3}$$

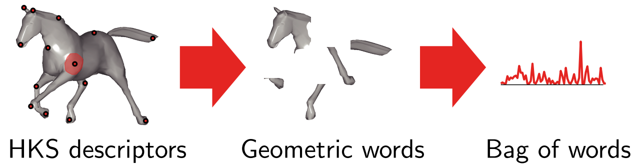
Manifold X

Color $\alpha : X \to \mathbb{R}^3$

Manifold $\mathbb{R}^3 \times \mathbb{R}^3$

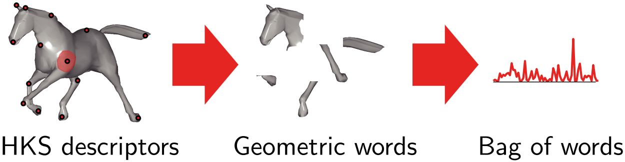
Euclidean metric

ShapeGoogle



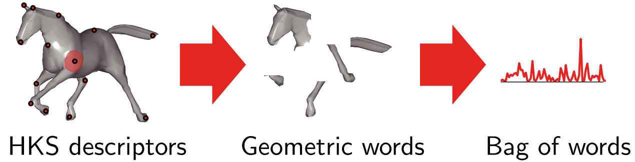
- Given point-wise descriptor $\mathbf{h}(x)$

ShapeGoogle



- Given point-wise descriptor $\mathbf{h}(x)$
- *Quantize* each $\mathbf{h}$ in a fixed vocabulary $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$

ShapeGoogle



- Given point-wise descriptor $\mathbf{h}(x)$
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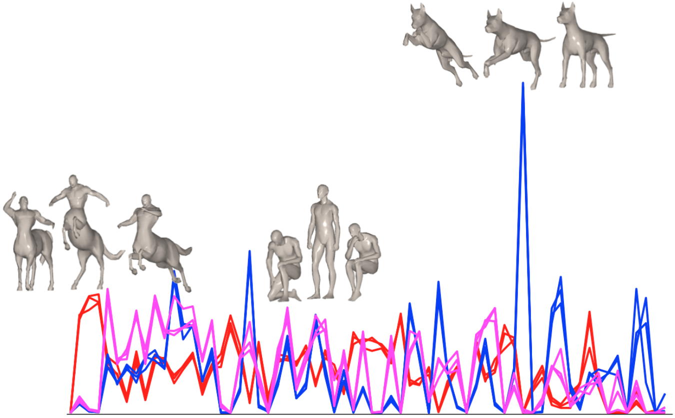
- *Bag-of-words* shape descriptor

$$\mathbf{H} = \int_X \mathbf{v}(x) da(x)$$

ShapeGoogle

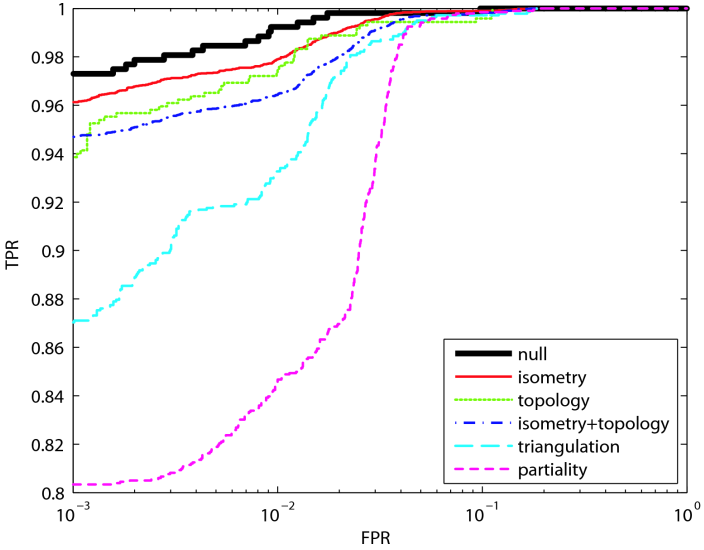


ShapeGoogle

HKS descriptors | Geometric expressions | Spatially-sensitive bag of words

- *Spatially-sensitive* bags of pairs of words

$$\mathbf{H} = \int_{X \times X} \mathbf{v}(x)\mathbf{v}(y) h_t(x,y) da(x) da(y)$$


1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
0.82
0.8
TPR
10^{-3}
10^{-2}
10^{-1}
10^{0}
FPR
null
isometry
topology
isometry+topology
triangulation
partiality

Partial matching

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- Regularize part *boundary length* a lá Mumford-Shah

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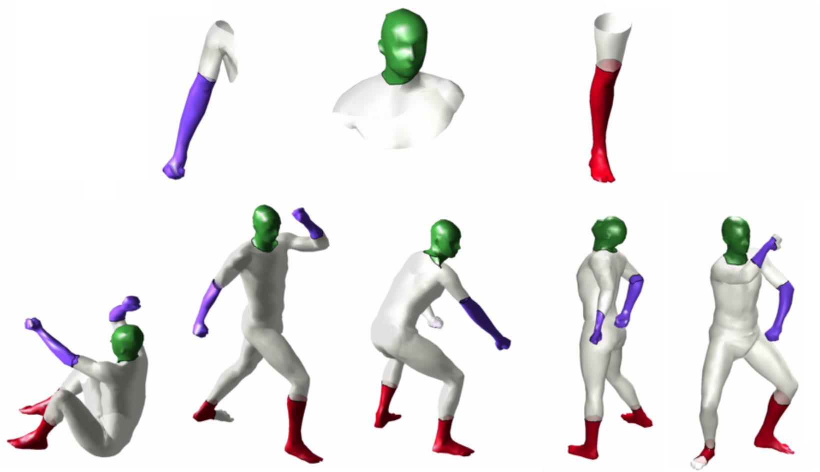
- *Ambrosio-Tortorelli* approximation

