Advanced Methods for ODEs 2015 – Projects

Project 1. Korteweg–de Vries equation and solitons

1. Show that
$$u(t,x) = 3c\,\mathrm{sech}^2\Big(\tfrac{\sqrt{c}}{2}(x - x_0 - ct)\Big), \qquad \mathrm{sech}\,x = \frac{1}{\cosh x}$$
is a solution of the Korteweg–de Vries (KdV) equation
$$u_t + u_{xxx} + uu_x = 0, \qquad x \in \mathbb{R}.$$
Give an interpretation of the real parameters $c > 0$ and x_0. The solution is called a *soliton* (solitary wave). What happens for $c \le 0$?

2. Construct the above soliton in the following way.

   (a) Let $\xi = x - ct$ and determine a function $u(t,x) = w(\xi)$ that solves the KdV equation. You will find $w_{\xi\xi\xi} + ww_\xi - cw_\xi = 0$.

   (b) Integrate this equation twice und use that w, w_ξ and $w_{\xi\xi}$ vanish for $x \to \infty$. (After the first integration, you should multiply the resulting equation by the integrating factor w_ξ.) This gives $w_\xi^2 + w^3/3 - cw^2 = 0$.

   (c) The last equation can be integrated by a separation of variables.

3. Solve the KdV equation with periodic boundary conditions on the interval $[0, 2\pi]$ for $0 \le t \le 1$. Use as initial value a soliton with appropriately chosen $c > 0$ and x_0. Employ a spectral discretization (based on FFT) for the linear part and the method of characteristics for the Burgers' nonlinearity. Study the error as a function of the time step size τ, the spatial resolution $h = \Delta x$ and the employed interpolation procedure in the method of characteristics.

4. Repeat the above experiment with two solitions and study their interaction. (The initial values of the two solitons must not overlap - why?)

Project 2. Exponential quadrature rules

1. Solve the one-dimensional heat equation
$$u_t(t,x) = u_{xx}(t,x) + (2 + (x(1-x))\,\mathrm{e}^t, \qquad 0 \le x \le 1,$$
subject to homogeneous Dirichlet boundary conditions. Use the initial value $u(0,x) = x(1-x)$ and integrate the equation up to $t = 1$.

   For the space discretization, use standard finite differences (i.e., the stencil $\frac{1}{(\Delta x)^2}[1, -2, 1]$). This results in a system of linear ODEs of the form
$$U' = AU + f(t).$$
For its time integration, employ the exponential Euler method. The required matrix functions can be computed by diagonalizing A.

2. Compute the weights of the Gaussian exponential quadrature methods with 1 and 2 stages. Show that for $A = 0$, the weights reduce to the standard weights $b_1 = 1$ and $b_1 = b_2 = \frac{1}{2}$, respectively.

3. Repeat the above integration with the Gaussian exponential quadrature methods and study the order of convergence for fixed Δx. You might use, e.g., $\Delta x = 1/100$ and $1/256 \le \tau \le 1$. The observed order of convergence for the two-stage method in the L^p-norm should be $3 + \frac{1}{2p}$, i.e. 3 in the infinity norm, 3.25 in the Euclidian norm, and 3.5 in the 1-norm.

4. Which order do you observe in the H^1-norm, i.e. $\|u\|^2 = u^\mathsf{T} Au$?

Project 3. Reaction-diffusion splitting

Consider the one-dimensional heat equation

$$u_t(t,x) = u_{xx}(t,x) + f(u(t,x)), \quad u(0,x) = u_0(x),$$

subject to Dirichlet boundary conditions $u(t,0) = b_0(t)$ and $u(t,1) = b_1(t)$.

1. Find the solution of $z_{xx} = 0$ that satisfies the boundary conditions b_0 and b_1 and transform the above problem to homogeneous Dirichlet boundary conditions $\widetilde{u} = u - z$

$$\widetilde{u}_t = \widetilde{u}_{xx} + f(\widetilde{u} + z) - z_t. \tag{1}$$

Henceforth, we consider
(i) the *standard splitting* of (1) into

$$\widetilde{v}_t = \widetilde{v}_{xx} - z_t, \quad \widetilde{v}(t,0) = \widetilde{v}(t,1) = 0 \tag{2}$$

and

$$\widetilde{w}_t = f(\widetilde{w} + z); \tag{3}$$

(ii) the *modified splitting* of (1) into

$$\widetilde{v}_t = \widetilde{v}_{xx} + f(z) - z_t, \quad \widetilde{v}(t,0) = \widetilde{v}(t,1) = 0 \tag{4}$$

and

$$\widetilde{w}_t = f(\widetilde{w} + z) - f(z). \tag{5}$$

The first step of the standard [resp. modified] Stang splitting takes the form:

(a) Compute the initial value $\widetilde{v}(0) = u_0 - z_0$.
(b) Compute the solution of (2) [resp. (4)] with initial value $\widetilde{v}(0)$ to obtain $\widetilde{v}(\frac{\tau}{2})$.
(c) Compute the solution of (3) [resp. (5)] with initial value $\widetilde{w}(0) = \widetilde{v}(\frac{\tau}{2})$ to obtain $\widetilde{w}(\tau)$.
(d) Compute the solution of (2) [resp. (4)] with initial value $\widetilde{v}(0) = \widetilde{w}(\tau)$ to obtain $\widetilde{v}(\frac{\tau}{2})$.
(e) Set $u_1 = \widetilde{v}(\frac{\tau}{2}) + z(\tau)$, where $\widetilde{v}(\frac{\tau}{2})$ is taken from step (d).

Write down the general step of the standard and the modified Strang splitting, respectively. That is, given an approximation u_n to the exact solution at time t_n, find the approximation u_{n+1} to the solution at time $t_{n+1} = t_n + \tau$.

2. Solve the one-dimensional heat equation

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)^2, \qquad 0 \le x \le 1$$

with initial value $u(0,x) = 1 + \sin^2(\pi x)$ and boundary conditions $b_0(t) = b_1(t) = 1$. For the spatial discretization, choose 500 grid points. Study the order of convergence of the standard and the modified Strang splitting at $t = 0.1$ in different norms (e.g., 1-norm, Euclidian norm and maximum norm).

3. Repeat the above experiment with time dependent $b_0(t) = b_1(t) = 1 + \sin 5t$.

4. Repeat the above experiments for Lie and modified Lie splitting.