Integration of Vlasov-type equations

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Joint work with
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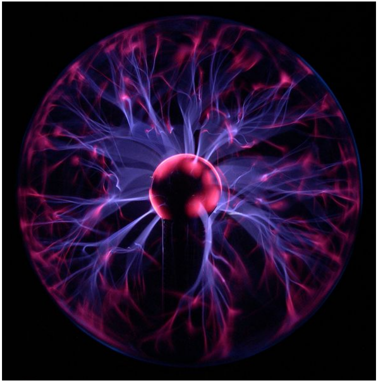
Plasma – the fourth state of matter

99% of the visible matter in the universe is made of plasma

Source: Hubble telescope



Source: Luc Viatour / www.Lucnix.be

Vlasov equation

Evolution of (collisionless) plasma is described by

$$\partial_t f(t, \boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f(t, \boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f(t, \boldsymbol{x}, \boldsymbol{v}) = 0,$$

where

f	...	particle density function	$\boldsymbol{x}$	...	position
$\boldsymbol{v}$	...	velocity	$\boldsymbol{F}$	...	force field

Coupled to the self-consistent electric (and magnetic) field via

$$\boldsymbol{F} = \frac{e}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

yields the Vlasov–Poisson (Vlasov–Maxwell) equations.

Applications: Tokamaks (fusion), plasma-laser interactions, ...

Vlasov equation, cont.

Anatoly Alexandrovich **Vlasov** (1908–1975), Moscow State U

The vibrational properties of an electron gas (1938, 1968)

533.9

О ВИБРАЦИОННЫХ СВОЙСТВАХ ЭЛЕКТРОННОГО ГАЗА *)

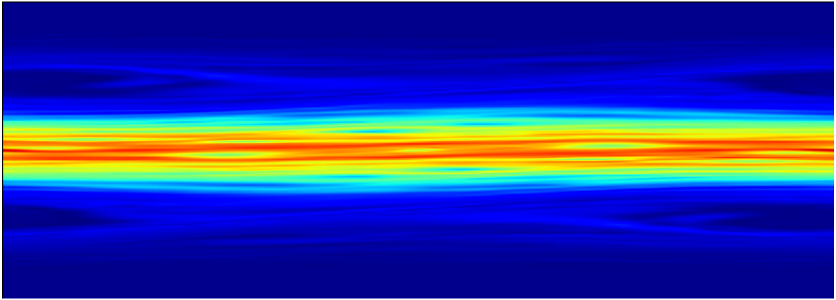
А. А. Власов

1. Постановка задачи. 2. Исходные уравнения и их упрощение. 3. Решение линеаризованных уравнений. 4. Дисперсия продольных волн. 5. Дисперсия продольных волн в электронном газе с функцией распределения по Ферми. 6. Дисперсия поперечных волн. 7. Резюме и заключение.

$$\frac{\partial f}{\partial t} + \mathrm{div}_{\mathbf{r}}\, \mathbf{v} f + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\,[\mathbf{vH}]\right)\mathrm{grad}_{\mathbf{v}} f = 0$$

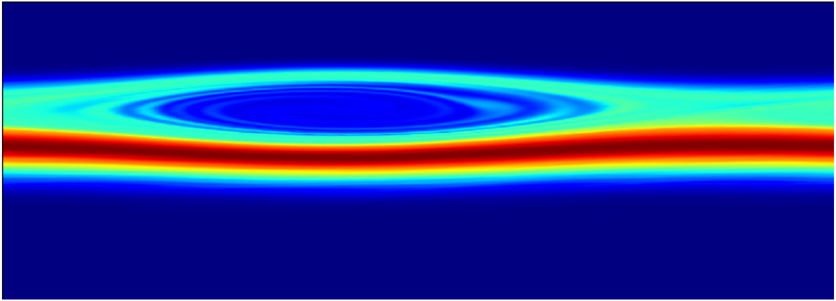
$$\partial_t f(t,\boldsymbol{x},\boldsymbol{v}) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f(t,\boldsymbol{x},\boldsymbol{v}) + \frac{e}{m}\Big(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\Big)\cdot\nabla_{\boldsymbol{v}} f(t,\boldsymbol{x},\boldsymbol{v}) = 0$$

Filamentation of phase space



Simulation of the filamentation of phase space as is studied, for example, in the context of Landau damping.
The horizontal axis represents space and the vertical one velocity.

