Vortex reconnections in atomic condensates at finite temperature

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The study of vortex reconnections is an essential ingredient of understanding superfluid turbulence, a phenomenon recently also reported in trapped atomic Bose–Einstein condensates. In this work we show that, despite the established dependence of vortex motion on temperature in such systems, vortex reconnections are actually temperature independent on the typical length/time scales of atomic condensates. Our work is based on a dissipative Gross-Pitaevskii equation for the condensate, coupled to a semiclassical Boltzmann equation for the thermal cloud (the Zaremba–Nikuni–Griffin formalism). Comparison to vortex reconnections in homogeneous condensates further show reconnections to be insensitive to the inhomogeneity in the background density.

In classical hydrodynamics, reconnections of stream lines, vortex lines and magnetic flux tubes change the topology of the flow and contribute to energy dissipation. In quantum fluids, vorticity is not a continuous field, but is concentrated in discrete vortex lines of quantised circulation; their reconnections are therefore isolated, dramatic events. Individual quantum reconnections have been recently visualized [1] in superfluid ^{4}He. The role of vortex reconnections in the dynamics of quantum turbulence [2] in superfluid ^{4}He, superfluid ^{3}He and atomic Bose–Einstein condensates (BECs) is currently debated. For example, one would like to understand their contribution to acoustic dissipation of kinetic energy [3], their role in the proposed bottleneck [4] between a semi–classical Kolmogorov cascade at small wavenumbers and a quantum Kelvin wave cascade at large wavenumbers, and the possibility of a cascade of vortex rings scenario [5, 6]. The increasing ability to imprint [7], generate [8–11], manipulate [12] and directly image [11, 13] vortices makes atomic condensates ideal systems to study vortex reconnection events [14]. This problem is of particular interest in light of recent experiments regarding quantum turbulence and vortex dynamics in both two [15–17] and three dimensions [18] (for a review on progress in two and three dimensions see, e.g. [19]).

Since many experiments are performed at relatively high temperatures, i.e. large fractions of the critical temperature, T_c, a natural question to ask is whether thermal excitations affect vortex reconnections. A recent experiment visualising vortex reconnections in superfluid ^{4}He, suggests that this is not the case [1]. However, previous investigations have shown that the presence of a thermal cloud significantly changes the motion of vortices in harmonically trapped condensates [20–25].

In this paper we present results of an investigation of vortex reconnections in finite–temperature trapped Bose–Einstein condensates. We model the problem in the context of the Zaremba–Nikuni–Griffin (ZNG) formalism [26, 27], where the Gross-Pitaevskii equation (GPE) is generalized by the inclusion of the thermal cloud mean field, and a dissipative or source term which is associated with a collision term in a semiclassical Boltzmann equation for the thermal cloud. The main feature of this model is that the condensate and thermal cloud interact with each other self–consistently; for a strongly nonlinear dynamical event like a vortex reconnection, a simpler and less accurate approach may give misleading answers.

The governing ZNG equations are

$$i\hbar\frac{\partial\phi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}} + g\left[n_c + 2\tilde{n}\right] - iR\right)\phi\,, \quad (1)$$

and

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}f - (\nabla_{\mathbf{r}}U_{\text{eff}})\cdot(\nabla_{\mathbf{p}}f) = C_{12} + C_{22}. \quad (2)$$

In this formalism $\phi = \phi(\mathbf{r}, t)$ is the condensate wavefunction, $f = f(\mathbf{r}, \mathbf{p}, t)$ is the phase-space distribution function of thermal atoms, $V_{\text{ext}} = m\omega^2 r^2/2$ is the harmonic potential which confines the atoms (assumed, for simplicity, to be spherically-symmetric), ω is the trapping frequency, m the atomic mass, and $g = 4\pi\hbar^2 a_s/m$, with a_s being the s-wave scattering length. Equation (1) generalises the GPE for a $T = 0$ condensate by the addition of the thermal cloud mean–field potential $2g\tilde{n}$ and the dissipation/source term $-iR(\mathbf{r}, t)$. The condensate density is $n_c(\mathbf{r}, t) = |\phi(\mathbf{r}, t)|^2$ and the thermal cloud density is recovered from $f(\mathbf{r}, \mathbf{p}, t)$ via an integration over all momenta, $\tilde{n}(\mathbf{r}, t) = (2\pi\hbar)^{-3}\int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t)$. The mean-field potential acting on the thermal cloud is $U_{\text{eff}} = V_{\text{ext}}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)]$. The quantities $C_{22}[f]$ and $C_{12}[\phi, f]$ are collision integrals defined in

