A PRAGMATIC INTERPRETATION OF SUBSTRUCTURAL LOGICS

GIANLUIGI BELLIN AND CARLO DALLA POZZA

Abstract. Following work by Dalla Pozza and Garola [2, 3] on a pragmatic interpretation of intuitionistic and deontic logics, which has given evidence of their compatibility with classical semantics, we present sequent calculus system **ILP** formalizing the derivation of assertive judgements and obligations from *mixed contexts* of assertions and obligations and we prove the cut-elimination theorem for it. For the formalization of real-life *normative systems* it is essential to consider inferences from mixed contexts of assertions and obligations, and also of assertions justifiable *relatively to a given state of information* and obligations valid *in a given normative system.* In order to provide a formalization of the notion of *causal implication* and its interaction with obligations, the sequents of **ILP** have *two areas* in the antecedent, expressing the relevant and the ordinary intuitionistic consequence relation, respectively. To provide a pragmatic interpretation of reasoning with the *linear consequence relation* we consider the deductive properties of pragmatic schemes where the operators of illocutionary force are unknown (*free logic of pragmatic force*). We introduce an auxiliary system **ILLP** which formalizes such a logic, and has *substitution rules* that allow us to derive the desired mixed sequents. It is shown that in order to permit *non-uniform susbtitutions* and to preserve *Hume's law* (the underivability of obligations from truth-asserting judgements only and the potential *ineffectuality* of norms) **ILLP** must indeed be based on the *linear consequence relation.* It is also argued that the above uses of linear and relevant logics are perfectly compatible with a theory of pragmatics in a classical semantical setting and immune from popular confusions about the intuitive interpretations of substructural logics.

§1. Preface. Most forms of human reasoning can be formalized and those which deserve such a treatment also benefit from it. Not all philosophers may agree with such a statement, but never mind: the most distinguished opponent of the formalization of mathematical reasoning in the 20th century, Brouwer, was luckily contradicted by his followers and sympathisers, to whom we owe the beautifully insightful constructions of intuitionistic logic, from Heyting's axiomatization and informal semantics of proofs to Gentzen and Prawitz's natural deduction **NJ**, from Curry and

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Howard's correspondence between **NJ** and the typed lambda calculus to categorical logic, and so on and on. Today formal logic has been developed to represent a large variety of forms of reasoning; even such branches as *deontic logic*, which in the 1950s was the territory of a very small group of logicians, have now become popular and found applications, e.g., in Artificial Intelligence. Does this lush development threaten the idea(l) of the unity of logic?

1.1. A simple observation about the development of formal logic may give reassurances about "the unity of logic". Many mathematical systems receiving the name "logic" *prima facie* do not deserve it. But the considerations which justify regarding such systems as logics also show their compatibility with the main body of logical theory. A formal system may be properly regarded as a logic (within the framework of a commonly accepted logical theory) only if it adequately represents pre-formal forms of reasoning which actually occur in human practice and can be shown to be correct and compatible with the standard forms of reasoning (as defined by the logical framework in question). For instance, in the predominant framework of *classical* logical theory, logic has to do with propositions, and propositions are objects which a truth-value can be assigned to: this does not mean that the only possible logic is a theory of propositions, but rather that other aspects of logical theory, e.g., a theory of judgements, must be shown to be in harmony with the logic of propositions classically understood.

There are forms of reasoning, e.g., reasoning with defaults, whose formalization is convenient for applications to Artificial Intelligence, although strictly speaking they ought to be regarded as fallacious and would indeed lead to fallacy if their range of application wasn't clearly defined. The "user manual" for such logics (implicitly or explicitly) contains a translation into the syntax and the semantics of ordinary propositional or first-order theories. In this way, "default logics" are shown to be no threat to the "unity of logic".

1.2. There are good reasons to believe that the garden of logic is not in danger of degenerating into a wild and impenetrable jungle. For instance, only those who see an irreconciliable opposition between classical logic and "deviant logics" may be tempted to mount crusades against the intuitionistic troublemakers or the classical imperialists, respectively, in the name of "the unity of logic". But there is a more interesting way of looking at this opposition, namely, to ask whether classical and intuitionistic logic are about the same aspects of reasoning and, if not, what they are about. An answer may be given by recalling Frege's distinction between *propositions* and *judgements* (or *assertions*): in [2] the second author has argued that Heyting's interpretation of intuitionistic connectives applies not to propositions but to the relations between illocutionary acts, such as assertions, commands, etc. Propositions are *true* or *false*. Illocutionary acts are *justified* or *unjustified*. But the illocutionary act of assertion can be justified only by a *proof* of the proposition expressed in it: a mathematical proof in the case of a mathematical assertion, or some kind or empirical evidence in the case of an empirical assertion. Proofs in classical logic use semantical properties of propositions given by the truth-functional semantics of connectives as a method of justification. This method of proof is not allowed intuitionistically: here the justification of an intuitionistic formula is built up according to Heyting's interpretation of intuitionistic connectives starting from proofs of atomic propositions, thus from non-logical proofs. But if we restrict the content of assertions to atomic propositions, then truth-functional interpretation of the connectives is never used as a tool for the justification of judgements; thus the restriction on the propositional content of judgements is sufficient to guarantee that restricted expressions are constructively interpreted. If this is correct, then intuitionistic logic is a special case of the logic of judgement.

Intuitionistically minded philosophers question the distinction between *truth* and *provability*. The point of view expressed in [2] is that a proof can only be a proof *of the truth* of a proposition: here "truth" is meant in the sense of Tarski's semantics as "truth-as-correspondence", and is regarded as an ontologically neutral notion.[1] This amounts to saying that the classical notion of truth is prior to the notions of proof and hence of justification and, moreover, that the logic of judgement is an extension of the classical logic of propositions.

1.3. A similar treatment can be given of other illocutionary acts: for instance, *prescriptions* are regarded as *propositions* pragmatically used in a prescriptive way. Among these, *norms* are *impersonal prescriptions* [3]. Indeed when we speak of illocutionary acts as assertions and norms, we regard them abstractly, i.e., *impersonally*: we are not considering an assertion made by Bill in a given situation with obvious physical limitations on information retrieval and on the capacity of inferring the consequences of a given body of knowledge, but the assertion of a proposition which could or could not be justifiedly made, given existing scientific information. Similarly, we are not interested in a command given in a given situation by Bill, a person in position of legitimate authority, but limited in his capacity of judgement and subject to "weakness of the will"; rather we are interested in the command that Bill should have given in the framework of the normative theory he operates in. A logic of assertions and norms along these lines is an attempt to extend the domain of logic beyond the class of expressions to which a truth-value can be assigned and to create a *formal pragmatics*, i.e., a logic of illocutionary acts.

Extending Frege's theory (later developed by Reichenbach), in Section 2 we define a formal language for pragmatics $\mathcal{L}^P$, containing not only formulas $\vdash \alpha$ ("α is assertible") and $\circ\!\!- \alpha$ ("α is obligatory") but also formulas $\vdash_k \alpha$ and $\circ\!\!-_N \alpha$, which express assertions justifiable *depending on* a system of knowledge k and norms valid *relative to* a normative system N, respectively. The main novelty of the language $\mathcal{L}^P$ is the presence of connectives representing relations between illocutionary acts, in addition to the connectives of classical logic. As indicated above, the pragmatic connectives

[1] The philosophical position adopted here seems to us in agreement with the point of view expressed by Professor Solomon Feferman in many occasions, in particular in his review of Prawitz [12] in the *J.S.L.*[5].

are given Heyting's "semantics of proofs". A formula is called *pragmatically valid* (or *p-valid*) if it represents relations between illocutionary acts which hold in all circumstances. Like the notion of provability in Heyting's semantics, the notion of justifiability is informal.

In previous papers [2, 3] Hilbert-style deductive systems for the pragmatic language $\mathcal{L}^P$ have been given, in which classic, intuitionistic and deontic logics are representable. In particular the pragmatic operator of obligation has the following properties:

(i) It satisfies the axiom of *classical deontic necessitation*:

$$\text{if} \quad \beta_1, \dots, \beta_n \models \alpha \quad \text{then} \quad \circ\!\!-\, \beta_1, \dots, \circ\!\!-\, \beta_n \;\Rightarrow\; \circ\!\!-\, \alpha$$

namely, *"if α is a truth-functional consequence of $\beta_1, \dots, \beta_n$, then $\circ\!\!-\, \alpha$ follows from $\circ\!\!-\, \beta_1, \dots, \circ\!\!-\, \beta_n$"*.

(ii) The class of obligations is consistent:

$$\circ\!\!-\, (p \wedge \neg p) \Rightarrow \;\vdash (p \wedge \neg p)$$

namely, *"if an absurdity is an obligation then an absurdity is justifiably assertible"*.

(iii) The principle *"$\circ\!\!-\, \alpha$ or not $\circ\!\!-\, \alpha$"* is a fact about normative systems, *not a law of logic*.

Notice that (iii) is a departure from the tradition of systems such as **KD** in harmony with our Heyting-style interpretation of the pragmatic connectives.

Notice also that, unlike the case of assertions, it is not assumed that the justification of an obligation ultimately lies in the recognition of the *truth* of the proposition expressing the content of the obligation. Even in a cognitivist approach to ethics where *"bonum"* and *"verum"* ultimately meet it would be counterintuitive to postulate that the class of propositions expressing moral obligations *coincides* with the class of true propositions: the former is rather be *a proper subclass* of the latter. It is the task of ethical theory to characterize these nonlogical axioms and what counts as a justification for regarding these axioms as obligations. The task of logic is to guarantee that logical inferences may be correctly performed from the axioms.