Bump-on-tail instability



The bump-on-tail instability leads to a traveling vortex in phase space. The horizontal axis represents space and the vertical one velocity.

Outline

Strang splitting for Vlasov-type equations

dG space discretization

Numerical examples

References

Abstract formulation

Abstract initial value problem

$$\partial_t f(t) = (A + B) f(t), \quad f(0) = f_0,$$

where

- A is a *linear* differential operator;
- the *nonlinear* operator B has the form $Bf = B(f)f$, where $B(f)$ is an (unbounded) linear operator.

Formulation comprises the Vlasov–Poisson and the Vlasov–Maxwell equations as special cases:

take $A = -\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}$ and choose B appropriately

Example: Vlasov–Poisson equation in 1+1 d

Recall the abstract formulation

$$\partial_t f(t) = (A + B) f(t), \quad f(0) = f_0.$$

Example: Vlasov–Poisson equations in 1+1 dimensions

$$\begin{aligned} \partial_t f(t,x,v) &= -v\partial_x f(t,x,v) - E(f(t,\cdot,\cdot),x)\partial_v f(t,x,v) \\ \partial_x E(f(t,\cdot,\cdot),x) &= \int_{\mathbb{R}} f(t,x,v)\,\mathrm{d}v - 1, \qquad E = -\partial_x \Phi \\ f(0,x,v) &= f_0(x,v) \end{aligned}$$

with periodic boundary conditions in space

$$f(t,x,v) = f(t,x+L,v).$$

Strang splitting

Problem: $\partial_t f(t) = (A + B)f(t), \quad Bf = B(f)f.$

Denote $f_k \approx f(t_k)$ at time $t_k = k\tau$ with step size τ.

Solve

$$\partial_t f = Af, \qquad \partial_t g = B_{k+1/2}g,$$

where $B_{k+1/2} \approx B\left(f(t_k + \frac{\tau}{2})\right)$ and let

$$f_{k+1} = S_k f_k,$$

where the (nonlinear) splitting operator S_k is given by

$$S_k = \mathrm{e}^{\frac{\tau}{2}A}\mathrm{e}^{\tau B_{k+1/2}}\mathrm{e}^{\frac{\tau}{2}A}.$$

For grid-based Vlasov solvers:
Cheng and Knorr 1976; Mangeney et al. 2002

Computational efficiency and stability

Splitting is very efficient in our application:

$$\partial_t f = -v\partial_x f$$
$$f(\tau, x, v) = \mathrm{e}^{-\tau v \partial_x} f(0, x, v) = f(0, x - \tau v, v)$$

and

$$\partial_t g = -E_{k+1/2}(x)\partial_v g$$
$$g(\tau, x, v) = \mathrm{e}^{-\tau E_{k+1/2}(x)\partial_v} g(0, x, v) = g(0, x, v - \tau E_{k+1/2}(x))$$

imply $\|S_k\| = 1$. Moreover, we have

$$\left\| \left(\mathrm{e}^{\tau B_{k+1/2}} - \mathrm{e}^{\tau B(f(t_k+\tau/2))} \right) \mathrm{e}^{\tau A} f(t_k) \right\| \le C\tau \| f_{k+1/2} - f(t_k+\tau/2)\|$$

(requires some smoothness of f).

Consistency – technical tools

Lemma [Gröbner–Alekseev formula] Consider

$$f'(t) = G(f(t)) + R(f(t)), \quad f(0) = f_0$$
$$g'(t) = G(g(t)) \quad \text{with solution } E_G(t, g_0)$$

Then

$$f(t) = E_G(t, f_0) + \int_0^t \partial_2 E_G\left(t - s, f(s)\right) R\left(f(s)\right) \mathrm{d}s.$$

Proof. Fundamental theorem of calculus + trick.