Refs. [26, 27] (which contain further details of the model). C_{22} describes the redistribution of thermal atoms as a result of two–atom collisions (as in the usual Boltzmann equation), while C_{12}, which is related to $-iR$ via $R(\mathbf{r}, t) = \hbar/(2n_c(\mathbf{r}, t)) \int d\mathbf{p}/(2\pi\hbar)^3 C_{12}[\phi(\mathbf{r}, t), f(\mathbf{p}, \mathbf{r}, t)]$, describes the change in the phase-space distribution function $f(\mathbf{p}, \mathbf{r}, t)$ resulting from particle-exchange collisions between thermal atoms and condensate atoms.

Before solving the ZNG equations (1) and (2) numerically, we write them in dimensionless form, using the harmonic oscillator's characteristic length $\ell = \sqrt{\hbar/m\omega}$ as the unit of distance, the inverse trapping frequency ω^{-1} as the unit of time, and $\hbar\omega$ as the unit of energy. We choose experimentally realistic parameters: $\omega = 2\pi \times 150$Hz, $\tilde{\mu} = \mu/(\hbar\omega) \approx 18$ where μ is the chemical potential, and $\tilde{g} = g/(\ell\hbar\omega) \approx 6230$. This corresponds to $\ell = 0.88\mu$m, $\omega^{-1} = 1.06$ms and $N_c \approx 75000$ [87]Rb atoms. Throughout this work, in order to facilitate more direct comparisons, we keep this value of N_c approximately constant even at temperatures above zero so that the effect of increasing temperature is to increase the number of thermal atoms in the system rather than depleting the condensate. As well as solving the ZNG equations, we use another model to describe finite temperature effects. A simple phenomenological extension of the GPE known as the dissipative Gross–Pitaevskii equation (DGPE, (Eq. 3)), is used for temperatures below $0.4T_c$, which is thought to be reasonable limit for its validity [28], while in the trapped case, for temperatures higher than this we use the ZNG equations.

First we consider what happens in the limit of zero temperature, for which Eq. (1) with $iR = 0$ and $\tilde{n} = 0$ reduces to the GPE, a model known to provide an accurate description of condensate dynamics for $T \ll T_c$, including collective modes and vortex dynamics [29, 30]. For the quantities considered here, we have confirmed that the results of the ZNG equations in the absence of any significant thermal cloud, agree with the results given by the GPE. The initial state of our simulation is the condensate containing a pair of straight line anti-parallel vortices aligned in the z–direction at locations $(x_0/\ell, y_0/\ell) = (-1, \pm 1)$. This state is formed by imaginary time propagation of the GPE while enforcing a 2π winding of the phase of ϕ around the location of the cores.

In the GPE model, the vortex core size is of the order of the condensate healing length, $\xi = \hbar/\sqrt{2m\mu}$. For our assumed parameters, $\xi/\ell \approx 0.167$. This is much smaller than the condensate radius, given approximately by the Thomas-Fermi radius, $R_{TF}/\ell = \sqrt{2\tilde{\mu}} \approx 6$. To ensure that the vortices reconnect in the central region of the condensate and away from its boundary, we perturb their y position along z according to $A[\cos(2\pi z/\lambda)]^6$ where $A/\ell = 0.5$ and $\lambda/\ell = 20$. This initial condition is shown in the top left panel of Fig. 1, where we have plotted a series of snapshots of the reconnection for the case of $T = 0$. It is apparent that the two vortices initially move as a pair in the xy plane, traveling in the x direction. The slight initial curvature enhances the Crow instabil-
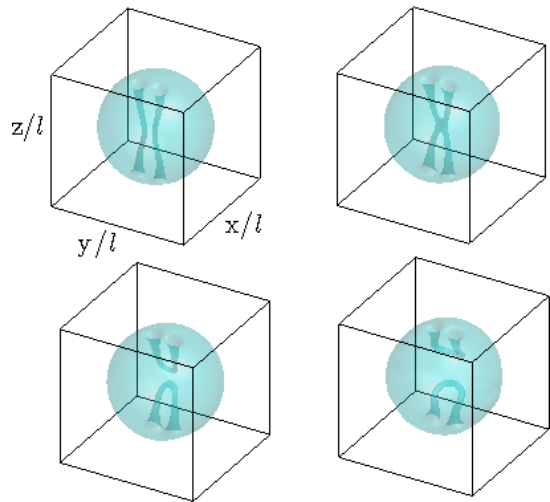