1.4. There is a hitch in our treatment of pragmatic obligation: in reality, *legal obligations* are always expressed as conditional statements of the form: *"if β is the case, then α is obligatory"*. In coherence with the above discussion, here conditional normative expressions must have the form *"if β is assertible, then α is obligatory"*, where *"β is assertible"* means *"there is a proof of β"* rather than *"it is the case that β"*. It is not a problem to express such statements as *non-logical axioms* within the frame of the existing systems for $\mathcal{L}^P$ However, as soon as *mixed contexts* are introduced, it becomes natural to ask whether in a *sequent calculus* for $\mathcal{L}^P$ we need *bridging principles* for assertive and deontic judgements such as

$$(*0) \qquad \circ\!-\alpha,\ \vdash(\alpha\rightarrow\beta)\ \Rightarrow\ \circ\!-\beta.$$

or

$$(*\S) \qquad \circ\!-\alpha,\ (\vdash\alpha\ \supset\ \vdash\beta)\ \Rightarrow\ \circ\!-\beta.$$

Notice that $(*0)$ is problematic, because a proof in classical logic of $\alpha\rightarrow\beta$ may be too weak to convert a justification of $\circ\!-\alpha$ into a justification of $\circ\!-\beta$: from evidence that *"if Hitler is dead, then the Titanic has sunk"* and the assumption *"Hitler ought to have died"* we do not intuitively infer the conclusion *"the Titanic ought to have sunk"*. On the other hand, let us consider $(*\S)$: intuitionistically, $\vdash\alpha\supset\vdash\beta$ is justified if there is a method to convert any proof of the truth of α into a proof of the truth of β. But what kind of method is needed to transform a justification of $\circ\!-\alpha$ into a justification of $\circ\!-\beta$ according to our intuitions about justified obligations? It is not difficult to show that in $\mathcal{L}^P$ the formula

$$\vdash(\alpha\rightarrow\beta)\Rightarrow(\vdash\alpha\supset\vdash\beta)$$

is p-valid. It follows that, *if we accept classical logic*, we cannot accept $(*\S)$ without also accepting $(*0)$. (Of course, if $\alpha\rightarrow\beta$ is a tautology, then also $\vdash\alpha\supset\vdash\beta$ and $\circ\!-\alpha\supset\circ\!-\beta$ are p-valid, see [3], so the problem does not arise.)

However, there is a sense in which principle $(*\S)$ *is plausible* (at least to some logicians in this area): this is the case whenever the method that converts any proof of the truth of α into a proof of the truth of β is based upon a principle of *causality*. Suppose α

causes β; then from a proof of this relation of causality and from the obligation that α we conclude that β is obligatory. Let us use the special sign "$\supset\!\!\!\cdot$" to indicate this special (*causal*) case of the intuitionistic implication. The principle ($*$§) becomes

$$(\S)\qquad \circ\!\!-\alpha, (\vdash\alpha \supset\!\!\!\cdot \vdash\beta) \Rightarrow \circ\!\!-\beta.$$

which is intuitively plausible.

In our framework we would like to have rule of deontic necessitation so as to derive sequents of the form (§). Clearly, it should not be possible to derive

$$(*1)\qquad \vdash p \Rightarrow \circ\!\!-p$$

because of a formal constraint which we may call *Hume's Law*: *"we cannot derive an* ought *from an* is*"*.[2] Indeed stronger formal constraints are needed: *"we cannot derive an* ought *from an* is *and from* ought*s that are irrelevant"*. Otherwise, we could derive the undesirable sequent

$$(*2)\qquad \circ\!\!-q,\ \vdash p \ \Rightarrow\ \circ\!\!-p$$

where p and q are atomic. Moreover the following sequent is counterituitive:

$$(*3)\qquad \vdash\alpha, (\circ\!\!-\alpha \supset \circ\!\!-\beta) \Rightarrow \vdash\beta.$$

as obligations may be ineffective. In general, *"we cannot derive an* is *from a relevant* ought*"*.

We have extended the language $\mathcal{L}^P$, by introducing a new connective $\supset\!\!\!\cdot$ of *causal implication*. We need to show that this extension is compatible with our framework of logical theory of propositions and judgements. An expression $\vdash\alpha \supset\!\!\!\cdot \vdash\beta$ is justified if and only if $\vdash\alpha \supset \vdash\beta$ is justified as an instantiation of a first order expression $\vdash \forall x(\phi(x) \rightarrow \chi(x))$ which expresses a scientific law, where $\alpha \equiv \phi(t)$ and $\beta \equiv \chi(t)$ for some objects t. Of course, it is the task of epistemology to identify the universal statements which express scientific laws.

[2]It should be clear from the discussion in section **1.3.** above that such a formal constraint does not rule out the possibility of expressing a cognitivist approach to ethics within $\mathcal{L}^P$.

Now the rule

$$\frac{\Gamma \Rightarrow \vdash \alpha}{\Gamma \Rightarrow \vdash \beta \mathbin{\supset\!\!\!\circ} \vdash \alpha}$$

is clearly inadmissible for *causal implication* (when $\vdash \beta$ does not occur in the contex Γ) and this show that a sequent calculus for causal implication cannot admit the unrestricted rule of Weakening.

1.5. We need a deductive system in which the logical properties of *causal implication* and of *ordinary intuitionistic implication* can be presented in the same context. This system may be regarded as a framework where new deductive principles, corresponding to intutively p-valid expressions, can be safely added. Here we present a new sequent calculus $\mathbf{ILP}^-$ for the intuitionistic fragment of $\mathcal{L}^P$, inspired by Girard's **LU** [7]. Sequents have *two areas* in the antecedent:

$$\Gamma\ ;\ \Delta \Rightarrow \delta$$

In the *internal area* (where Δ lies) and in the succedent formulas are introduced according to the familiar rules of inference of Gentzen's intuitionistic sequent calculus **LJ**. In the *external area* (where Γ lies) Weakening is not allowed; formulas can be introduced here by the axiom *"ex falso quodlibet"* or by the logical rule *left causal implications*; moreover, the rule *right causal implications* is correct only with the crucial restriction that the internal area should be empty. In particular, it is incorrect to infer a causal implication through the use of the axiom *ex falso quodlibet"*:

$$\frac{\delta_1, \Gamma\ ;\ \bigwedge, \Delta \Rightarrow \delta_2}{\Gamma\ ;\ \bigwedge, \Delta \Rightarrow \delta_1 \mathbin{\supset\!\!\!\circ} \delta_2}$$

(where $\bigwedge$ is the symbol for absurdity, δ_1, δ_2 are formulas and Γ, Δ sequences of formulas of $\mathcal{L}^P$).)

In the setting of $\mathbf{ILP}^-$ we can safely introduce the deductive principles characteristic of the system **ILP**:

causal implication - obligation

$$\frac{\vdash \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \vdash \beta}{\circ\!\!- \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \circ\!\!- \beta}$$

where $\vdash \mathbf{A}$ is a sequence of assertions of atomic formulas $\vdash \alpha$ and $\mathbf{Z}$ is a sequence of formulas of the form $(\vdash \alpha_1 \supset \ldots (\vdash \alpha_n \supset \vdash \beta)\ldots)$, i.e., built from atomic assertions using *positive* causal implications only.

On one hand it is easy to see that the system **ILP** solves the problem of deriving the new deductive principles, without deriving the undesirable ones. Other extensions are possible along the same lines, as we shall see below. On the other hand the use of *relevant logic* (or any other *substructural logic*) may be regarded with suspicion by those who believe that no satisfactory account has been given so far of substructural logics in the framework of logical theory: in this case substructural logics themselves may be regarded as a threat to the "unity of logic". For this reason we shall consider reasoning with the *linear consequence relation*, where not only *Weakening* but also *Contraction* is in general not allowed, and try to provide an interpretation of such forms of reasoning that shows their compatibility with the framework of the logical theory of propositions and assertions sketched above.

1.6. Linear logic is presented as a "refinement" of both *intuitionistic* and *classical* logic. It is based on the *linear inference relation* which rules out the structural rules of *Contraction* and *Weakening*, except for formulas prefixed with special modalities, the *exponential operators*. Many features of intuitionistic logic are extended in a very interesting way to linear logic: the denotational semantics of coherent spaces, the categorical semantics of intuitionistic linear logic, the game-theoretic semantics are all important "refinements" of ideas developed for intuitionistic logic and the typed λ calculus. Moreover, classical logic can be represented within linear logic in many ways, but in such representations the proof-theory of classical logic acquires the properties of strong normalization and convergence of normalization which are notoriously lacking in Gentzen's classical sequent calculus **LK**. The list may be extended with several significant items, but all of them are *mathematical* properties which do not give an answer to the question: *what is linear logic about?*

In [8] the following explanations are given:

> "classical and intuitionistic logic deal with stable truths:
>
> if A and $A \Rightarrow B$, then B, *but A still holds.*
>
> This is perfect in mathematics, but wrong in real life, since real implication is *causal.* A causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises (conditions) is known in physics as *reaction.*"

The suggestion is that in everyday reasoning the use of a *causal implication* modifies the deductive context in the same way as in a physical process the application of a causal law modifies the preconditions. The notion of "causal implication" is related here to the notion of assumptions as "deductive resources". Now in proof-theory we may give a technical sense to the expression "deductive resources", but it is less clear how an act of assertion may be "consumed" during an argument. What is worse, the popular examples commonly cited to suggest a natural language interpretation seem to imply a confusion between logic and its domain of application: for instance, the formula *"two atoms of hydrogen and one of oxygen yield a molecule of water"* is quoted to suggest that the meaning of "yield" in the chemical sense could help clarify the logical notion of causal implication and could give an example of reasoning where the rules of Weakening and Contraction do not apply. It is therefore a lucky circumstance that an interesting example of the role of linear logic as an auxiliary system in philosophical logic may come from the development of a formal pragmatics for *mixed assertive and deontic deductive contexts*: we shall show that in this theory it becomes natural to prohibit the unrestricted rules of Weakening and Contraction, without making appeal to the notion of "deductive resource", but preserving Girard's intuition of linear implication as causal implication. Such an example should give reassurances about the status of linear logic in the general framework of logical theory.