Lemma. Let E generate a (semi)group. Then

$$\mathrm{e}^{\tau E} = \sum_{k=0}^{m-1} \frac{\tau^k}{k!} E^k + \tau^m E^m \int_0^1 \mathrm{e}^{(1-s)\tau E} \frac{s^{m-1}}{(m-1)!} \mathrm{d}s$$

Proof. Integration by parts.

Proof of the Gröbner–Alekseev formula

Fix t and let $u(t)$ be a solution of $u'(t) = G(u(t))$.

Next differentiate the identity

$$E_G(t-s, u(s)) = u(t)$$

with respect to s to get

$$-\partial_1 E_G(t-s, u(s)) + \partial_2 E_G(t-s, u(s))\, G(u(s)) = 0.$$

The initial value of u is now chosen such that $u(s) = f(s)$ which implies

$$-\partial_1 E_G(t-s, f(s)) + \partial_2 E_G(t-s, f(s))\, G(f(s)) = 0.$$

Proof of the Gröbner–Alekseev formula, cont.

Let $\varphi(s) = E_G(t-s, f(s))$. By the fundamental theorem of calculus

$$\begin{aligned} f(t) - E_G(t, f_0) &= \int_0^t \varphi'(s)\,\mathrm{d}s \\ &= \int_0^t \Big(-\partial_1 E_G(t-s, f(s)) + \partial_2 E_G(t-s, f(s)) f'(s)\Big)\,\mathrm{d}s \\ &= \int_0^t \partial_2 E_G\,(t-s, f(s))\, R\,(f(s))\,\mathrm{d}s, \end{aligned}$$

where we have used $f'(s) = G(f(s)) + R(f(s))$ and

$$-\partial_1 E_G\,(t-s, f(s)) + \partial_2 E_G\,(t-s, f(s))\, G\,(f(s)) = 0.$$

Consistency, cont.

Expansion of the exact solution

$$f(\tau) = E_B(\tau, f_0) + \int_0^\tau \partial_2 E_B(\tau - \tau_1, f(\tau_1)) A E_B(\tau_1, f_0)\, \mathrm{d}\tau_1$$

$$+ \int_0^\tau \int_0^{\tau_1} \partial_2 E_B(\tau - \tau_1, f(\tau_1)) A \partial_2 E_B(\tau_1 - \tau_2, f(\tau_2)) A E_B(\tau_2, f_0) \mathrm{d}\tau_2 \mathrm{d}\tau_1$$

$$+ \int_0^\tau \int_0^{\tau_1} \int_0^{\tau_2} \left(\prod_{k=0}^{2} \partial_2 E_B(\tau_k - \tau_{k+1}, f(\tau_{k+1})) A \right) f(\tau_3)\, \mathrm{d}\tau_3 \mathrm{d}\tau_2 \mathrm{d}\tau_1,$$

where $\tau_0 = \tau$.

Expansion of the numerical solution $S_k f_0 = \mathrm{e}^{\frac{\tau}{2}A} \mathrm{e}^{\tau B_{k+1/2}} \mathrm{e}^{\frac{\tau}{2}A} f_0$

$$S_0 f_0 = \mathrm{e}^{\tau B_{1/2}} f_0 + \frac{\tau}{2} \left\{ A, \mathrm{e}^{\tau B_{1/2}} \right\} f_0 + \frac{\tau^2}{8} \left\{ A, \left\{ A, \mathrm{e}^{\tau B_{1/2}} \right\} \right\} f_0 + R_3 f_0$$

Consistency, cont.