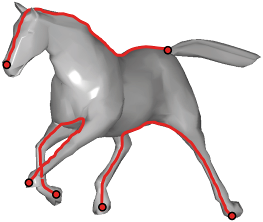
$$\min_{u,\rho} \left\| \int_{X} \mathbf{v} u da - \mathbf{H}' \right\|^2 + \mu_1 \int_X \rho^2 \|\nabla u\|^2 da + \mu_2 \epsilon \int_X \|\nabla \rho\|^2 da$$

$$+ \frac{\mu_2}{4\epsilon} \int_X (1-\rho)^2 da \ \text{ s.t. } \int_X u da = A(Y')$$

Partial matching

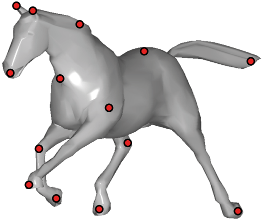


Global + local structures



Global structure

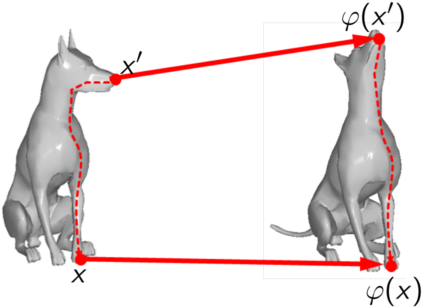
Metric space



Local structure

Point descriptors

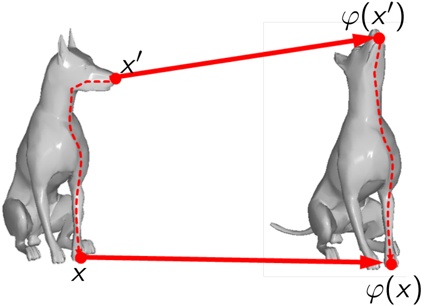
Correspondence *encore*



- Find *minimum distortion* correspondence

$$\min_{\varphi:X\to Y} \|d_X - d_Y \circ (\varphi \times \varphi)\|$$

Correspondence *encore*



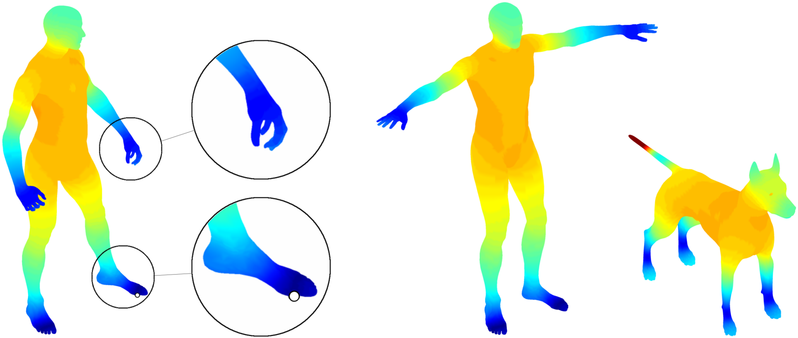
- Find *minimum distortion* correspondence

$$\min_{\varphi:X\to Y} \|d_X - d_Y \circ (\varphi \times \varphi)\| + \mu \|\mathbf{h}_X - \mathbf{h}_Y \circ \varphi\|$$

d_X, d_Y – global structures
$\mathbf{h}_X, \mathbf{h}_Y$ – local structures

- Combine local and global structure distortion

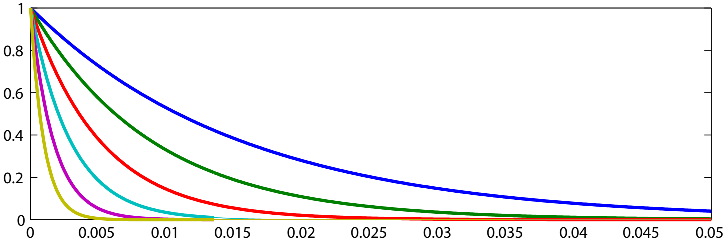
Heat kernel signature *encore*



- Poor spatial feature localization!

Aubry *et al.*, CVPR'11; B, PAMI'11

Heat kernel signature *encore*



- Collection of *low pass filters*

$$\mathbf{p}(x) = \sum_{k \geq 0} \begin{pmatrix} p_1(\lambda_k) \\ \vdots \\ p_n(\lambda_k) \end{pmatrix} \phi_k^2(x)$$

$$p_i(\lambda) = \exp(-\lambda t_i)$$

Wave kernel signature

- Different physical model: a quantum particle with initial energy distribution $f(\nu)$.