1.7. The main idea of this paper is that *the linear consequence relation expresses the deductive properties of pragmatic schemes where the operators of illocutionary force are unknown and their specification is not required to be uniform.* Let us write $\bullet\,\alpha$ for an illocutionary act whose propositional content is α and whose force is unspecified. We want to define a formal theory of illocutionary acts under such restrictions, a system which may be called the *free logic of pragmatic force*. Notice that in the popular interpretations, linguistic acts interpreting atomic expressions of linear logic are *single* events, while those interpreting the atoms of intuitionistic logic are *repeatable* events: in some sense the former are more *concrete* entities than the latter. On the contrary the atoms of linear logic are interpreted here more *abstractly* than the assertions and obligations denoted by the atoms of the intuitionistic logic of pragmatic force. Therefore in our "intended interpretation", linear logic is about more "abstract" patterns of reasoning than those of standard logical systems.

To see that Contraction is not generally valid in such a free logic, consider the rule

$$(*) \qquad \frac{\bullet\alpha,\ \bullet\alpha,\ \Gamma \ \Rightarrow\ \Delta}{\bullet\alpha,\ \Gamma \ \Rightarrow\ \Delta}$$

and notice that in the premise we may want to replace the first $\bullet\,\alpha$ with $\vdash\alpha$ and the second with $\circ\!\!-\,\alpha$. (Of course, the inference $(*)$ would be valid if the substitution $[\vdash\alpha/\bullet\alpha]$ or $[\circ\!\!-\,\alpha/\bullet\alpha]$ was uniform in both the sequent-premise and the sequent-conclusion!)

To see that Weakening is not generally valid, consider that from

$$\frac{\bullet\, p \Rightarrow \bullet\, p}{\bullet\, q, \bullet\, p \Rightarrow \bullet\, p}$$

through *non-uniform substitutions* we could derive $(*2)$, which is unacceptable by Hume's law. Also we could derive

$$\bullet\, p \Rightarrow \bullet\, q \multimap \bullet\, p$$

and this is inadmissible, if from the free logic by suitable substitutions we want to recover the notion of *causal implication*. But then the free logic of pragmatic force must also take into accout the relation of relevance between assumptions and conclusion, and thus it cannot allow unrestricted uses of Weakening.

1.8. Next we must show how intuitionistic forms of reasoning of the system $\mathbf{ILP}^-$ can be represented in the context of the linear consequence relation and how the deductive principles characteristic of **ILP** can be implemented. This task may be achieved in several ways (for a possible alternative, see [13]). Here we take the familiar system **ILL** of *intuitionistic linear logic* and extend it with an exponential "$!_c$", loosely inspired by B. Jacob [10]: to expressions of the form $!_c\alpha$ the rule of *contraction* applies, but not the rule of *Weakening*. We define the language $\mathcal{L}_i^\ell$ and the sequent calculus **ILLP** of Intuitionistic Linear Logic for Pragmatics.

The language $\mathcal{L}_i^\ell$ is that of intuitionistic linear logic **ILL** where atoms are of the form $\bullet\,\alpha$, extended with *implicit substitution operators*, regerded as "pragmatic modalities". Such an operator, e.g., $(\lambda)^{\vdash}$, turns a linear formula λ into a *pre-sentence* ς and indicates a possible replacement of the symbol "$\bullet$" with the actual sign of illocutionary force "$\vdash$" (similarly for the other operators $(\lambda)^{\vdash_k}$, $(\lambda)^{\mathcal{O}\!-}$ and $(\lambda)^{\mathcal{O}\!-_N}$). Now *given any "suitable" translation of* **IL** *into* **ILL**, (where the symbol "$!_c$" is used instead of "$!$" in the translation of causal implication) we consider the resulting map

$$[\quad]^\bullet : \mathcal{L}^P \to \mathcal{L}_i^\ell$$

and we stipulate that an *implicit substitution* indicated by our operators is realized *only for those expressions of* $\mathcal{L}_i^\ell$ *that are translations of formulas of* $\mathcal{L}^P$. We indicate the result of an *actual* substitution with $\overline{(\lambda^{\vdash})}$. For instance, a "suitable translation" yields

$$[\quad]^\bullet : \quad \vdash\alpha,\ \vdash\alpha \supset \vdash\beta \Rightarrow \vdash\beta \quad\longmapsto\quad !_c \bullet\alpha, !_c(!_c\bullet\alpha \multimap\, !_c\bullet\beta) \Rightarrow\, !_c\bullet\beta$$

and the principle (§) becomes

$$\overline{(!_c\bullet\alpha)^{\mathcal{O}\!-}}, \overline{(!_c(!_c\bullet\alpha \multimap\, !_c\bullet\beta))^{\vdash}} \Rightarrow \overline{(!_c\bullet\beta)^{\mathcal{O}\!-}}$$

Moreover the rules of **ILLP** are the familiar ones for **ILL**, extended with the obvious ones for "$!_c$" and also with *substitution rules*: the latter are similar to the *exponential* rules of linear logic and allow us to derive sequents $\Sigma \Rightarrow \varsigma$ where Σ is a sequence of pre-sentences, possibly containing different substitution operators. The map $[\,]^\bullet$: $\mathcal{L}^P \to \mathcal{L}_i^\ell$ extends to a map

$$[\,]^\bullet : \mathbf{ILP} \to \mathbf{ILLP},$$

thus if $\Sigma \Rightarrow \varsigma$ is the translation into **ILLP** of a sequent in the language of $\mathcal{L}^P$, then the transformation of the pre-sentences into

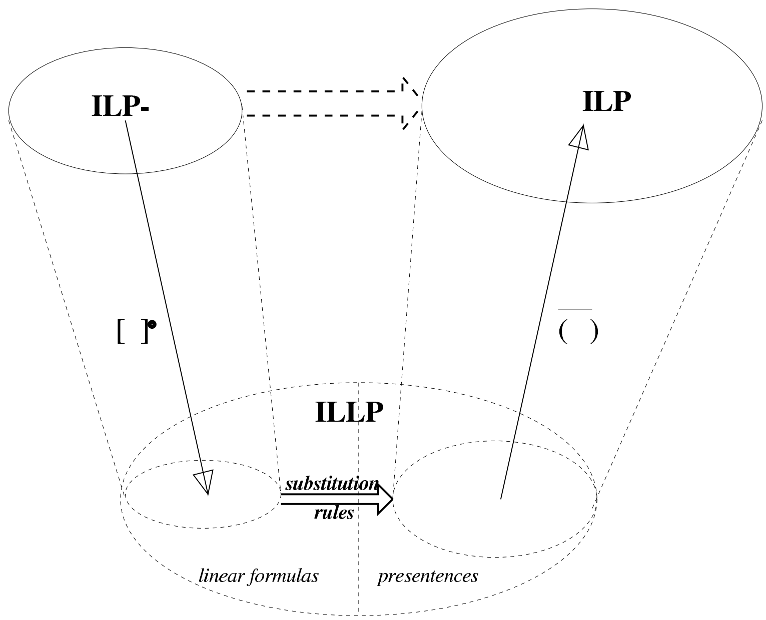

FIGURE 1

sentences can be realized, the resulting sequent is taken as a non-logical axiom of **ILP** and the derivation in **ILLP** may be regarded as a justification for such an axiom. Therefore as represented in Figure 1, we have

$$\mathbf{ILP} \xrightarrow{[\ ]^{\bullet}} \mathbf{ILLP} \xrightarrow{(\ )^{\vdash}} \mathbf{ILLP} \xrightarrow{\overline{(\ )}} \mathbf{ILP}.$$

We claim that the systems **ILP** and **ILLP** have the cut-elimination property, so that the above translations are sound.

1.9. We have not produced a formal semantics for the extended system **ILLP**, just proved its consistency through cut-elimination. But the proposed formalism well reflects the intuitive properties of the illocutionary forces under consideration and their ranking.

The interpretation of the atoms of linear logic as *unspecified illocutionary acts* provides an argument for considering the use of the *linear consequence relation* as an *extension*, rather than an *alternative* to the ordinary intuitionistic consequence relation. However, we do not have a clear and convincing interpretation of Girard's operator "**!**". The mapping $\overline{(\ )}$ does provide a pragmatic interpretation of linear formulas through some translation of **IL** into **ILL**, some of which make a rather unparsimonious use of the operator "**!**". Moreover, only a few linear formulas of $\mathcal{L}^\ell_i$ are translated into formulas of the pragmatic language $\mathcal{L}^P$ through the map $\overline{(\ )}$, namely, only those which result from formulas of $\mathcal{L}^P$ *relatively to the given translation* $[\ ]^\bullet$. All other formulas of $\mathcal{L}^\ell_i$ are *uninterpreted* and play only a *syntactic* role in the calculus, similar to the role played in scientific theories by *theoretic formulas* which lack a direct observative meaning. Thus, the possibility remains open of new interpretations of the formal calculus, which may capture new forms of reasoning, different from those considered above.