- Compare the terms by employing quadrature rules (Jahnke, Lubich 2000).
- Treat τ term with trapezoidal rule

$$\int_0^\tau \partial_2 E_B(\tau-\tau_1, f(\tau_1)) A E_B(\tau_1, f_0)\,\mathrm{d}\tau_1 =$$

$$\frac{\tau}{2}\partial_2 E_B(\tau, f_0) A f_0 + \frac{\tau}{2} A E_B(\tau, f_0) f_0 + R_1$$

and compare with

$$\frac{\tau}{2}\left\{A, \mathrm{e}^{\tau B_{1/2}}\right\} f_0 = \frac{\tau}{2} A\,\mathrm{e}^{\tau B_{1/2}} f_0 + \frac{\tau}{2}\mathrm{e}^{\tau B_{1/2}} A f_0$$

Consistency, cont.

- Estimate the difference

$$\left(\mathrm{e}^{\tau B_{1/2}} - \partial_2 E_B(\tau, f_0)\right) A f_0 = \partial_2 \left(\mathrm{e}^{\tau B_{1/2}} - E_B(\tau, f_0)\right) A f_0.$$

- Apply again the variation of constants formula

$$E_B(\tau, f_0) - \mathrm{e}^{\tau B_{1/2}} f_0 = \int_0^\tau \mathrm{e}^{(\tau - s) B_{1/2}} (B - B_{1/2}) E_B(s, f_0) \,\mathrm{d}s.$$

- Gives condition on $B_{1/2}$.
- Treat all terms in this way and estimate remainders.

Regularity of exact solution

Consider Vlasov–Poisson problem as before.

Theorem (Glassey 1996)

Assume that $f_0 \in \mathcal{C}^m_{\mathrm{per,c}}$ is non-negative, then

$$f \in \mathcal{C}^m(0, T; \mathcal{C}^m_{\mathrm{per,c}}),$$
$$E(f(t,\cdot,\cdot), x) \in \mathcal{C}^m(0, T; \mathcal{C}^m_{\mathrm{per}}).$$

Moreover, there exists $Q(T)$ such that for all $t \in [0, T]$ and $x \in \mathbb{R}$ it holds that

$$\operatorname{supp} f(t, x, \cdot) \subset [-Q(T), Q(T)].$$

Important result: Regularity of initial data is preserved.

Abstract convergence result

Theorem (L. Einkemmer, A.O., *SIAM J. Numer. Anal.*, 2014a)

Let

$$f_0 \in \mathcal{C}^3_{\mathrm{per,c}} \text{ be non-negative}$$

$$\|f_{k+1/2} - f(t_k + \tfrac{\tau}{2})\| \le C\tau$$

Then Strang splitting for the Vlasov–Poisson equations is second-order convergent.

Possible choices for $f_{k+1/2}$:

first-order Lie–Trotter splitting $f_{k+1/2} = \mathrm{e}^{\frac{\tau}{2}B(f_k)}\mathrm{e}^{\frac{\tau}{2}A} f_k$

due to the special structure of the E field $f_{k+1/2} = \mathrm{e}^{\frac{\tau}{2}A} f_k$

linear extrapolation using f_{k-1} and f_k

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Strang splitting for Vlasov-type equations

dG space discretization

Numerical examples

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dG approximation in space and velocity

- Choose grid in (x, v) plane with cells R_{ij}.
- Project onto space of Legendre polynomials of order ℓ.

  The projection is given by

$$Pg|_{R_{ij}} = \sum_{k=0}^{\ell} \sum_{m=0}^{\ell} b_{km}^{ij} P_k^{(1)}(x) P_m^{(2)}(v)$$

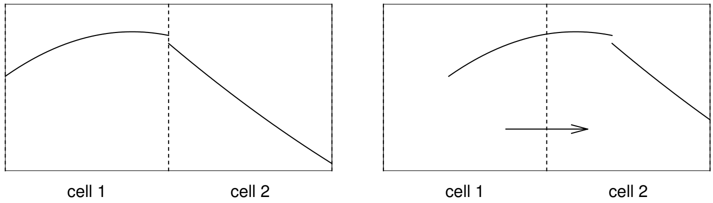
  with coefficients

$$b_{km}^{ij} = \frac{(2k+1)(2m+1)}{h_x h_v} \int_{R_{ij}} P_k^{(1)}(x) P_m^{(2)}(v) g(x, v) \, \mathrm{d}(x, v),$$

  where $P_k^{(1)}$ and $P_m^{(2)}$ are translated and scaled Legendre polynomials.