Aubry *et al.*, CVPR'11

Wave kernel signature

- Different physical model: a quantum particle with initial energy distribution $f(\nu)$.
- Described by the *Schrödinger equation*

$$\left(i\Delta + \frac{\partial}{\partial t}\right)\psi(x,t) \quad = \quad 0$$

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- Solution in spectral domain

$$\psi(x,t) \quad = \quad \sum_{k\geq 0} e^{i\nu_k t} f(\nu_k)\phi_k(x)$$

Aubry *et al.*, CVPR'11

Wave kernel signature

- Family of *log-normal* initial energy distributions

$$f_e(\nu) \propto \exp\left(-\frac{(\log e - \log \nu)^2}{2\sigma^2}\right)$$

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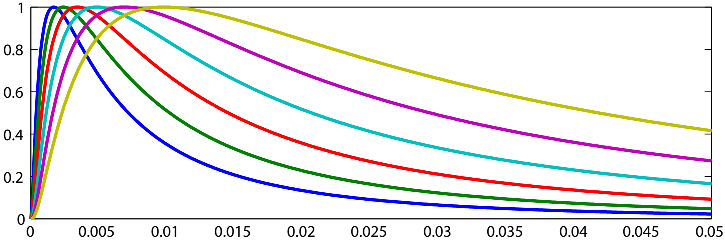
- Each point is associated the *wave kernel signature*

$$\mathbf{p}(x) : x \quad \mapsto \quad (p_{e_1}(x), \ldots, p_{e_n}(x))$$

Aubry *et al.*, CVPR'11

Wave kernel signature



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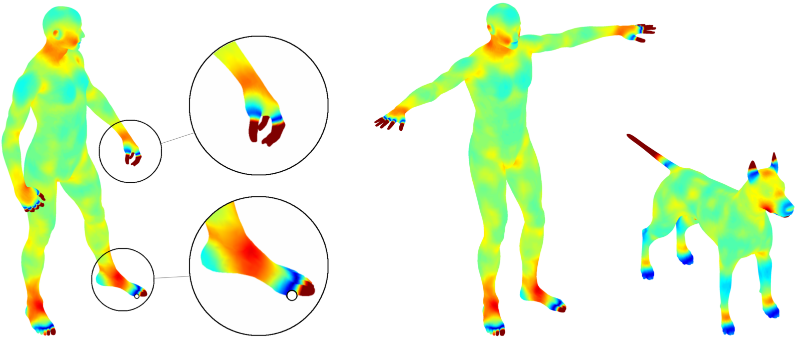
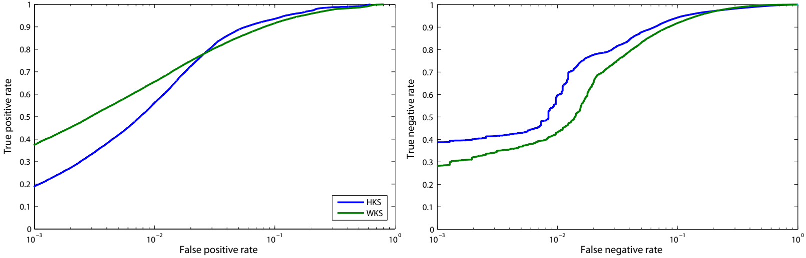


- Better spatial feature localization
- Lower discriminativity

Aubry *et al.*, CVPR'11; B, PAMI'11

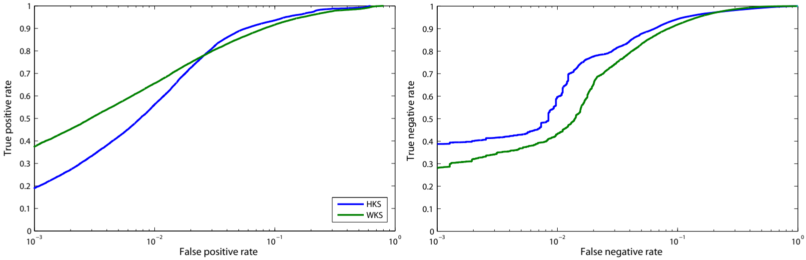
Waves vs Heat



- WKS has higher *sensitivity* than HKS (better at low FPR)

B, PAMI'11

Waves vs Heat



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- Yet has lower *specificity* (worse at low FNR)

B, PAMI'11

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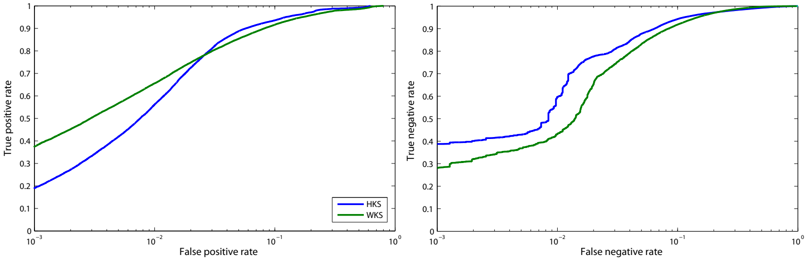

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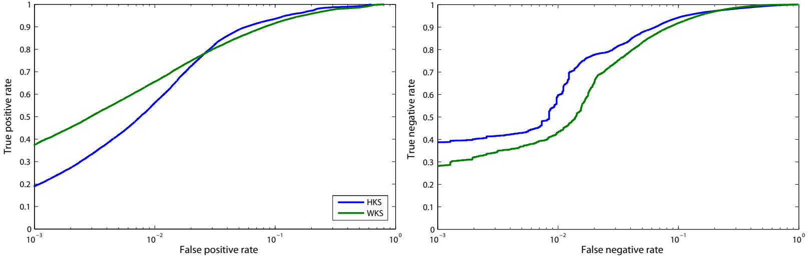
- WKS is better for *correspondence* problems

B, PAMI'11

Waves vs Heat



- WKS has higher *sensitivity* than HKS (better at low FPR)
- Yet has lower *specificity* (worse at low FNR)
- WKS is better for *correspondence* problems
- HKS is better for shape *retrieval* problems

B, PAMI'11

An ideal descriptor

- **Localization**: changes a lot under a small displacement on the surface.