A satisfactory feature of our system is that in the object language we may formalize *normative orders* as *dynamic systems*, as they are indeed, by means of a sequence of operators $\multimapinv_N$, $\multimapinv_{N'}$, $\ldots$, where N, N' $\ldots$ are indices for *normative systems.*

In the same way *cognitive dynamic systems* can also be formalized by a sequence of operators $\vdash_k$, $\vdash_{k'}$, $\ldots$ where k, k', $\ldots$ are (indices for) *non-logical* theories. Here we may consider also situations where the language of such a theory is not included in the metalanguage, so that $\vdash_k \alpha$ may not be expressible as $K \vdash \alpha$. The logics of Gödel's provability predicate *Bew* provide interesting mathematical examples along these lines.

It is therefore clear that the cases discussed here are only instances of a large family, where the combined use of several modalities requires restrictions on the inference relation in the metalanguage according to specific properties of the modalities under consideration. The axiomatic method in logic has successfully described the properties of modal operators in isolation. New challenges now arise from the formal study of several modalities in the same context, an inevitable step towards a better understanding of the complexities of informal reasoning. It should be clear that this paper is a tentative incursion in a new territory, and that further work will certainly simplify and improve the technical development.

§2. The pragmatic language $\mathcal{L}^P$.

DEFINITION 1. (*Syntax*) (i) The language $\mathcal{L}^P$ is built from an infinite set of *propositional letters* p, p_0, p_1 ... using the *propositional connectives* $\neg$, $\wedge$, $\vee$, $\rightarrow$; these expressions are called *radical formulas*. The *elementary formulas* of the pragmatic language are obtained by prefixing a radical formula with a sign of *illocutionary force* "$\vdash$", "$\vdash_k$", "$\multimapinv$" or "$\multimapinv_N$". There is only one elementary constant for absurdity, namely $\bigwedge$. Finally, the *sentential formulas* of $\mathcal{L}^P$ are built from the elementary formulas and the constant $\bigwedge$, using the *pragmatic connectives* $\cap$, $\cup$ and $\supset$.

(ii) (*Formation Rules*) The *pragmatic* language $\mathcal{L}^P$ is the union of the sets **Rad** of *radical formulas* and **Sent** of *sentential formulas*. These sets are defined inductively as follows:

$$\alpha \ := \ p \mid \neg\alpha \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \alpha_1 \rightarrow \alpha_2 \mid$$

$$\eta \ := \ \bigwedge \mid \ \vdash \alpha \mid \ \vdash_k \alpha \mid \ \multimapinv \alpha \mid \ \multimapinv_N \alpha \mid$$

$$\zeta \ := \ \vdash \alpha \mid \ \vdash \alpha \Supset \zeta \mid$$

$$\delta \ := \ \eta \mid \zeta \mid \delta_1 \supset \delta_2 \mid \delta_1 \cap \delta_2 \mid \delta_1 \cup \delta_2 \mid$$

(iii) The *intuitionistic fragment* of the language $\mathcal{L}^P$ is obtained by restricting the class of elementary sentences to those with *atomic radical* only:

$$\eta \ := \ \bigwedge \mid \ \vdash p \mid \ \vdash_k p \mid \ \multimapinv p \mid \ \multimapinv_N p \mid$$

We use the letters α, β, α_1, ... to denote *radical* formulas, η, η_1, ... to denote *elementary sentential* formulas, ζ, and δ, δ_1, ... to denote *sentential* formulas. The letters ζ_1, ... denote expressions built from assertions and positive occurrences of *casual implication*. The negation of δ is defined as $\sim \delta =_{df} \delta \supset \bigwedge$.

(iii) The symbol $\bigwedge$ stands for an illocutionary act which is justifiable in no situation, e.g., asserting that "$0 = 1$". (Such an act is called *p-invalid*, see the next definition.) Let α be "$0 = 1$". We postulate

$$\bigwedge \ = \ \vdash \alpha \ = \ \multimapinv \alpha \ = \ \vdash_k \alpha \ = \ \multimapinv_N \alpha$$

We will not write down the formal axioms expressing this equivalence.

Definition 2. (*Informal Interpretation*) (i) *Radical* formulas are interpreted as propositions, with the Tarskian classical semantics, as usual.

(ii) *Sentential expressions* are interpreted as follows:

1. $\vdash \alpha$ and $\vdash_k \alpha$ are interpreted as illocutionary acts of assertion; $\circ\!- \alpha$ and $\circ\!-_N \alpha$ are interpreted as illocutionary acts of prescription (or norms). All such acts are regarded as *impersonal*, i.e., making abstraction from the specific qualities of the subjects of such acts. Illocutionary acts can be *"justified"* (**J**) or *"unjustified"* (**U**); by extension, so are also the corresponding elementary sentential expressions.
2. $\vdash \alpha$ is *justified* if and only if *there is a proof that α is true*; it is *unjustified* otherwise.
3. Let k denote a state of information (regarded as equal to its deductive closure). Then $\vdash_k \alpha$ is *justified* if and only if it is provable that α is true given the state of information k; $\vdash_k \alpha$ is *unjustified* otherwise.
4. Let N be a *normative system* (regarded as equal to its deductive closure). Then $\circ\!-_N \alpha$ is *justified* if and only if there is a proof that α is obligatory in the normative system N; $\circ\!-_N \alpha$ is *unjustified* otherwise.
5. $\circ\!- \alpha$ is *justified* if and only if $\circ\!-_N \alpha$ is justified for all normative systems N; $\circ\!- \alpha$ is *unjustified* otherwise.
6. $\delta_1 \cap \delta_2$ is *justified* if and only if δ_1 is justified *and* δ_2 is justified; it is *unjustified* otherwise.
7. $\delta_1 \cup \delta_2$ is *justified* if and only if δ_1 is justified *or* δ_2 is justified; it is *unjustified* otherwise.
8. $\delta_1 \supset \delta_2$ is *justified* if and only if there is a proof that a justification of δ_1 can be transformed into a justification of δ_2; it is *unjustified*, otherwise.
9. $\vdash \alpha \mathbin{\dot\supset} \vdash \beta$ is justified if and only if $\vdash \alpha \supset \vdash \beta$ is justified as an instantiation of a first order formula $\vdash \forall x.\phi(x) \rightarrow \chi(x)$ which expresses a scientific law, where $\alpha \equiv \phi(t)$ and $\beta \equiv \chi(t)$ for some (sequence of) objects t.

(iii) Let η be an elementary (sentential) formula of $\mathcal{L}^P$: we denote the maximal radical subformula of η by $|\alpha|$. A pragmatic interpretation π gives a *justification value* **J** (justified) or **U** (unjustified) to the sentential formulas $\mathcal{L}^P$. Let σ be a truth-value assignment to the radicals of $\mathcal{L}^P$. We say that π *depends on* σ if for every elementary formulas η, $\pi(\eta) = \mathbf{J}$ implies $\sigma(|\eta|) = T$. We write π_σ to indicate that the pragmatic interpretation π depends on the truth-value assignment σ. A sentential formula δ of $\mathcal{L}^P$ is called *pragmatically valid* (or *p-valid*) if and only if for all σ and for all π_σ we have $\pi_\sigma(\delta) = \mathbf{J}$. Similarly, given a sentential formula δ and a set of sentential formulas Γ, we say that δ is a *pragmatically valid consequence* (or a *p-consequence*) of Γ if for every σ and every π_σ, if π_σ makes all formulas in Γ justified, then π_σ makes δ justified as well. Since the notion of pragmatic interpretation is still informal, the definitions of p-validity and p-consequence are informal as well. To give a precise mathematical model for these notions goes beyond the scope of this paper: our task here is to identify some principles which seem to characterize these notions in an intuitively compelling way.

Examples. (i) A state of information k may be formalized as an axiomatic theory $\mathbf{T}_k$, say, in first-order logic. Let $\mathcal{A}_k$ be the set of axiom of $\mathbf{T}_k$ and let α^k be the translation of α in the language of $\mathbf{T}_k$. Then $\vdash_k \alpha$ is justified if and only if *there is a deduction of* α^k *from* $\mathcal{A}_k$ in first-order logic.

(ii) Extend the set of radical formulas of $\mathcal{L}^P$ with a modal operator $\Box$

$$\alpha \ := \ p \mid \neg\alpha \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \alpha_1 \rightarrow \alpha_2 \mid \Box\alpha$$

and let $\mathbf{T_{GL}}$ be the classical propositional modal theory with axioms and rules of the system **GL** (Gödel-Löb) formalizing the properties of the predicate *Bew* of arithmetic provability. Let $\mathbf{T}_k$ be first-order Peano Arithmetic together with a sequence of closed sentences, one for each propositional letter of $\mathbf{T_{GL}}$. Any such $\mathbf{T}_k$ determines an interpretation $(\ )^k$ of the formulas of $\mathbf{T_{GL}}$ where $\Box$ is interpreted by the arithmetic predicate *Bew* and the interpretation commutes with classical connectives. Soloway's theorem [15] states that $\mathbf{T_{GL}} \vdash \alpha$ if and only if for all k, $\mathbf{PA} \vdash_k (\alpha)^k$. In propositional $\mathcal{L}^P$ we can express the infinite set of equivalences $\vdash_{\mathbf{GL}} \alpha \ \equiv \vdash_k \alpha$, for every α and k, which we may regard as non-logical axioms.

(iii) Consider the same setting of (ii) for *intuitionistic* logic,[3] i.e., let $\mathbf{T_{IGL}}$ be the modal theory with axioms and rules of the system Gödel-Löb on *intuitionistic propositional logic* and let $\mathbf{T}_k$ be first-order Heyting Arithmetic together with a sequence of closed sentences, one for each propositional letter of $\mathbf{T_{IGL}}$. As Solovay's theorem does not hold for Heyting arithmetic, in $\mathcal{L}^P$ there will be nontrivial logical relations between $\vdash_{\mathbf{IGL}} \alpha$ and all the expressions $\vdash_k \alpha$.