FIG. 1: Vortex reconnection in the $T = 0$ trapped condensate. Density isosurfaces of the dynamics of the initial anti–parallel vortex pair, at times $t\omega = 0$ (top left), $t\omega = 1.19$ (top right), $t\omega = 1.80$ (bottom left) and $t\omega = 2.5$ (bottom right), according to the GPE. The isosurfaces are plotted at 40% of the peak density. The range of the axes is $-6\ell \leq x, y, z \leq 6\ell$.

ity [31]: the vortices approach each other and reconnect, creating two U–shaped cusps which lift and move away from each other above and below the xy plane (bottom right panel).

Simulations at various temperatures show that the vortex reconnection proceeds essentially unaffected by the presence of the thermal cloud, consistent with the findings of Ref. [1]. To illustrate this we focus on the relatively high temperature case of $T/T_c = 0.62$. Figure 2 compares density slices of the $T = 0$ condensate (left panel) and the $T/T_c = 0.62$ condensate and thermal cloud (middle and right panels), both before reconnection (top row) and after reconnection (middle and bottom rows). It is apparent that thermal atoms are concentrated at the edge of the condensate and inside the vortex cores [25], an effect of the mean–field repulsion from the condensate. Importantly, over the time scale for the reconnection, we observe no difference in the vortex dynamics between the $T = 0$ and the $T > 0$ cases.

The above numerical simulations refer to the typical experimental situation where the condensate is larger, but not much larger, than the vortex core size ($R_{TF} \approx 36\xi$). In this case, and as evident in Figs. 1 and 2, the reconnection region is not far from the condensate outer surface. This surface region can undergo oscillations, particularly in a turbulent condensate [32], and interact with the vortices. Moreover the surface region is where thermal atoms accumulate, and is likely to introduce relatively large frictional effects upon the vortices. In order to more clearly extract out the role of finite temperature on reconnections it is therefore instructive to consider reconnections in a homogeneous (boundary-free) condensate.
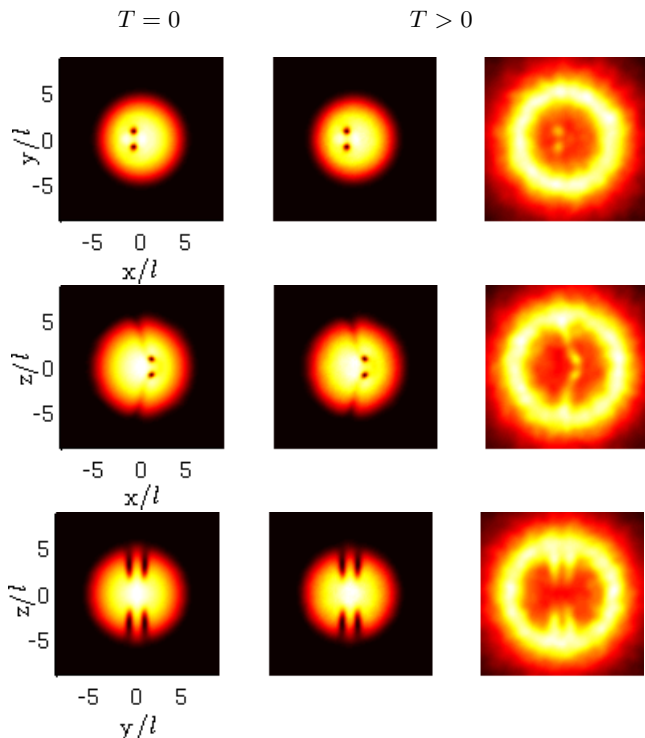
FIG. 2: Vortex reconnection in the trapped condensate at $T = 0$ (left column) and $T/T_c = 0.62$ (middle and right columns). 2D density plots from the ZNG model of the condensate (left and middle) and thermal cloud (right). Top: Before the reconnection, $t\omega = 0.3$, on the $z/\ell = -2.5$ plane (rather than the $z/\ell = 0$ plane, in order to better distinguish the two vortex cores). Middle and bottom: After the reconnection, $t\omega = 2$, slices through the xz plane for $y/\ell = 0$ (middle row) and the yz plane for $x/\ell = 0$ (bottom row). In the density scale, white corresponds to peak density and black to zero density. Note the vanishing condensate density in the vortex cores (left and middle column) and the concentration of thermal atoms at the edge of the condensate and inside the cores (right column).