B, PAMI'11

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B, PAMI'11

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B, PAMI'11

An ideal descriptor

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- **Sensitivity**: a small set of best matches should contain a correct match with high probability.
- **Discriminativity**: distinguish between shapes belonging to different classes.
- **Invariance**: invariant (insensitive) to a class of transformations.

B, PAMI'11

An ideal descriptor

- **Localization**: changes a lot under a small displacement on the surface.
- **Sensitivity**: a small set of best matches should contain a correct match with high probability.
- **Discriminativity**: distinguish between shapes belonging to different classes.
- **Invariance**: invariant (insensitive) to a class of transformations.
- **Efficiency**: maximum information per number of dimensions.

B, PAMI'11

Optimal spectral descriptors

- Build *optimal spectral descriptor* of the form

$$\mathbf{p}(x) = \sum_{k \geq 0} \begin{pmatrix} f_1(\nu_k) \\ \vdots \\ f_n(\nu_k) \end{pmatrix} \phi_k^2(x)$$

B, PAMI'11

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B, PAMI'11

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  - attenuate frequencies with large noise content (deformation)
  - pass frequencies with large signal content (discriminative geometric features)

B, PAMI'11

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B, PAMI'11

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- Similar in spirit to *Wiener filter*:
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  - pass frequencies with large signal content (discriminative geometric features)
- Hard to model axiomatically
- ...yet easy to *learn* from examples!

B, PAMI'11

Parametrization

- To do learning, we need a *finite* set of parameters

B, PAMI'11

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- Problem: each shape has *distinct eigen-frequencies* $\{\nu_i\}$.

B, PAMI'11

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spanning a sufficiently wide interval of frequencies $[0, \nu_{\max}]$

B, PAMI'11

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- Represent responses as

$$\begin{pmatrix} f_1(\nu) \\ \vdots \\ f_n(\nu) \end{pmatrix} = \mathbf{A} \begin{pmatrix} b_1(\nu) \\ \vdots \\ b_m(\nu) \end{pmatrix}$$

with the matrix of parameters **A**

B, PAMI'11

Parametrization

- Select sufficiently large s for which $\nu_s \geq \nu_{\max}$

B, PAMI'11

Parametrization

- Select sufficiently large s for which $\nu_s \geq \nu_{\max}$
- Represent the point descriptor

$$
\begin{aligned}
\mathbf{p}(x) &= \sum_{k \geq 0} \begin{pmatrix} f_1(\nu_k) \\ \vdots \\ f_n(\nu_k) \end{pmatrix} \phi_k^2(x) \\
&\approx \mathbf{A} \begin{pmatrix} b_1(\nu_1) & \cdots & b_1(\nu_s) \\ \vdots & \ddots & \vdots \\ b_m(\nu_1) & \cdots & b_m(\nu_s) \end{pmatrix} \begin{pmatrix} \phi_1^2(x) \\ \vdots \\ \phi_s^2(x) \end{pmatrix} = \mathbf{Ag}(x)
\end{aligned}
$$

B, PAMI'11

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- *Geometry vector* $\mathbf{g}$ consistently represents all geometric information at point x.

B, PAMI'11

Learning

- x a point, x_+ a knowingly similar point (*positive*), x_- a knowingly dissimilar point (*negative*).

B, PAMI'11

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B, PAMI'11

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B, PAMI'11

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$$
\begin{aligned}
d_\pm^2 &= \|\mathbf{p} - \mathbf{p}_\pm\|^2 = \|\mathbf{A}(\mathbf{g} - \mathbf{g}_\pm)\|^2 \\
&= (\mathbf{g} - \mathbf{g}_\pm)^{\mathrm{T}} \mathbf{A}^{\mathrm{T}} \mathbf{A} (\mathbf{g} - \mathbf{g}_\pm).
\end{aligned}
$$

B, PAMI'11

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B, PAMI'11

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- *Mahalanobis distance* on geometry vectors space.
- Mahalanobis *metric learning* problem.

B, PAMI'11

Learning

- Taking expectation over positive/negative pairs

$$\begin{aligned}\mathbb{E}\{d_\pm^2\} &= \mathbb{E}\left\{\|\mathbf{p}-\mathbf{p}_\pm\|^2\right\} = \mathbb{E}\left\{(\mathbf{g}-\mathbf{g}_\pm)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}(\mathbf{g}-\mathbf{g}_\pm)\right\} \\ &= \mathrm{tr}\left\{\mathbf{A}\mathbb{E}((\mathbf{g}-\mathbf{g}_\pm)(\mathbf{g}-\mathbf{g}_\pm)^{\mathrm{T}})\mathbf{A}^{\mathrm{T}}\right\} \\ &= \mathrm{tr}\left\{\mathbf{A}\mathbf{C}_\pm\mathbf{A}^{\mathrm{T}}\right\}\end{aligned}$$

Strecha,Fua,BB, PAMI'11; B, PAMI'11

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&= \mathrm{tr}\left\{\mathbf{A}\mathbf{C}_{\pm}\mathbf{A}^{\mathrm{T}}\right\}
\end{aligned}$$

- $\mathbf{C}_{\pm}$ is the *covariance matrix* of positives/negatives $\mathbf{g}-\mathbf{g}_{\pm}$.