Space discretization

$$Pe^{\frac{\tau}{2}A}g(x,v) = Pg\left(x - \tfrac{\tau}{2}v, v\right)$$



- Polynomial approximation of functions with a small jump discontinuity;
- Jump heights $\varepsilon^{(k)} = g^{(k)}(x_0+) - g^{(k)}(x_0-)$ satisfy $|\varepsilon^{(k)}| \le ch^{\ell-k+1}$ for all $k \in \{0,\ldots,\ell\}$;
- Then $\left\| g^{(k)} - (Pg)^{(k)} \right\| \le Ch^{\ell-k+1}, \quad 0 \le k \le \ell.$

Space and velocity discretization

- $P\mathrm{e}^{\frac{\tau}{2}A} g(x,v) = Pg\left(x - \frac{\tau}{2}v, v\right)$
- $P\mathrm{e}^{\tau B(Pf_{k+1/2})} g(x,v) = Pg\left(x, v - \tau E_{k+1/2}(x)\right)$
- evaluate integrals exactly with Gauss–Legendre quadrature

Spatially discretized Strang splitting:

$$\widetilde{S}_k = P\mathrm{e}^{\frac{\tau}{2}A} P\mathrm{e}^{\tau B(Pf_{k+1/2})} P\mathrm{e}^{\frac{\tau}{2}A}$$

Convergence of full discretization

Theorem (L. Einkemmer, A.O., *SIAM J. Numer. Anal.*, 2014b)
Suppose that $f_0 \in \mathcal{C}^{\max\{\ell+1,3\}}$, nonnegative and compactly supported. Then

$$\left\| \left(\prod_{k=0}^{n-1} \widetilde{S}_k \right) P f_0 - f(n\tau) \right\| \leq C \left(\tau^2 + \frac{h^{\ell+1}}{\tau} + h^{\ell+1} \right).$$

Proof. Combine time and space error

Stability, revisited

In advection dominated problems instabilities can occur if the numerical integration is not performed exactly.

Example (Molenkamp–Crowley test problem)

$$\begin{aligned} \partial_t f(t,x,y) &= 2\pi(y\partial_x - x\partial_y) f(t,x,y) \\ f(0,x,v) &= f_0(x,y), \end{aligned}$$

with off-centered Gaussian as initial value.
Initial value turns around origin with period 1.

Finite element schemes turn out to be unstable for most quadrature rules [Morton, Priestley, Süli (1988)].

Molenkamp–Crowley test problem, cont.

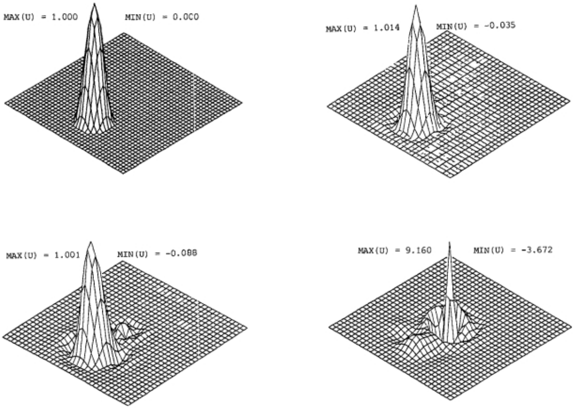
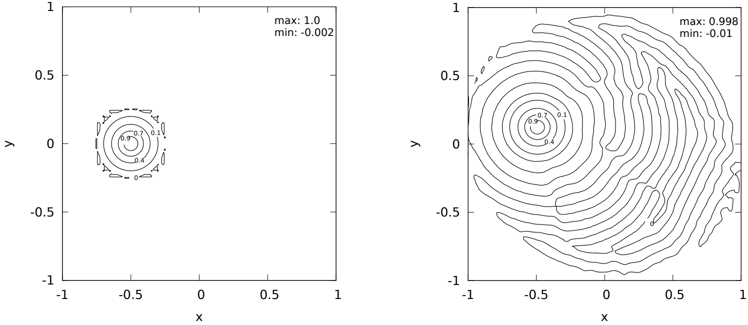


Figure 4.1. — Rotating cone problem with three interior point quadrature on triangles after 0, 1, 5 and 20 revolutions.

