Forward Prices

The forward contract with price at delivery of $F(S,T)$ where S = current price (at time 0) and T is the time to expiry is equivalent to having an option with payoff $(S_T) = S_T - K$, with $K = F(S,T)$.

Since there is no money exchanged when signing this contract, the value of this option at time 0 is $V=0$.

But,

$$
\begin{aligned}
V_0 &= e^{-rT}\,\mathbb{E}(S_T - K)\\
&= e^{-rT}\left\{\mathbb{E}(S_T) - K\right\}\\
&= 0 \iff K = \mathbb{E}(S_T).
\end{aligned}
$$

Hence $F(S,T) = \mathbb{E}(S_T)$ (with the mean taken with respect to the risk-neutral measure)

Now the function $e^{-r(T-t)}\,\mathbb{E}(S_T)$ is a solution of the PDE for option prices. Hence let $V(S,t) = e^{-r(T-t)}F(S,t) = e^{-r(T-t)}\,\mathbb{E}_t(S_T)$. It is a solution of the option value PDE.

Now,
$$\frac{\partial V}{\partial t} = r e^{-r(T-t)} F + e^{-r(T-t)}\frac{\partial F}{\partial t}$$

$$\frac{\partial V}{\partial S} = e^{-r(T-t)}\frac{\partial F}{\partial S} \quad \text{and} \quad \frac{\partial^2 V}{\partial S^2} = e^{-r(T-t)}\frac{\partial^2 F}{\partial S^2}.$$

Substituting into the PDE, we get

$$e^{-r(T-t)}\left\{rF + \frac{\partial F}{\partial t} + \left((\mu - \hat{\lambda}) - \log(S)\right)S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2} - rF\right\} = 0$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial t} + \alpha\left((\mu-\hat{\lambda}) - \log(S)\right) S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} = 0}.$$

Writing $\tau = T - t$ as the time to expiry, we get

$$\boxed{-\frac{\partial F}{\partial \tau} + \alpha\left((\mu-\hat{\lambda}) - \log(S)\right) S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} = 0,} \qquad (3)$$

To solve this, try the ansatz of

$$F(S,\tau) = \exp\left(a(\tau)\log(S) + b(\tau)\right),$$

so that

$$\frac{\partial F}{\partial \tau} = F \cdot \left\{ a'(\tau)\log(S) + b'(\tau) \right\}.$$

$$\frac{\partial F}{\partial S} = F \cdot \frac{a(\tau)}{S}$$

$$\frac{\partial^2 F}{\partial S^2} = \frac{\partial F}{\partial S} \cdot \frac{a(\tau)}{S} - F\frac{a(\tau)}{S^2}$$

$$= F \cdot \left(\frac{a(\tau)}{S}\right)^2 - F\frac{a(\tau)}{S^2}.$$

and substituting into the PDE for F, (3), we obtain

$$-F\left\{a'(\tau)\log(S) + b'(\tau)\right\} + \alpha\left((\mu-\hat{\lambda}) - \log(S)\right) S\frac{a(\tau)}{S} \cdot F$$
$$+ \frac{1}{2}\sigma^2 S^2 \left\{\frac{a^2(\tau) - a(\tau)}{S^2}\right\} \cdot F = 0$$

$$\Leftrightarrow \quad a'(\tau)\log(S) + b'(\tau) = \alpha\left(\mu - \hat{\lambda} - \log(S)\right) a(\tau) + \frac{1}{2}\sigma^2\left(a^2(\tau) - a(\tau)\right).$$

$\Leftrightarrow$ ① $a'(\tau) = -\alpha a(\tau)$ ($\log(S)$ terms)

and

② $b'(\tau) = \alpha(\mu-\hat{\lambda})a(\tau) + \frac{1}{2}\sigma^2\left(a^2(\tau) - a(\tau)\right)$

("instant" terms).

Since $F(S,0) = S$, we also must have $a(0)=1$, $b(0)=0$.

0 time to expiry

Solving (1) with initial condition $a(0)=1$, we obtain

$$\boxed{a(\tau) = e^{-\alpha\tau}} \qquad (4)$$

Substituting this into (2), we get

$$b'(\tau) = \alpha(\mu-\hat{\lambda})e^{-\alpha\tau} + \frac{1}{2}\sigma^2\left(e^{-2\alpha\tau} - e^{-\alpha\tau}\right)$$

with $b(0)=0$.