(iv) Let $\mathbf{T}_0, \ldots, \mathbf{T}_\alpha, \ldots$ be a transfinite recursive sequence of systems where $\mathbf{T}_0$ is first-order Peano Arithmetic, $\mathbf{T}_{\alpha+1}$ is $\mathbf{T}_\alpha$ extended with the Gödel sentence $\neg Bew(0 = 1)$ for $\mathbf{T}_\alpha$, as in Feferman [4].

(v) A normative system could be formalized as an axiomatic theory $\mathbf{T}_N$ possibly with mixed modalities. Let $\mathcal{A}_N$ be the set of axioms of such a theory and let α^N be the translation of α into the language of $\mathbf{T}_N$. Then $\circ\!\!-_N \alpha$ is justified if and only if α^N *is derivable from* $\mathcal{A}_N$. Let $\mathbf{T}_N, \mathbf{T}_{N'}, \ldots$ be a sequence of such theories. Then in the language of $\mathcal{L}^P$ one may hope to express the evolution of a normative system, the relative ranking of its principles, the metarules which govern the abrogation of laws when a set of laws has became inconsistent with more fundamental principles, etc.

Remark. (i) Clauses 2, $5 - 7$ in Definition 2.(ii) are those of Heyting's semantics of proofs for intuitionistic logic, but the notion of *proof* is not restricted here to that of a mathematical proof. The claim implicit in Definition 2 is that Heyting's "semantics" belongs to the logic of judgement and is rather the beginning of a formal pragmatics: it characterizes the notion of justification of pragmatics sentences, as inductively defined relations between illocutionary acts.

(ii) Suppose the language of $\mathcal{L}^P$ is extended as in Example (*ii*) above, but with the axioms of the modal system **S4**. Then one could follow Gödel [9], McKinsey and Tarski [11] and interpret $\Box\alpha$ as "*there is a proof that* α *is true*"; thus we obtain the following modal translation of the assertive fragment of $\mathcal{L}^P$ into its radical

[3] We thank Giovanni Sambin for suggesting this example.

part thus extended:

$$(\vdash \alpha)^m = \Box\alpha \qquad (\delta_1 \supset \delta_2)^m = \Box(\delta_1^m \to \delta_2^m)$$
$$(\delta_1 \cup \delta_2)^m = \delta_1^m \vee \delta_2^m \qquad (\delta_1 \cap \delta_2)^m = \delta_1^m \wedge \delta_2^m$$

Kripke's semantics for the system **S4** applies here. Notice that this is a *descriptive interpretation* of the pragmatic sign of assertive force, a *reflection* of the pragmatic metalanguage into the set of radical modal formulas. The pragmatic and the descriptive approaches may be very closely related from a mathematical point of view, but cannot be identified: for instance, in the descriptive interpretation nested occurrences of the sign $\Box$ may occur, while as a sign of pragmatic force $\vdash$ cannot be iterated.

(iii) A similar remark applies for the sign of deontic force, which could be interpreted descriptively within some system of deontic logic, from which we would also obtain a form of Kripke's semantics. The challenge of this paper is to provide a Gentzen-style system for the language $\mathcal{L}^P$ extended with *mixed contexts*. We do not consider the descriptive interpretation of pragmatic forces here and do not provide a Kripke-style semantics for our system.

2.1. Sequent calculi for intuitionistic $\mathcal{L}^P$. The sequent calculus $\mathbf{ILP}^-$ formalizes derivations of sentential formulas of $\mathcal{L}^P$. No rule is given for the derivation of elementary sentences, there is a connective expressing "causal implication" which requires a *relevant* consequence relation and the other connectives represent relations between sentential formulas informally characterized by Heyting's "semantics of proofs". The calculus therefore resembles the intuitionistic part of Girard's **LU** system (with a *relevant* rather than *linear* area for causal implication).

DEFINITION 3. Let Γ, Δ denote finite sequences of sentential formulas. The sequent calculus $\mathbf{ILP}^-$ is defined in Tables 1 and 2.

The proof of the following theorem is similar to that Girard's system **LU**:

THEOREM 1. *The sequent calculus* $\mathbf{ILP}^-$ *has the cut-elimination property, thus it is consistent.*

We have the following *symmetric reduction for causal implication*:

identity rules

logical axiom:

$$\delta \;;\Rightarrow \delta$$

cut$_1$:

$$\frac{\Gamma \;;\; \Delta \Rightarrow \delta \qquad \delta, \Gamma' \;;\; \Delta \Rightarrow \delta'}{\Gamma, \Gamma' \;;\; \Delta \Rightarrow \delta'}$$

cut$_2$:

$$\frac{;\; \Delta \Rightarrow \delta \qquad \Gamma \;;\; \delta, \Delta \Rightarrow \delta'}{\Gamma \;;\; \Delta \Rightarrow \delta'}$$

structural rules

exchange:

$$\frac{\Gamma, \delta_1, \delta_2, \Gamma' \;;\; \Delta \Rightarrow \delta}{\Gamma, \delta_2, \delta_1, \Gamma' \;;\; \Delta \Rightarrow \delta}$$

exchange:

$$\frac{\Gamma \;;\; \Delta, \delta_1, \delta_2, \Delta' \Rightarrow \delta}{\Gamma \;;\; \Delta, \delta_2, \delta_1, \Delta' \Rightarrow \delta}$$

contraction:

$$\frac{\delta, \delta, \Gamma \;;\; \Delta \Rightarrow \delta'}{\delta, \Gamma \;;\; \Delta \Rightarrow \delta'}$$

contraction:

$$\frac{\Gamma \;;\; \delta, \delta, \Delta \Rightarrow \delta'}{\Gamma \;;\; \delta, \Delta \Rightarrow \delta'}$$

permeability:

$$\frac{\Gamma, \delta \;;\; \Delta \Rightarrow \delta'}{\Gamma \;;\; \delta, \Delta \Rightarrow \delta'}$$

weakening:

$$\frac{\Gamma \;;\; \Delta \Rightarrow \delta'}{\Gamma \;;\; \delta, \Delta \Rightarrow \delta'}$$

TABLE 1. The sequent calculus $\mathbf{ILP}^{-}$, structural rules

$$\dfrac{right \supset\ \dfrac{\Gamma, \vdash\alpha \;;\; \Rightarrow \zeta}{\Gamma \;;\; \Rightarrow \vdash\alpha \supset \zeta} \qquad \dfrac{\Gamma' \;;\; \Rightarrow \vdash\alpha \qquad \zeta, \Gamma'' \;;\; \Delta \Rightarrow \delta}{\alpha \supset \zeta, \Gamma', \Gamma'' \;;\; \Delta \Rightarrow \delta}\ left \supset}{\Gamma, \Gamma', \Gamma'' \;;\; \Delta \Rightarrow \delta}\ cut_1$$

reduces to

$$\dfrac{\dfrac{\Gamma' \;;\; \Rightarrow \vdash\alpha \qquad \Gamma, \vdash\alpha \;;\; \Rightarrow \zeta}{\Gamma, \Gamma' \;;\Rightarrow \zeta}\ cut_1 \qquad \zeta, \Gamma'' \;;\; \Delta \Rightarrow \vdash\delta}{\Gamma, \Gamma', \Gamma'' \;;\; \Delta \Rightarrow \delta}\ cut_1$$

The reductions strategies for the cut-elimination process are subject to constraint resulting from the restrictions on the context for the *causal implication* rules. Consider the case of a cut where the right sequent-premise is the conclusion of *right* $\supset$:

logical rules

right $\supset\!\!\!\cdot$:
$$\frac{\Gamma, \vdash \alpha\, ; \Rightarrow \zeta}{\Gamma\, ; \Rightarrow\, \vdash \alpha \supset\!\!\!\cdot\, \zeta}$$

left $\supset\!\!\!\cdot$:
$$\frac{\Gamma;\ \Rightarrow\, \vdash \alpha \quad \zeta, \Gamma'\, ; \Delta \Rightarrow \delta}{\vdash \alpha \supset\!\!\!\cdot\, \zeta, \Gamma, \Gamma'\ ;\ \Delta \Rightarrow \delta}$$

right $\cap$:
$$\frac{\Gamma\ ;\ \Delta \Rightarrow \delta_1 \quad \Gamma\ ;\ \Delta \Rightarrow \delta_2}{\Gamma\ ;\ \Delta \Rightarrow \delta_1 \cap \delta_2}$$

left $\cap$:
$$\frac{\Gamma\ ;\ \delta_i\ \Delta \Rightarrow \delta}{\Gamma\ ;\ \delta_0 \cap \delta_1, \Delta \Rightarrow \delta} \quad \text{for } i = 0, 1.$$

right $\supset$:
$$\frac{\Gamma\ ;\ \Delta, \delta_0 \Rightarrow \delta_1}{\Gamma\ ;\ \Delta \Rightarrow \delta_0 \supset \delta_1}$$

left $\supset$:
$$\frac{;\ \Delta \Rightarrow \delta_0 \quad \Gamma\, ; \delta_1, \Delta \Rightarrow \delta}{\Gamma\ ;\ \delta_0 \supset \delta_1, \Delta \Rightarrow \delta}$$

right $\cup$:
$$\frac{\Gamma\, ; \Delta \Rightarrow \delta_i}{\Gamma\, ; \Delta \Rightarrow\ \delta_0 \cup \delta_1} \quad \text{for } i = 0, 1.$$

left $\cup$:
$$\frac{\Gamma\ ;\ \delta_0, \Delta \Rightarrow \delta \quad \Gamma\ ;\ \delta_1, \Delta \Rightarrow \delta}{\Gamma\ ;\ \delta_0 \cup \delta_1, \Delta \Rightarrow \delta}$$

absurdity axiom:
$$\Gamma\ ;\ \bigwedge, \Delta \Rightarrow \delta$$
for any Γ, Δ and δ.