In the interest of computational ease and efficiency, we can use a simpler extension to the GPE which simulates thermal effects by the inclusion of a phenomenological damping parameter, called the dissipative Gross–Pitaevskii equation (DGPE) [28, 33],

$$(i - \gamma)\hbar\frac{\partial\phi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + g|\phi|^2 - \mu\right)\phi. \qquad (3)$$

The phenomenological damping parameter γ, which we assume to be constant, has been used in a variety of systems to simulate thermal effects (see e.g. [28, 34–36]). For $\gamma = 0$ this model reduces to the GPE. We solve Eq. (3) in a periodic box in the absence of an external potential. It must be stressed that, unlike the ZNG model, the DGPE does not include the dynamics of the thermal cloud.

Before solving Eq. (3) we write it in dimensionless form using natural units based on the healing length $\xi = \hbar/\sqrt{2m\mu}$ and the time unit ξ/c. As done by Zuc-
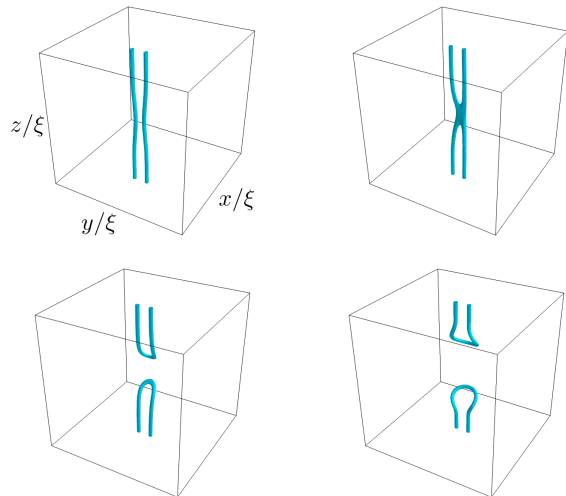

FIG. 3: Vortex reconnection in the homogeneous condensate at $T = 0$. Density isosurfaces showing the reconnection of the vortex-antivortex pair, according to the GPE, at times $t/(\xi/c) = 0$ (top left), $t/(\xi/c) = 20$ (top right), $t/(\xi/c) = 40$ (bottom left) and $t/(\xi/c) = 60$ (bottom right). The isosurfaces are plotted at 20% of the peak density. The range of the axes is $-30\xi \leq x, y, z \leq 30\xi$.

cher et $al.$ [37], we use a Fourier-splitting scheme where the Laplacian part is trivially solved in spectral space, whereas the remaining part is exactly solved in physical space as suggested by Bao et $al.$ [38]. We place a pair of anti–parallel vortex lines in a computational box of size $-30\xi \leq x, y, z \leq 30\xi$ at position $(x_0/\xi, y_0/\xi) = (10, \pm 3)$. Again, to accelerate the reconnection and ensure it occurs away from boundaries, the vortex lines are initially perturbed according to $A[\cos(2\pi z/\lambda)]^6$, now with $A/\xi = 1$ and $\lambda/\xi = 120$ (chosen such that the the vortices are unperturbed at $z_{\max}$ and $z_{\min}$). The box size is choosen such that the vortex length is comparable to the vortex length in the ZNG simulation.

The initial configuration and subsequent evolution of the vortex pair is depicted in Fig. 3 for $\gamma = 0$, corresponding to $T = 0$. The vortex reconnection proceeds in much the same way as for the trapped condensate shown in Fig. 1. We repeat the simulation for $\gamma = 0.03$ (corresponding to a temperature of approximately $0.4T_c$ [28, 39, 40]) and again note that the reconnection proceeds essentially unchanged despite the presence of dissipation in the system.