Strecha,Fua,BB, PAMI'11; B, PAMI'11

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- Minimize weighted difference

$$\min\ (1-\alpha)\mathbb{E}(d_{+}^2) - \alpha\mathbb{E}(d_{-}^2)$$

Strecha,Fua,BB, PAMI'11; B, PAMI'11

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Strecha,Fua,BB, PAMI'11; B, PAMI'11

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Strecha,Fua,BB, PAMI'11; B, PAMI'11

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$$\min \ \mathrm{tr}\left\{\mathbf{A}\mathbf{D}_{\alpha}\mathbf{A}^{\mathrm{T}}\right\}$$

- α controls tradeoff between *sensitivity* ($\alpha \to 1$) and *specificity* ($\alpha \to 0$).

Strecha,Fua,BB, PAMI'11; B, PAMI'11

Learning

- Scale of $\mathbf{A}$ is arbitrary!

B, PAMI'11

Learning

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- Recall *efficiency*: dimensions of $\mathbf{p}$ should be *statistically independent*

B, PAMI'11

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B, PAMI'11

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$$\mathbf{I} = \mathbb{E}(\mathbf{p}\mathbf{p}^{\mathrm{T}}) = \mathbf{A}\mathbb{E}(\mathbf{g}\mathbf{g}^{\mathrm{T}})\mathbf{A}^{\mathrm{T}} = \mathbf{A}\mathbf{C}\mathbf{A}^{\mathrm{T}}$$

B, PAMI'11

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B, PAMI'11

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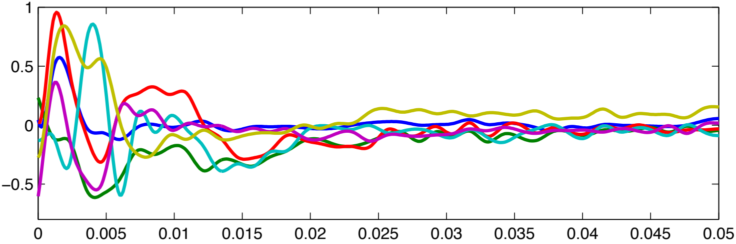
- Solution: $\mathbf{A} = \mathbf{U}_n^{\mathrm{T}}\mathbf{C}^{-1/2}$ where $\mathbf{U}_n$ are the n smallest eigenvectors of $\mathbf{C}^{-1/2}\mathbf{D}_{\alpha}\mathbf{C}^{-1/2}$.

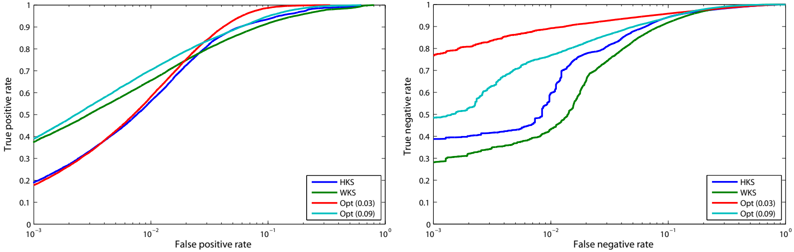
B, PAMI'11

Optimal spectral descriptor



- Optimal filters (invariance learned from positive and negative examples)

Optimal spectral descriptor



- Better specificity and sensitivity than HKS and WKS

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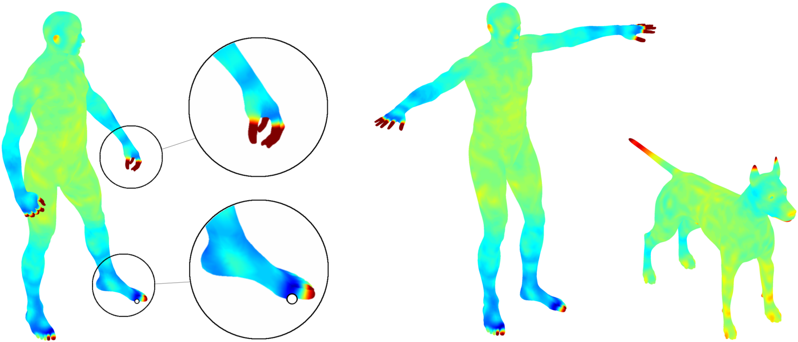
Optimal spectral descriptor



- Better spatial feature localization than WKS
- Better discriminativity than HKS

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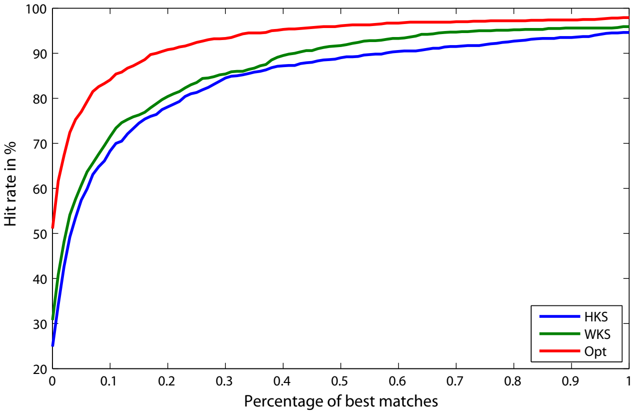
Optimal spectral descriptor



- Better performance in correspondence problems

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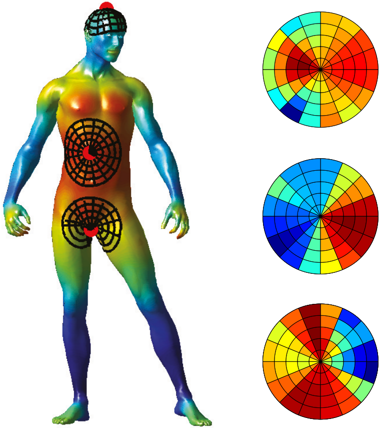
Intrinsic shape contexts