TABLE 2. The sequent calculus $\mathbf{ILP}^-$, logical rules

$$\dfrac{\begin{matrix} d_1 \\ \Gamma\ ;\ \Delta \Rightarrow\ \vdash \zeta \end{matrix} \qquad \dfrac{\begin{matrix} d_2 \\ \vdash \beta, \zeta, \Gamma';\ \Rightarrow \zeta' \end{matrix}}{\zeta, \Gamma'\ ;\ \Rightarrow\ \vdash \beta \supset\!\!\!\cdot\, \zeta'}\ \textit{right}\ \supset\!\!\!\cdot}{\Gamma, \Gamma'\ ;\ \Delta \Rightarrow\ \vdash \beta \supset\!\!\!\cdot\, \zeta'}\ \textit{cut}_1$$

here the $\textit{cut}_1$ can always be permuted with the inferences of d_1. Since ζ has one of the forms $\vdash \gamma$ or $\vdash \gamma \supset\!\!\!\cdot\, \zeta'$, eventually one of the following is the case:

(i) the left premise of the cut is an *absurdity axiom*, and the cut is eliminated by taking its conclusion as a single absurdity axiom;
(ii) the left premise of the cut is a *logical axiom*, and the usual axiom reduction applies;

(iii) the left premise of the cut is also the conclusion of a *right* $\supset_c$ and in both sequent-premises the area to the right of ";" is empty; henceforth the cut-elimination procedure continues without further constraints in the relevant area only.

Similarly, if the right sequent-premise is the conclusion of a *left* $\supset_c$. Other details of the proof are left to the reader.

The above calculus can be extended in various ways, to formalize arguments that are p-valid according to the informal interpretation of the language $\mathcal{L}^P$ but cannot be represented in $\mathbf{ILP}^-$. We extend $\mathbf{ILP}^-$ to an intuitionistic system **ILP** that represent reasoning with *mixed context*. A first set of *mixed rules* connects *indexed assertions* with *assertions*, on one hand, and *indexed obligations* with *obligations*, on the other. A second kind of mixed rules connects *obligations* (indexed or not) with *causal implication*. We write $\mathbf{Z}$ for $\zeta_1, \ldots, \zeta_n$; also we write $\vdash \mathbf{A}$ for $\vdash \alpha_1, \ldots, \vdash \alpha_k$ and similarly $\circ\!- \mathbf{A}$.

mixed rules

k-assertability	*N-obligation*
$\vdash \delta\ ; \Rightarrow\ \vdash_k \delta$	$\circ\!- \delta;\ \Rightarrow\ \circ\!-_N \delta$

for any k and N.

causal implication - obligation

$$\frac{\vdash \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \vdash \beta}{\circ\!- \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \circ\!- \beta}$$

causal implication - N-obligation

$$\frac{\vdash \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \vdash \beta}{\circ\!-_N \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \circ\!-_N \beta}$$

TABLE 3. The sequent calculus **ILP**, mixed rules

DEFINITION 4. Let **ILP** be $\mathbf{ILP}^-$ with the addition of the axioms *K-assertability*, *N-obligation*, *causal implication - obligation* and *causal implication - N-obligation* (see Table 3).

THEOREM 2. *The system* **ILP** *emjoys the cut-elimination property* (modulo atomic cuts with k-assertability and N-obligation axioms).

We have the following reductions:

i. causal implication - obligation / causal implication - obligation reduction:

$$\dfrac{\supset - \circ\!- \ \dfrac{\vdash \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \vdash \beta}{\circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \circ\!- \beta} \qquad \dfrac{\vdash \beta,\ \vdash \mathbf{A}',\mathbf{Z}'\ ;\ \Rightarrow\ \vdash \gamma}{\circ\!- \beta,\ \circ\!- \mathbf{A}',\mathbf{Z}'\ ;\ \Rightarrow\ \circ\!- \gamma}\ \supset - \circ\!-}{\circ\!- \mathbf{A},\ \circ\!- \mathbf{A}',\mathbf{Z},\mathbf{Z}'\ ;\ \Rightarrow\ \circ\!- \gamma}\ cut$$

reduces to

$$\dfrac{\dfrac{\vdash \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \vdash \beta \qquad \vdash \beta,\ \vdash \mathbf{A},\mathbf{Z}'\ ;\ \Rightarrow\ \vdash \gamma}{\vdash \mathbf{A},\ \vdash \mathbf{A}',\mathbf{Z},\mathbf{Z}'\ ;\ \Rightarrow\ \vdash \gamma}\ cut}{\circ\!- \mathbf{A},\ \circ\!- \mathbf{A}',\mathbf{Z},\mathbf{Z}'\ ;\ \Rightarrow\ \circ\!- \gamma}\ \supset - \circ\!-$$

ii. causal implication / causal implication - obligation reduction:

$$\dfrac{right\ \supset\ \dfrac{\overset{d_1}{\vdash \alpha,\Gamma;\Rightarrow \zeta}}{\Gamma;\Rightarrow\ \vdash \alpha \supset \zeta} \qquad \dfrac{\overset{d_2}{\vdash \alpha \supset \zeta,\ \vdash \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \vdash \gamma}}{\vdash \alpha \supset \zeta,\ \circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \circ\!- \gamma}\ \supset - \circ\!-}{\Gamma,\ \circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \circ\!- \gamma}\ cut$$

reduces to

$$\dfrac{\dfrac{right\ \supset\ \dfrac{\overset{d_1}{\vdash \alpha,\Gamma;\Rightarrow \zeta}}{\Gamma;\Rightarrow\ \vdash \alpha \supset \zeta} \qquad \overset{d_2}{\vdash \alpha \supset \zeta,\ \vdash \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \vdash \gamma}}{\Gamma,\ \vdash \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \vdash \gamma}\ cut}{\Gamma,\ \circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \circ\!- \gamma}\ \supset - \circ\!-$$

iii. causal implication - obligation permutation:

$$\dfrac{\overset{d_1}{\Gamma\ ;\ \Delta \Rightarrow\ \vdash \alpha \supset \zeta} \qquad \dfrac{\overset{d_2}{\vdash \alpha \supset \zeta,\ \vdash \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \vdash \beta}}{\vdash \alpha \supset \zeta,\ \circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \circ\!- \beta}\ \supset - \circ\!-}{\Gamma,\ \circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Delta \Rightarrow\ \circ\!- \gamma}\ cut$$

The cut can always be permuted with the other inferences of d_1.

iv. causal implication - obligation permutation:

$$\dfrac{\overset{d_1}{\Gamma\ ;\ \Delta \Rightarrow\ \circ\!- \alpha} \qquad \dfrac{\overset{d_2}{\vdash \alpha,\ \vdash \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \vdash \beta}}{\circ\!- \alpha,\ \circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Rightarrow\ \circ\!- \beta}\ \supset - \circ\!-}{\Gamma,\ \circ\!- \mathbf{A},\mathbf{Z}\ ;\ \Delta \Rightarrow\ \circ\!- \gamma}\ cut$$

The *cut* can always be permuted with the inferences of d_1.

v. causal implication - obligation permutation

$$\dfrac{\mathord{\supset\!\!\!\!\cdot} - \circ\!\!-\ \dfrac{\overset{d_1}{\vdash \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \vdash \alpha}}{\circ\!\!-\ \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \circ\!\!-\ \alpha} \qquad \overset{d_2}{\circ\!\!-\ \alpha, \Gamma\ ;\ \Delta \Rightarrow\ \circ\!\!-\ \beta}}{\circ\!\!-\ \mathbf{A}, \mathbf{Z}, \Gamma\ ;\ \Delta \Rightarrow\ \circ\!\!-\ \beta}\ cut$$

The *cut* can always be permuted with the inferences of d_2.

The formal details of the proof are left to the reader.

Another extension of $\mathbf{ILP}^-$ is obtained by allowing *classical* methods of inference: in our framework this is done by removing the restriction that the radical $|\eta|$ of any elementary formula η should be atomic and by introducing all axioms of the form

classical assertability	*classical obligation*
$\dfrac{\text{if } \alpha_1, \dots, \alpha_n \models \alpha, \text{ then}}{;\ \vdash \alpha_1, \dots, \vdash \alpha_n \Rightarrow \vdash \alpha}$	$\dfrac{\text{if } \alpha_1, \dots, \alpha_n \models \alpha, \text{ then}}{;\ \circ\!\!-\ \alpha_1, \dots, \circ\!\!-\ \alpha_n \Rightarrow \circ\!\!-\ \alpha}$

where $\alpha_1, \dots, \alpha_n, \alpha$ are arbitrary radical formulas.

Definition 5. Consider the language $\mathcal{L}^P$ *without the connective* $\supset\!\!\!\!\cdot$ *of causal implication* and let $\mathbf{LP}^-$ be the sequent calculus obtained by adding the *classical assertability* and *classical obligation* rules to the system $\mathbf{ILP}^-$ *without the rules for* $\supset\!\!\!\!\cdot$ *and the causal implication-obligation rule.*

Corollary. *The system* $\mathbf{LP}^-$ *is consistent.*

Indeed, by Theorem 1 we may reduce all cuts, except those where both cut-formulas occur in *classical assertability* axioms or both in *classical obligation* axioms and similarly for indexed assertability and obligation. But these cuts can also be reduced in an obvious way, since $\alpha_1, \dots, \alpha_m \models \alpha$ and $\alpha, \beta_1, \dots, \beta_n \models \beta$ yield $\alpha_m \models \alpha, \beta_1, \dots, \beta_n \models \beta$.