Morton, Priestley, Süli: Stability of the Lagrange–Galerkin method with non-exact integration, RAIRO (1988)

Molenkamp–Crowley test problem, cont.



Projected initial value (left); numerical solution after 60 revolutions with $\tau = 0.02$, $N_x = N_v = 40$, and $\ell = 2$ (right). Our dG method is stable.

Outline

Example: nonlinear Landau damping

Vlasov–Poisson equations in 1+1 dimensions on $[0, 4\pi] \times \mathbb{R}$ together with the initial value

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-v^2/2} \left(1 + \alpha \cos \frac{x}{2}\right).$$

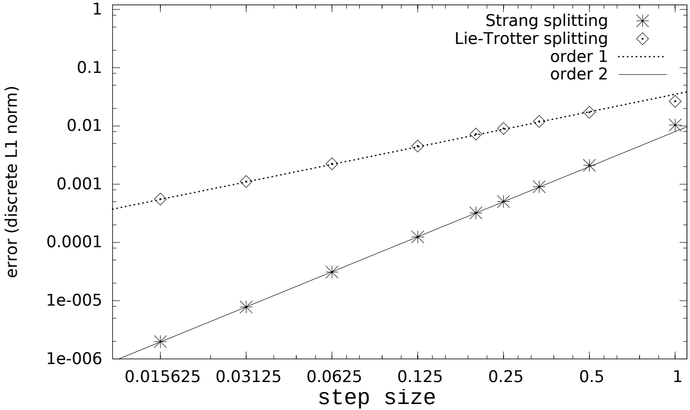
Landau damping is a popular test problem.

- weak or linear Landau damping $\alpha = 0.01$
- strong or nonlinear Landau damping $\alpha = 0.5$

Consider Strang and Lie–Trotter splitting

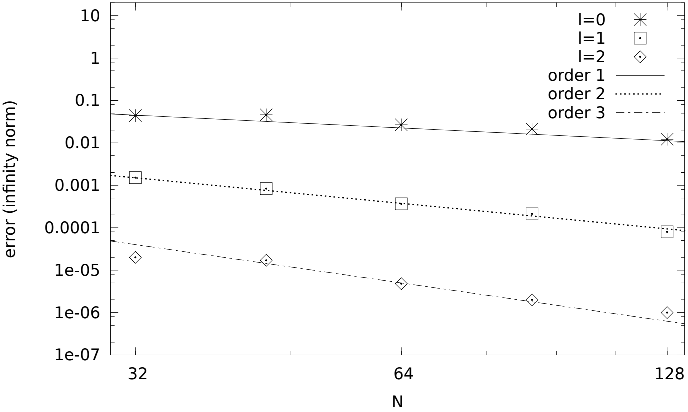
$$S_{L,k} = \mathrm{e}^{\tau B_k} \mathrm{e}^{\tau A}.$$

Nonlinear Landau damping; time error



Error (in discrete L^1 norm) of the particle density function $f(1,\cdot,\cdot)$ for Strang and Lie–Trotter splitting, respectively.

Nonlinear Landau damping; space error



Spatial error (in maximum norm) of the particle density function for dG discretizations of Strang splitting.