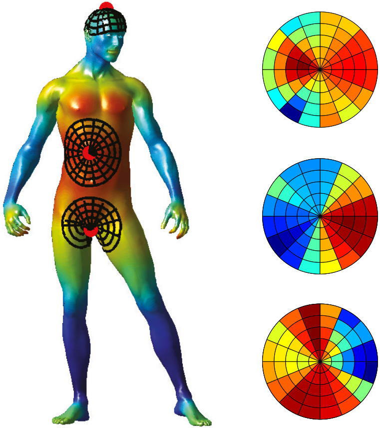
- Spectral descriptors lack *orientation* information



Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts

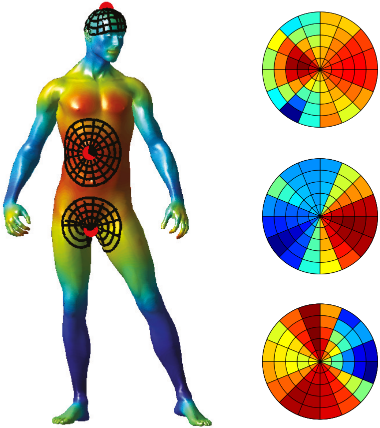
- Spectral descriptors lack *orientation* information
- Given a vector field **p** on surface, compute its distribution over a local polar system of coordinate



Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts

- Spectral descriptors lack *orientation* information
- Given a vector field **p** on surface, compute its distribution over a local polar system of coordinate
- *Intrinsic shape context* (ISC) – a meta-descriptor



Kokkinos,Litman,BB, CVPR'12

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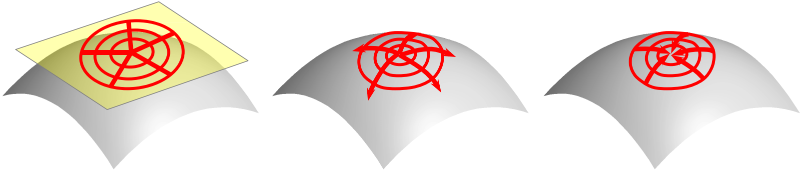
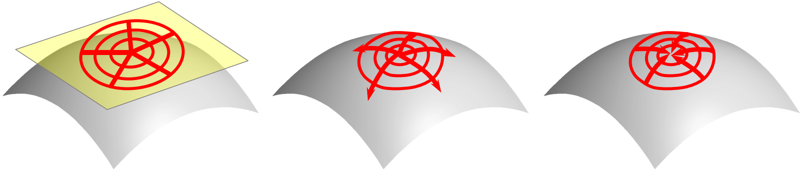


Tangent plane map | Inward shooting | Outward shooting

- Problem I: no global coordinate system

Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts



Tangent plane map Inward shooting Outward shooting

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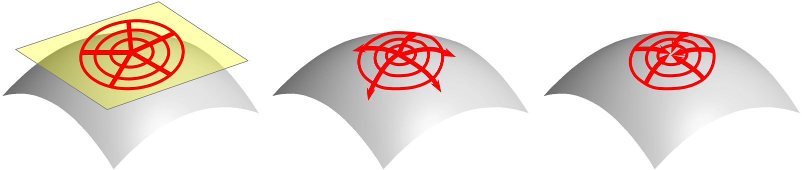
Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts



Tangent plane map   Inward shooting   Outward shooting

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Kokkinos,Litman,BB, CVPR'12

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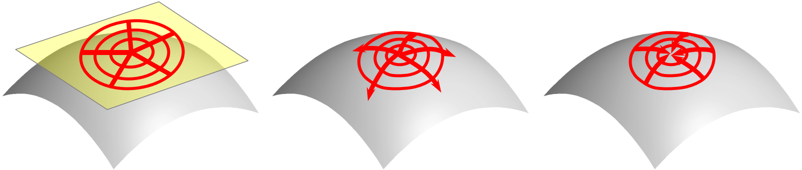
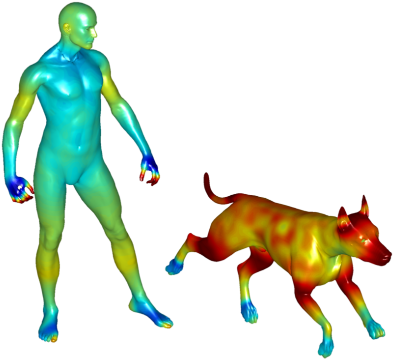


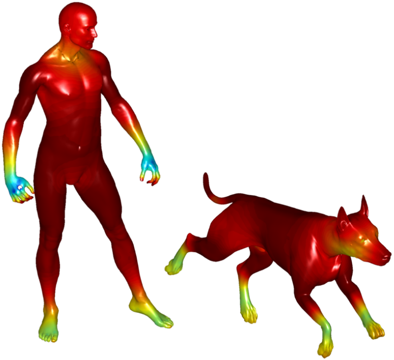
- Problem I: no global coordinate system
- Local chart has to be created
- Problem II: arbitrary angular coordinate
- Undone using Fourier transform modulus

Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts



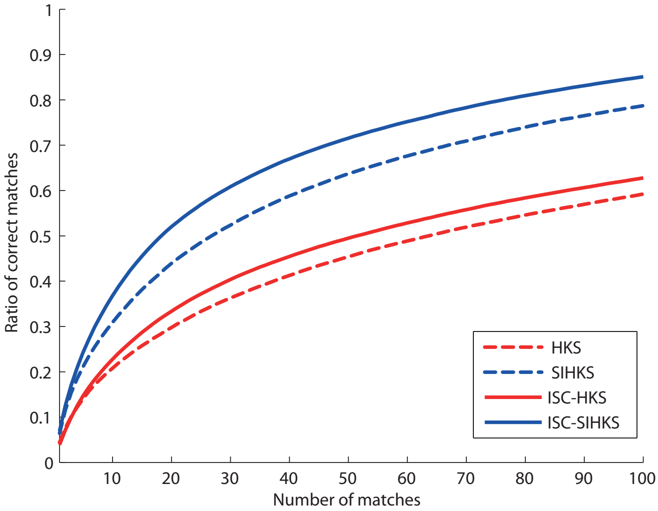
Scale Invariant HKS



Intrinsic Shape Context

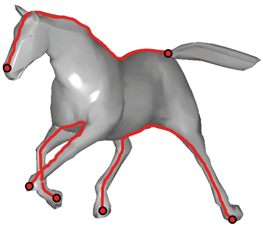
Kokkinos,Litman,BB, CVPR'12

Intrinsic shape contexts

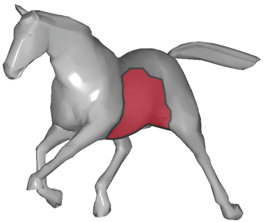


Kokkinos,Litman,BB, CVPR'12

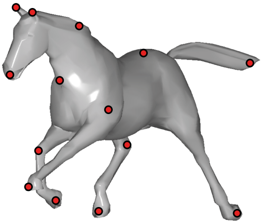
Glocal structure



Global structure
Metric space



Glocal structure
Stable regions



Local structure
Point descriptors

Component trees

- Measure proximity $d(x, y)$ in some local neighborhood $y \in \mathcal{N}(x)$

Component trees

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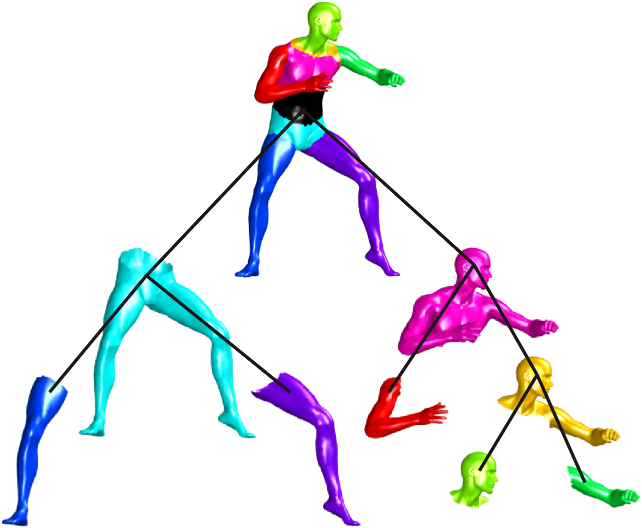
- Single-linkage *agglomerative clustering*
- Hierarchy described as *component tree*

Component trees

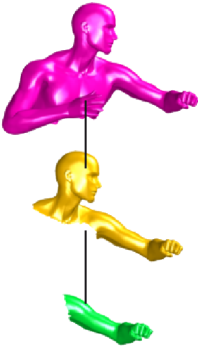


Maximally stable components

- *Stability* of a component

$$\sigma(C_t) = \frac{A(C_t)}{\frac{dA(C_t)}{dt}}$$

- Measures relative change of area as function of change of threshold
- (Better stability functions are available)



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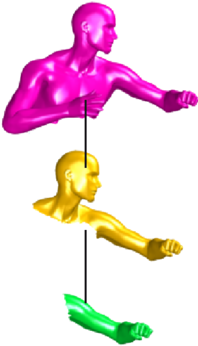
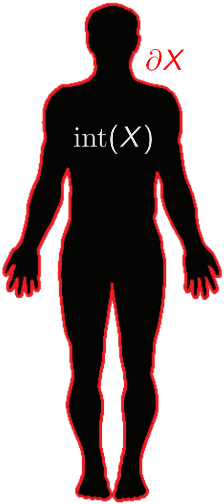
- (Better stability functions are available)
- *Maximally stable components*: local maximizers of σ



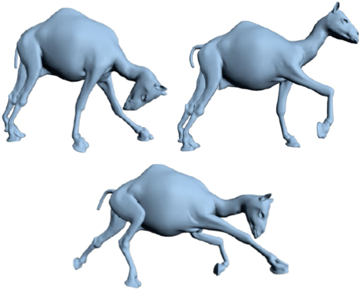
Maximally stable components

Is our model good?



Is our model good?

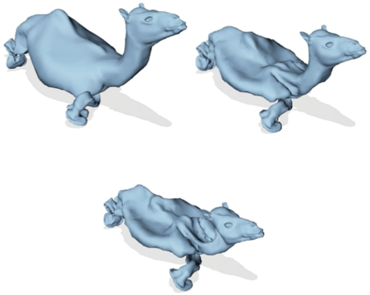
Volume isometry



Preserves geodesic distances
inside the volume

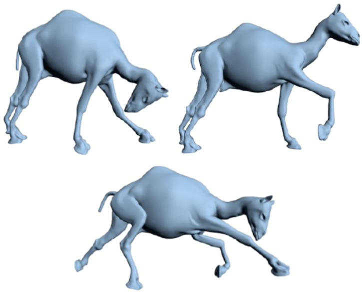
Is our model good?