The classical extension of **ILP** seems to be more problematic. We will not pursue this extension here.

§3. The language of intuitionistic linear $\mathcal{L}^\ell_i$. We present the language of the intuitionistic linear pragmatic language $\mathcal{L}^\ell_i$. We consider translations of the intuitionistic fragment of the $\mathcal{L}^P$ into $\mathcal{L}^\ell_i$ and we define the substitution operations, giving partial translations from $\mathcal{L}_i^\ell$ back to $\mathcal{L}^P$. We define the intuitionistic linear sequent calculus **ILLP** and the bridging rules that allow us to extend the deductive principles of $\mathbf{ILP}^-$ with rules for mixed assertive and deontic contexts.

3.1. Linear expressions, translations, substitutions. $\mathcal{L}_i^\ell$ is a formal language for pragmatic sentences with *unspecified signs of illocutionary force.*

DEFINITION 6. (*Syntax* (i) $\mathcal{L}_i^\ell$ is based on the language of *intuitionistic linear logic* **ILL**, namely, on the constants **0**, **1** and $\top$, the connectives $\multimap$, $\otimes$, & and $\oplus$ and the exponential **!** with the addition of the *relevant exponential* $\mathbf{!}_c$. Moreover, there are operators "$\bullet$", $(\;)^{\vdash}$, $(\;)^{\vdash_k}$, $(\;)^{\circ\!-}$ and $(\;)^{\circ\!-_N}$,

The elementary formulas are elementary pragmatic sentences, written $\bullet\,\alpha$, where "$\bullet$" is an operator capable of being substituted by any sign among $\vdash$, $\vdash_k$, $\circ\!-$ or $\circ\!-_N$ for some k and N.

The "modal" operators $(\lambda)^{\vdash}$, $(\lambda)^{\vdash_k}$, $(\lambda)^{\circ\!-}$ and $(\lambda)^{\circ\!-_N}$, whose occurrences cannot be nested, indicate the possibility, *but only the possibility*, of a substitution of the indicated operator of pragmatic force for every occurrence of $\bullet$ in the given formula λ, as explained above.

(ii) (*Formation Rules*) The set **Lin** of the formulas of $\mathcal{L}_i^\ell$ is defined inductively as follows:

$$\lambda \;:=\; \bullet\,\alpha \mid \mathbf{0} \mid \mathbf{1} \mid \top \mid \lambda_1 \multimap \lambda_2 \mid \lambda_1 \otimes \lambda_2 \mid \lambda_1 \& \lambda_2 \mid \lambda_1 \oplus \lambda_2 \mid \mathbf{!}\lambda \mid \mathbf{!}_c\lambda \mid$$

$$\varsigma \;:=\; \lambda^{\vdash} \mid \lambda^{\vdash_k} \mid \lambda^{\circ\!-} \mid \lambda^{\circ\!-_N} \mid$$

The expressions λ are the *linear formulas*, the expressions ς will be called *pre-sentences.* We use the letter ϑ to range over linear formulas and pre-sentences.

We are interested in using linear logic as a tool to expand the deductive power of the sequent calculus $\mathbf{ILP}^-$ so as to derive *mixed deontic and assertve* sequents. For this purpose we need translations $(\;)^\bullet : \mathbf{IL} \to \mathbf{ILL}$ of *intuitionistic logic* **ILL** into *intuitionistic linear logic* **ILL** which are *functorial* in the sense that if d is a derivation of $\Gamma \Rightarrow \delta$ in **IL**, then $(d)^\bullet$ is a derivation $(\Gamma)^\bullet \Rightarrow (\delta)^\bullet$

in **ILL** and, moreover, the translation respects cut. There exist in the literature a few translations of this kind.

For example, we may use the "*boring translation*" $[\quad]^+ : \mathbf{Sent} \to \mathbf{Lin}$ (see [6], p.81) which is related to the **S4** interpretation of intuitionistic logic:

i. $[\vdash \alpha]^+ \;=\; [\vdash_k \alpha]^+ \;=\; [\multimapinv \alpha]^+ \;=\; [\multimapinv_N \alpha]^+ \;=_{df}\; \mathbf{!} \bullet \alpha$;
ii. $[\delta_1 \cap \delta_2]^+ = \mathbf{!}[\delta_1]^+ \otimes \mathbf{!}[\delta_2]^+$;
iii. $[\delta_1 \cup \delta_2]^+ = \mathbf{!}[\delta_1]^+ \oplus \mathbf{!}[\delta_2]^+$;
iv. $[\delta_1 \supset \delta_2]^+ = \mathbf{!}\big([\delta_1]^+ \multimap [\delta_2]^+\big)$;
v. $[\vdash \alpha \supset \vdash \beta]^+ = \mathbf{!}_c(\mathbf{!}_c \bullet \alpha \multimap \mathbf{!}_c \bullet \beta)$;
$[\vdash \alpha \supset \zeta]^+ = \mathbf{!}_c(\mathbf{!}_c \bullet \alpha \multimap [\zeta]^+)$;
vi. $[\zeta \supset \delta]^+ = \mathbf{!}\big(\mathbf{!}[\zeta]^+ \multimap [\delta]^+\big)$;
$[\delta_1 \supset \zeta_2]^+ = \mathbf{!}\big([\delta]^+ \multimap \mathbf{!}[\zeta]^+\big)$;
vii. $[\bigwedge]^+ =_{df} \mathbf{0}$.

We could also use the more familiar translation $[\quad]^* : \mathbf{Sent} \to \mathbf{Lin}$ as follows:

i. $[\vdash \alpha]^* \;=\; [\vdash_k \alpha]^* \;=\; [\multimapinv \alpha]^* \;=\; [\multimapinv \alpha]^* \;=_{df}\; \bullet \alpha$;
ii. $[\delta_1 \cap \delta_2]^* = [\delta_1]^* \& [\delta_2]^*$;
iii. $[\delta_1 \cup \delta_2]^* = \mathbf{!}[\delta_1]^* \oplus \mathbf{!}[\delta_2]^*$;
iv. $[\delta_1 \supset \delta_2]^* = \mathbf{!}[\delta_1]^* \multimap [\delta_2]^*$;
v. $[\vdash \alpha \supset \vdash \beta]^* = \mathbf{!}_c \bullet \alpha \multimap \mathbf{!}_c \bullet \beta$;
$[\vdash \alpha \supset \zeta]^* = \mathbf{!}_c \bullet \alpha \multimap [\zeta]^*$;
v. $[\delta \supset \zeta]^\bullet = \mathbf{!}\big(\mathbf{!}[\zeta]^\bullet \multimap \mathbf{!}[\delta]^\bullet\big)$;
vi. $[\bigwedge]^* =_{df} \mathbf{0}$.

In this paper we need a *fully functorial* translation $(\ )^\bullet : \mathbf{IL} \to \mathbf{ILL}$ i.e., a translation having the additional property that *for any derivation* d^ℓ *of* $(\Gamma)^\bullet \Rightarrow (\delta)^\bullet$ *in* **ILL** *there exists a derivation* d *of* $\Gamma \Rightarrow \delta$ *in* **IL** *such that* $d^\ell = (d)^\bullet$ (modulo suitable equalities of derivations). This property can be assumed of the translation $(\ )^*$ (see [14], 1.3.).

First of all we must check that given one of our translations, we may indeed use them for our purpose, i.e., to substitute the signs of illocutionary force in the language of $\mathcal{L}^P$.

PROPOSITION 1. *(Partial interpretation of pre-sentences) Given a functorial translation* $[\quad]^\bullet : \mathbf{IL} \to \mathbf{ILL}$ *of intuitionistic logic* *within intuitionistic linear logic, there is a map* $\overline{(\ )} : (\mathcal{L}^P)^\bullet \Rightarrow \mathcal{L}^P$ *from a subset* $(\mathcal{L}^P)^\bullet$ *of the pre-sentences of* $\mathcal{L}^\ell_i$ *to the sentential formulas of* $\mathcal{L}^P$*, with the following properties:*

$$
\begin{array}{rcll}
\overline{(\zeta^\bullet)^{\vdash}} & = & \zeta & \\
\overline{(\delta^\bullet)^{\vdash}} & = & \delta[\vdash / \vdash_k, \vdash / \multimapinv, \vdash / \multimapinv_N], & \textit{for all } k,\ N; \\
\overline{(\delta^\bullet)^{\vdash_k}} & = & \delta[\vdash_k / \vdash, \vdash_k / \vdash_j, \vdash_k / \multimapinv, \vdash_k / \multimapinv_N], & \textit{for all } N,\ j \neq k; \\
\overline{(\delta^\bullet)^{\multimapinv}} & = & \delta[\multimapinv / \vdash, \multimapinv / \vdash_k, \multimapinv / \multimapinv_N] & \textit{for all } k,\ N; \\
\overline{(\delta^\bullet)^{\multimapinv_N}} & = & \delta[\multimapinv_N / \vdash, \multimapinv_N / \vdash_k, \multimapinv_N / \multimapinv, \multimapinv_N / \multimapinv_{N'}] & \textit{for all } k,\ N' \neq N.
\end{array}
$$

Expressions of the form $\lambda^{\vdash}$, $\lambda^{\vdash_k}$, $\lambda^{\multimapinv}$ *and* $(\lambda)^{\multimapinv_N}$ *are* uninterpreted, *if for all* δ, *we have* $\lambda \neq \delta^\bullet$.