Recurrence phenomenon

Most easily understood for a transport equation. Consider

$$\partial_t f = -v\partial_x f, \qquad f_0(x, v) = \frac{\mathrm{e}^{-v^2/2}}{\sqrt{2\pi}}\big(1 + 0.01\cos(0.5x)\big)$$

with exact solution

$$f(t, x, v) = \frac{\mathrm{e}^{-v^2/2}}{\sqrt{2\pi}}\big(1 + 0.01\cos(0.5x - 0.5vt)\big).$$

The electric energy

$$\mathcal{E}(t) = \int_0^{4\pi} E(t, x)^2\,\mathrm{d}x = \frac{\pi}{1250}\mathrm{e}^{-0.25t^2}$$

decays exponentially in time.

Recurrence, cont.

For the numerical approximation of

$$f(t,x,v) = \frac{\mathrm{e}^{-v^2/2}}{\sqrt{2\pi}}\big(1 + 0.01\cos(0.5x - 0.5vt)\big).$$

consider a piecewise constant approximation in v with grid size h_v.

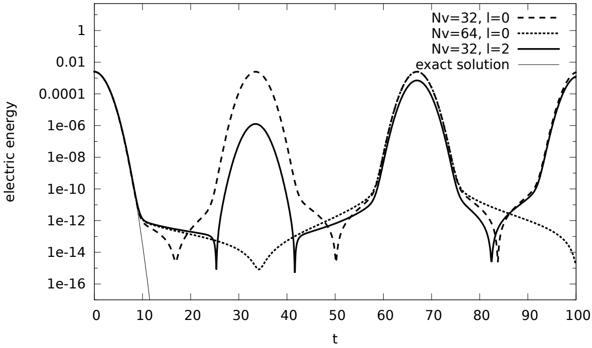
Let the time t_p be given by

$$h_v t_p = 4\pi, \qquad v_j t_p = j h_v t_p = 4j\pi.$$

Obviously, the numerical solution is periodic with period t_p.

The numerical approximation to the electric energy $\mathcal{E}$ is thus periodic in time as well.

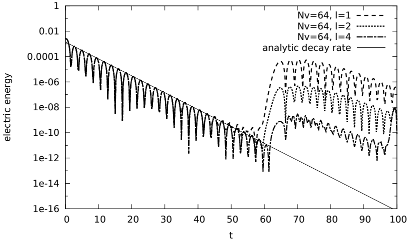
Recurrence for the advection equation

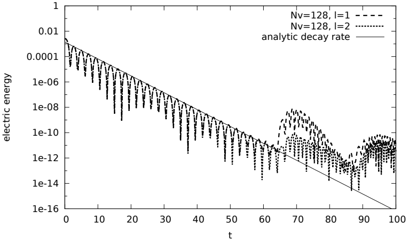


There is no periodicity in the approximation of degree 2.
Nevertheless, a recurrence-like effect is still visible.

Recurrence, weak Landau damping

The electric energy decays asymptotically as $\mathrm{e}^{-2\gamma t}$, $\gamma \approx 0.1533$. We compare our numerical solutions with this function.





The decay of the electric energy is shown for $N_x = N_v = 64$ (left) and $N_x = N_v = 128$ (right). In all cases a relatively large time step of $\tau = 0.2$ is employed.

A strategy to suppress the recurrence effect

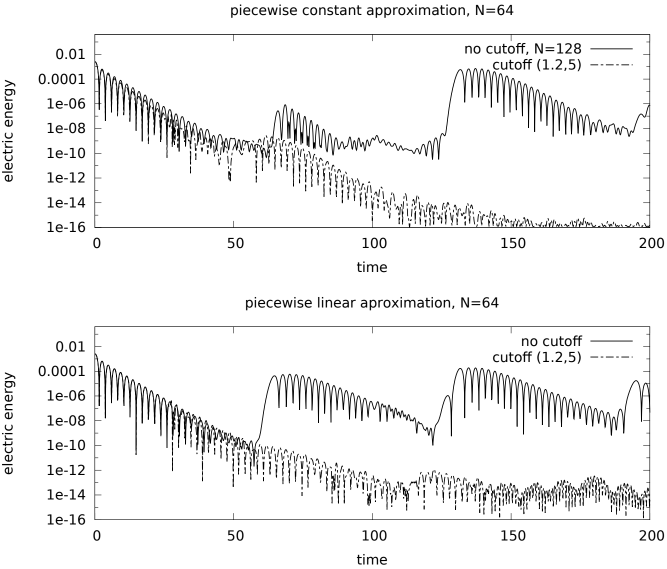
Recurrence is not a consequence of the lack of accuracy of the underlying space discretization, but a result of aliasing.