To monitor the vortex reconnections more precisely than "by eye", we consider the minimum distance between the vortex lines, $\delta(t)$. We extract the position of the vortex core by finding the grid points where the density is a local minimum and about which the phase changes by 2π [41]. The time–dependence of this quantity (before and after the reconnection) was experimentally observed in superfluid ^{4}He, and predicted theoretically for superfluids based on the GPE ($T = 0$) [37] and for ordinary viscous fluids based on the classical Navier–

Stokes equation [42]. To enable comparison of $\delta(t)$ between the homogeneous and trapped systems, we must convert between harmonic trap units (based on the harmonic oscillator length and frequency) and natural units (based on the healing length and the chemical potential). The conversion for length from harmonic oscillator units to natural units is given by $x' = \tilde{x}\ell/\xi$ and for time is $t' = c/(\omega\xi)\tilde{t}$ where we used a tilde to denote the quantity in harmonic trap units and a prime for natural units (see footnote [46] for more details). For the remainder of this article we express all quantities in natural units.

Figure 4 compares $\delta'(t')$ computed using the ZNG model (trapped condensate) and the DGPE (homogeneous condensate) before (top) and after the reconnection (bottom). For completeness, we have also carried out DGPE simulations, within the presence of a trap for the value $\gamma = 0.0012$ which corresponds to a temperature of approximately $0.4T_c$ (as used for the homogeneous case). We find that $\delta'(t')$ scales as

$$\delta'(t') = \kappa|t' - t'_0|^\nu \qquad (4)$$

where t_0 is the time at which the reconnection takes place (defined as when $\delta'(t') = 0$), and κ and ν are fitting parameters. It is apparent that the results are essentially independent of the model, the temperature and the presence/absence of trapping. The best fit to the parameter ν before the reconnection is $\nu = 0.41 \pm 0.02$ and after the reconnection is $\nu = 0.66 \pm 0.02$.

Our results compare well with the exponents $\nu = 0.39$ ($t < t_0$) and $\nu = 0.68$ ($t > t_0$) reported by Zuccher et al. [37] over a wide range of initial angles between the vortex lines, computed for $T = 0$ (GPE) in a homogeneous condensate. It is interesting to remark that viscous reconnections of the Navier–Stokes equation display a similar time asymmetry [42] (the largest ν being that after the reconnection, as in a quantum fluid). Zuccher et al. [37] argued that the time asymmetry is due to acoustic emission: part of the kinetic energy of the vortices is transformed into sound waves which radiate to infinity, in analogy with viscous dissipation in an ordinary fluid which turns kinetic energy into heat. Indeed, if one uses the Biot–Savart law (an incompressible model) to monitor the behaviour of vortices just before and after the reconnections, one finds $\nu = 0.5$ for both $t < t_0$ and $t > t_0$. Paoletti et al. [1] observed individual quantum reconnections in superfluid ^{4}He experiments, reporting the exponent $\nu = 0.5$ averaged over all $t < t_0$ and $t > t_0$ data. Above all, Paoletti et al. did not notice any temperature dependence, which is consistent with our findings.

We stress that, a priori, one would not apply the Biot–Savart model to a vortex in a small atomic condensate, as the vortex core is not negligible, particularly in a reconnection, when two vortex cores collide. However, we may gain some insight to the temperature–independence of the reconnecting behaviour from it. In the Biot–Savart model [43] the vortex is a three-dimensional space curve $\mathbf{s} \equiv \mathbf{s}(\varsigma, t)$ of infinitesimal thickness, where ς is
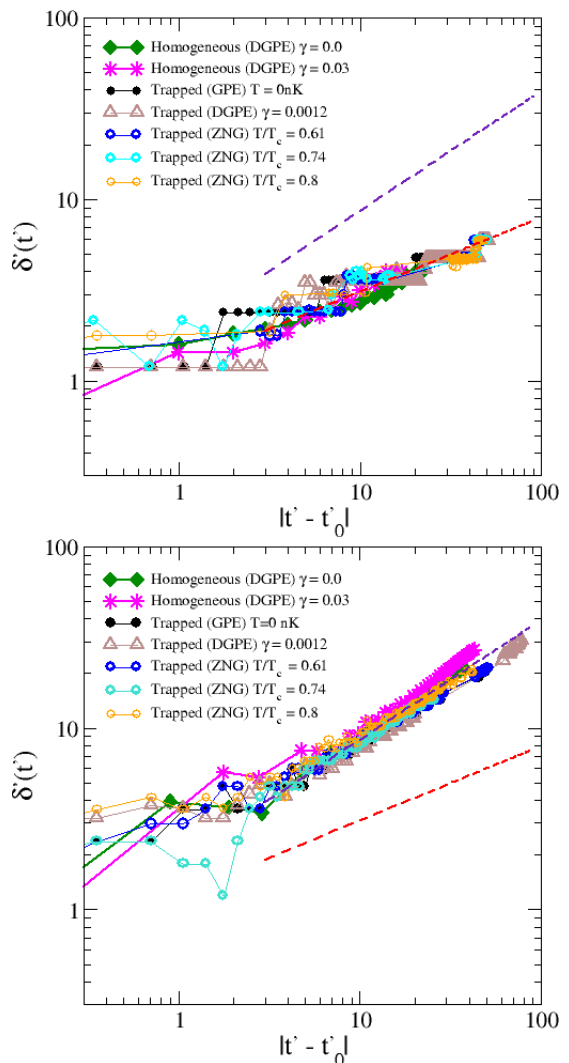