This is easily integrated to obtain

$$\boxed{b(\tau) = \left(\mu - \frac{\sigma^2}{2\alpha} - \hat{\lambda}\right)\left(1-e^{-\alpha\tau}\right) + \frac{\sigma^2}{4\alpha}\left(1-e^{-2\alpha\tau}\right)}.$$

For simplicity's sake we may define

$$\hat{\mu} = \mu - \frac{\sigma^2}{2\alpha} - \hat{\lambda}.$$

so that

$$\boxed{b(\tau) = \hat{\mu}\left(1-e^{-\alpha\tau}\right) + \frac{\sigma^2}{4\alpha}\left(1-e^{-2\alpha\tau}\right)} \qquad (5)$$

In conclusion, we have

$$F(S,\tau) = \exp\left(a(\tau)\log(S) + b(\tau)\right)$$

with $a(\tau)$ given by (4) and $b(\tau)$ by (5).

Calibration of the Schwartz Model

We use historical price data to find the parameters for the model

$$dS = \alpha(\mu - \log(S))\, S\,dt + \sigma S\,dW,$$

i.e., to estimate values for α, μ and σ.

Again, for $X = \log(S)$, we have

$$dX = \alpha(\mu^* - X)\,dt + \sigma\,dW$$

where

$$\alpha^* := \mu - \sigma^2/2\alpha.$$

for which

$$X_t = (1-e^{-\alpha t})\mu^* + X_0 e^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}\,dW_s.$$

Hence

$$X_{t+\Delta t} = (1-e^{-\alpha(t+\Delta t)})\mu^* + X_0 e^{-\alpha(t+\Delta t)} + \sigma e^{-\alpha(t+\Delta t)}\int_0^{t+\Delta t} e^{\alpha s}\,dW_s$$

$$= e^{-\alpha\Delta t}\left\{ X_0 e^{-\alpha t} + (e^{\alpha\Delta t} - e^{-\alpha t})\mu^* + \sigma e^{-\alpha t}\int_0^{t+\Delta t} e^{\alpha s}\,dW_s\right\}$$

$$= e^{-\alpha\Delta t}\left\{ X_0 e^{-\alpha t} + ((1-e^{-\alpha t}) + (e^{\alpha\Delta t}-1))\mu^* + \sigma e^{-\alpha t}\int_0^{t+\Delta t} e^{\alpha s}\,dW_s\right\}$$

$$= e^{-\alpha\Delta t}\left\{ X_0 e^{-\alpha t} + (1-e^{-\alpha t})\mu^* + \sigma e^{-\alpha t}\int_0^{t} e^{\alpha s}\,dW_s\right\}$$

$$+ (1-e^{-\alpha\Delta t})\mu^* + \sigma e^{-\alpha(t+\Delta t)}\int_t^{t+\Delta t} e^{\alpha s}\,dW_s$$

$$= e^{-\alpha\Delta t} X_t + (1-e^{-\alpha\Delta t})\mu^* + \sigma e^{-\alpha(t+\Delta t)}\int_t^{t+\Delta t} e^{\alpha s}\,dW_s.$$

$$= e^{-\alpha \Delta t} X_t + (1 - e^{-\alpha \Delta t})\mu^* + Z_t .$$

where $Z_t = \sigma e^{-\alpha(t+\Delta t)} \int_t^{t+\Delta t} e^{\alpha s} dW_s$.

is Normal with mean $= \underline{\underline{0}}$ and variance

$$Var = \sigma^2 e^{-2\alpha(t+\Delta t)} \int_t^{t+\Delta t} e^{2\alpha s} ds$$

$$= \sigma^2 e^{-2\alpha(t+\Delta t)} \frac{e^{2\alpha(t+\Delta t)} - e^{2\alpha t}}{2\alpha}$$

$$= \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha \Delta t})$$

If we write

$$X_{t+1} \approx c_1 X_t + c_2 + Z_t .$$

then we can use linear least squares to find an unbiased estimate for c_1 and for c_2.

From $c_1 = e^{-\alpha \Delta t}$, we find $\alpha = -\frac{\log(c_1)}{\Delta t}$.

An unbiased estimate of $var(Z_t)$ will be the sample

$$var(X_{t+1} - c_1 X_t - c_2).$$

From $\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha \Delta t}) = Var$

we get $\sigma^2 = \frac{2\alpha \, Var}{1 - e^{-2\alpha \Delta t}}$.