Boundary isometry



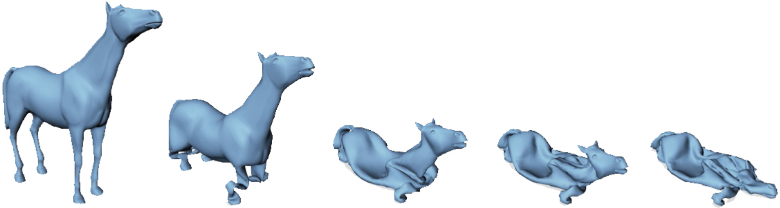
Preserves geodesic distances
on the boundary surface

Volume isometry



Preserves geodesic distances
inside the volume

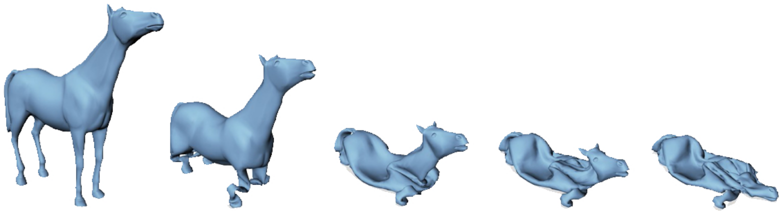
Volumetric diffusion geometry



- *Boundary isometry* does not always represent a realistic deformation

(image: Sumner)

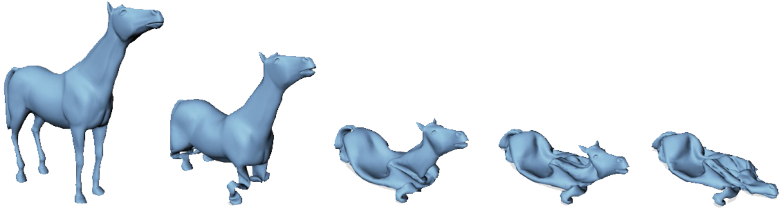
Volumetric diffusion geometry



- *Boundary isometry* does not always represent a realistic deformation
- May change volume of the solid

(image: Sumner)

Volumetric diffusion geometry



- *Boundary isometry* does not always represent a realistic deformation
- May change volume of the solid
- Solution: *volumetric* diffusion geometry

(image: Sumner)

Volumetric diffusion geometry

Boundary diffusion

$$\left(\Delta_{\partial X} + \frac{\partial}{\partial t}\right) u(x,t) \quad = \quad 0$$

where
$u : \partial X \times [0, \infty) \to \mathbb{R}$
$\Delta_{\partial X}$ - Laplace-Beltrami operator

Volumetric diffusion geometry

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$u : \partial X \times [0,\infty) \to \mathbb{R}$
$\Delta_{\partial X}$ - Laplace-Beltrami operator

Volumetric diffusion

$$\left(\Delta + \frac{\partial}{\partial t}\right) U(x,t) = 0,\; x \in \mathrm{int}(X)$$
$$\langle \nabla U(x,t), n(x) \rangle = 0,\; x \in \partial X$$

where
$U : X \times [0,\infty) \to \mathbb{R}$
Δ - Euclidean Laplace operator
n - normal to ∂X

Volumetric diffusion geometry

Boundary heat kernels

$$k_t(x, y) = \sum_{i \geq 0} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

where $\Delta_{\partial X} \phi_i(x) = \lambda_i \phi_i(x)$

Volumetric diffusion geometry

Boundary heat kernels

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Volumetric heat kernels

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$\langle \nabla \Phi_i(x), n(x) \rangle = 0$, $x \in \partial X$

Volumetric diffusion geometry

Boundary heat kernels

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Geometric interpretation
$k_t(x,x) \approx \frac{1}{4\pi t}\left(1 + \frac{1}{6}\kappa(x)\right)$

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Volumetric diffusion geometry

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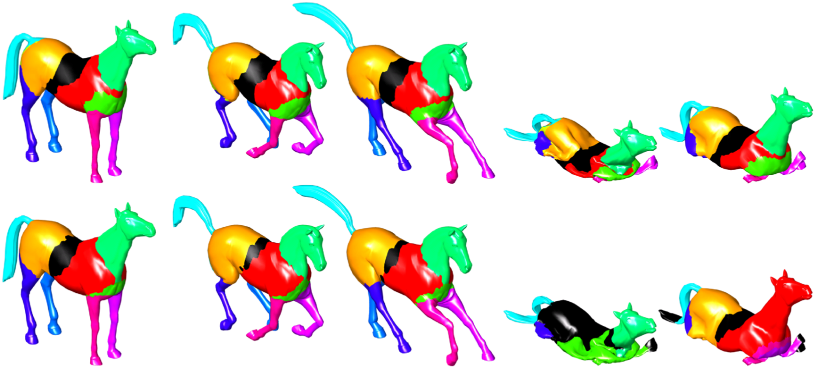
$\langle\nabla\Phi_i(x), n(x)\rangle = 0,\ x \in \partial X$

Geometric interpretation
$K_t(x,x) \approx \frac{1}{(4\pi t)^{3/2}}\left(1 + \frac{1}{6}s(x)\right)$

Volumetric diffusion geometry

Volumetric maximally stable components



Conclusion

- Diffusion processes on manifolds

Conclusion

- Diffusion processes on manifolds
- Spectrum of the Laplacian

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- *Global structure*: diffusion geometry

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- *Semi-local structure*: maximally stable components

Conclusion

- Diffusion processes on manifolds
- Spectrum of the Laplacian
- *Global structure*: diffusion geometry
- *Local structure*: diffusion kernel signatures
- *Semi-local structure*: maximally stable components
- Extensions

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Numerical Geometry of Non-Rigid Shapes

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