- higher and higher frequencies in phase space are created;
- aliasing of the high frequencies introduces an error in the macroscopic quantities (such as the electric energy).

Strategy to avoid recurrence (Einkemmer, A.O., EJPD 2014):

- damp highest frequencies;
- in our dG implementation: FFTs of coefficients of Legendre polynomials;
- after a cutoff time t_c, cut off the highest M frequencies cutoff(t_c, M)

Recurrence, weak Landau damping



Example: plasma echo phenomenon

Initial value

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-v^2/2} (1 + \alpha \cos(k_1 x))$$

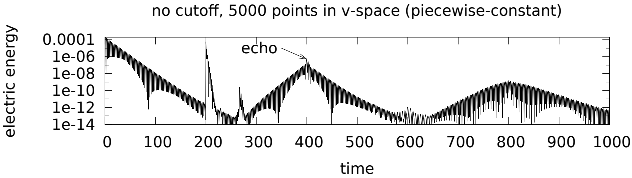
corresponds to an excitation with wavenumber k_1 at time $t = 0$. Furthermore, at time $t = t_2$ we excite a second perturbation with wavenumber k_2; that is, we superimpose

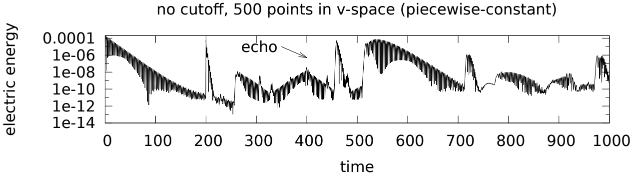
$$\frac{\alpha}{\sqrt{2\pi}} \mathrm{e}^{-v^2/2} \cos(k_2 x)$$

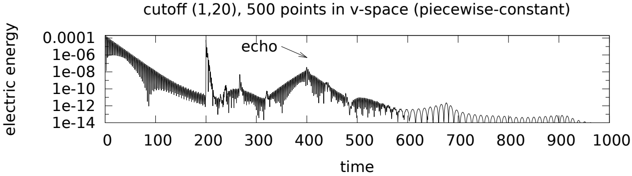
on the (numerical) solution at time t_2.

Data: $\alpha = 10^{-3}$, $k_1 = 12\pi/100$, $k_2 = 25\pi/100$, and $t_2 = 200$

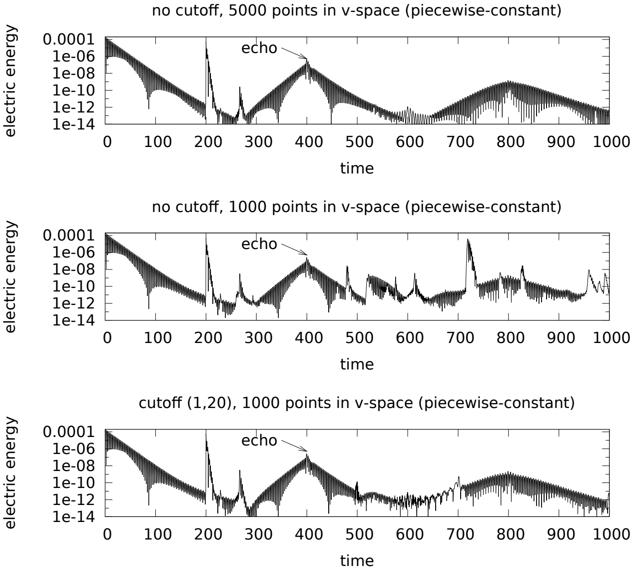
Echo, low resolution







Echo, higher resolution



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