FIG. 4: Minimum distance between vortex line, $\delta'(t')$, before (top) and after (bottom) the vortex-antivortex reconnection, computed for a harmonically trapped condensate and a homogeneous condensate at different values of T/T_c and γ (both the homogeneous and trapped values of γ correspond to $T \approx 0.4T_c$), including the limiting cases $T = 0$ and $\gamma = 0$. The dashed lines are fits according to Eq. (4) with parameters $\nu = 0.41 \pm 0.02$ (line of best fit before reconnection, red, dot-dashed) and $\nu = 0.66 \pm 0.02$ (line of best fit after reconnection, indigo, dashed) with corresponding κ values, for the data considered here, of 1.36 and 1.88 respectively.

arc length. The velocity of the curve at the point $\mathbf{s}$ is $\mathbf{v}_L = \mathbf{v}_{si} - \alpha\mathbf{s}' \times \mathbf{v}_{si}$, where $\mathbf{v}_{si}$ is the self–induced velocity (determined by a Biot–Savart integral over the entire vortex configuration), $\mathbf{s}' \equiv d\mathbf{s}'/d\varsigma$ is the unit tangent vector at $\mathbf{s}$, and α is a dimensionless temperature dependent friction parameter. In the full expression $\mathbf{v}_L$ there is a second friction parameter, α', which we have neglected here since it is much smaller than α [23, 44, 45]. In superfluid helium, outside the phase transition region (less than 1 percent from T_c), α is less than unity and positive [23, 44]. In atomic condensates, numerical sim-

ulations of vortex motion based on the ZNG model have shown that [23] $\alpha < 0.03$ for $T/T_c < 0.8$. The small value of α has been confirmed by an independent calculation based on a classical field approach [44]. The Biot–Savart model thus suggests that, instantaneously, the friction gives only a small contribution to the velocity of the vortex. One expects the friction to be significant only on large enough length scales and time scales, provided that its effects can accumulate. For example, in the case of a single off-centered vortex in a harmonic trap precessing at finite temperature, the interaction of the vortex with the thermal cloud causes it to lose energy and spiral out of the condensate, thus limiting its lifetime [20–25]. However, this decay may require many orbits in the trap [21, 23, 25, 36].

In conclusion, we have found that, on the typical short length scales and time scales relevant to a vortex reconnection in an atomic Bose–Einstein condensate, the reconnection is essentially temperature independent, despite the significant inhomogeneity of the thermal cloud in the vortex cores and near the boundary of the condensate. Since vortex reconnections are essential ingredients of turbulence, our result suggests that at least this rapid part of the dynamics is rather universal, and does not depend on T, although the large scale motion of vortices does depend on T.

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[46] For harmonic oscillator units, length and time are defined as $\tilde{x} = x/\ell$ and $\tilde{t} = \omega t$ respectively (where the tilde represents the quantity in harmonic oscillator dimensionless units). Similarly for natural units we define length and time as $x' = x/\xi$ and $t' = t/(\xi/c)$ respectively, where $c = \sqrt{\mu/m}$ is the sound speed (the prime denotes the quantity in natural dimensionless units)