Remark. The mapping $\overline{(\ )}$ provides a partial pragmatic interpretation of linear formulas by translating them into the language $\mathcal{L}^P$, which has an intended pragmatic interpretation. But only a few formulas of $\mathcal{L}^\ell_i$ are translated into formulas of $\mathcal{L}^P$ through the map $\overline{(\ )}$, namely, only those which result from formulas of $\mathcal{L}^P$ *relatively to a given translation* $[\ ]^\bullet$. All other formulas of $\mathcal{L}^\ell_i$ are *uninterpreted.* and play only a *syntactic* role in the calculus, similar to the role played in scientific theories by *theoretic formulas* which lack a direct observative meaning.

3.2. The sequent calculus ILLP.

DEFINITION 7. (i) (*Notation*) We use Greek capital letters Λ, Λ', ... to range over sequences of *linear formulas*, the capital letter Σ, Σ', ... to range over sequences of *pre-sentences* and lowercase and capital letters θ, Θ, Θ' to range over expressions and sequences of expressions consisting of either *linear formulas or pre-sentences.* As usual, if Λ is $\lambda_1, \ldots, \lambda_n$, then $!\Lambda$ is $!\lambda_1, \ldots, !\lambda_n$ and $\Lambda^{\vdash}$ is ${\lambda_1}^{\vdash}, \ldots, {\lambda_n}^{\vdash}$ and similarly for the other substitution operators.

(ii) The sequent calculus **ILLP** for intuitionistic linear logic for pragmatics is defined in Tables 4, 5, 6, 7 and 8.

PROPOSITION 2. *The fragment of sequent calculus* **ILLP** *without the* relevant implication/ $\multimapinv$ *rule has the cut-elimination property.*

identity rules

logical axiom:

$$\bullet\,\alpha \Rightarrow \bullet\,\alpha$$

cut :

$$\frac{\Theta \Rightarrow \vartheta \qquad \vartheta, \Theta' \Rightarrow \xi}{\Theta, \Theta' \Rightarrow \xi}$$

structural rule

exchange:

$$\frac{\Theta, \vartheta_0, \vartheta_1, \Theta' \Rightarrow \vartheta}{\Theta, \vartheta_1, \vartheta_0, \Theta' \Rightarrow \vartheta}$$

TABLE 4. The sequent calculus **ILLP**, structural part.

logical rules

1-*Right :*

$$\Rightarrow \mathbf{1}$$

1-*Left :*

$$\frac{\Lambda \Rightarrow \lambda}{\Lambda, \mathbf{1} \Rightarrow \lambda}$$

$\multimap$-*Right:*

$$\frac{\lambda_0, \Lambda \Rightarrow \lambda_1}{\Lambda \Rightarrow \lambda_0 \multimap \lambda_1}$$

$\multimap$-*Left:*

$$\frac{\Lambda \Rightarrow \lambda_0 \qquad \lambda_1, \Lambda' \Rightarrow \lambda}{\lambda_0 \multimap \lambda_1, \Lambda, \Lambda' \Rightarrow \lambda}$$

$\otimes$-*Right:*

$$\frac{\Lambda \Rightarrow \lambda_0 \qquad \Lambda' \Rightarrow \lambda_1}{\Lambda, \Lambda' \Rightarrow \lambda_0 \otimes \lambda_1}$$

$\otimes$-*Left:*

$$\frac{\lambda_0, \lambda_1, \Lambda \Rightarrow \lambda}{\lambda_0 \otimes \lambda_1, \Lambda \Rightarrow \lambda}$$

$\top$-*Right:*

$$\Lambda \Rightarrow \top$$

0-*Left:*

$$\Lambda, \mathbf{0} \Rightarrow \lambda$$

&-*Right:*

$$\frac{\Lambda \Rightarrow \lambda_0 \qquad \Lambda \Rightarrow \lambda_1}{\Lambda \rightarrow \lambda_0 \& \lambda_1}$$

&-*Left:*

$$\frac{\lambda_i, \Lambda \Rightarrow \lambda}{\lambda_0 \& \lambda_1, \Lambda \Rightarrow \lambda}$$

for $i = 0, 1$.

$\oplus$-*Right:*

$$\frac{\Lambda \Rightarrow \lambda_i}{\Lambda \Rightarrow \lambda_0 \oplus \lambda_1}$$

for $i = 0, 1$.

$\oplus$-*Left:*

$$\frac{\lambda_0, \Lambda \Rightarrow \lambda \qquad \lambda_1, \Lambda \Rightarrow \lambda}{\lambda_0 \oplus \lambda_1, \Lambda \Rightarrow \lambda}$$

TABLE 5. The sequent calculus **ILLP**, logic

exponentials

dereliction:
$$\frac{\lambda, \Lambda \Rightarrow \lambda'}{!\lambda, \Lambda \Rightarrow \lambda'}$$

weakening:
$$\frac{\Lambda \Rightarrow \lambda'}{!\lambda, \Lambda \Rightarrow \lambda'}$$

contraction:
$$\frac{!\lambda, !\lambda, \Lambda \Rightarrow \lambda'}{!\lambda, \Lambda \Rightarrow \lambda'}$$

promotion:
$$\frac{!\Lambda \Rightarrow \lambda}{!\Lambda, \Sigma \Rightarrow !\lambda}$$

dereliction:
$$\frac{\lambda, \Lambda \Rightarrow \lambda'}{!_c\lambda, \Lambda \Rightarrow \lambda'}$$

contraction:
$$\frac{!_c\lambda, !_c\lambda, \Lambda \Rightarrow \lambda'}{!_c\lambda, \Lambda \Rightarrow \lambda'}$$

promotion:
$$\frac{!_c\Lambda \Rightarrow \lambda}{!_c\Lambda \Rightarrow !_c\lambda}$$

TABLE 6. The sequent calculus **ILLP**, exponentials.

Indeed for the fragment of **ILLP** without substitution rules, the proof is essentially the same as for the system of intuitionistic linear logic **ILL** (cfr. [1] and the bibliography in it). Also the *left* substitution rules are analogous to the *dereliction* rule for the exponential operators; the *right* rules are analogous to the *promotion* rule for the exponential operator. Thus the cut reductions for the substitution rules (Table 7) are those for a *promotion* - *dereliction* pair in **ILL**, so the same argument applies *mutatis mutandis*.

Remark. Since the *substitution operators* can neither be nested nor occur in the scope of a linear operator, we may always assume that in a *cut-free* **ILLP** derivation quasi-sentences occur only in the terminal branch of the derivation.

DEFINITION 8. The sequent calculus $\mathbf{ILP}^+$ is $\mathbf{ILP}^-$, with the addition of all rules of the *bridging rules* of Table 9.

$\vdash$-*Subst Right:*
$$\frac{\Sigma \Rightarrow \lambda}{\Sigma \Rightarrow \lambda^{\vdash}}$$
$\vdash$-*Subst Left:*
$$\frac{\lambda, \Lambda, \Xi \Rightarrow \lambda'}{\lambda^{\vdash}, \Lambda, \Xi \Rightarrow \lambda'}$$
where all pre-sentences in Σ are of the form $(\lambda'')^{\vdash}$.

$\mathcal{O}\!-$-*Subst Right:*
$$\frac{\Sigma \Rightarrow \lambda}{\Sigma \Rightarrow \lambda^{\mathcal{O}\!-}}$$
$\mathcal{O}\!-$-*Subst Left:*
$$\frac{\lambda, \Lambda, \ \Xi \Rightarrow \lambda'}{\lambda^{\mathcal{O}\!-}, \Lambda, \Xi \Rightarrow \lambda'}$$
where all pre-sentences in Σ are of the form $(\lambda'')^{\mathcal{O}\!-}$

$\vdash$-*Subst Right:*
$$\frac{\Sigma \Rightarrow \lambda}{\Sigma \Rightarrow \lambda^{\vdash_k}}$$
$\vdash$-*Subst Left:*
$$\frac{\lambda, \Lambda, \Xi \Rightarrow \lambda'}{\lambda^{\vdash_k}, \Lambda, \Xi \Rightarrow \lambda'}$$
where all pre-sentences in Σ are of the form $(\lambda'')^{\vdash_k}$.

$\mathcal{O}\!-$-*Subst Right:*
$$\frac{\Sigma \Rightarrow \lambda}{\Sigma \Rightarrow \lambda^{\mathcal{O}\!-_N}}$$
$\mathcal{O}\!-$-*Subst Left:*
$$\frac{\lambda, \Lambda, \ \Xi \Rightarrow \lambda'}{\lambda^{\mathcal{O}\!-_N}, \Lambda, \Xi \Rightarrow \lambda'}$$
where all pre-sentences in Σ are of the form $(\lambda'')^{\mathcal{O}\!-_N}$

TABLE 7. Substitution rules - I

Given a fully functorial translation $[\ \ ]^{\bullet}$:

relevant implication / $\mathcal{O}\!-$*:*
$$\frac{\Lambda, \Lambda' \Rightarrow \lambda}{(\Lambda)^{\mathcal{O}\!-}, (\Lambda')^{\vdash} \Rightarrow \lambda^{\mathcal{O}\!-}}$$
where all formulas in Λ and λ are of the form $!_c \bullet \alpha$
and all the formulas in Λ' are of the form $\zeta^{\bullet}$.

relativized assertion:
$$([\vdash \alpha]^{\bullet})^{\vdash} \Rightarrow ([\vdash \alpha]^{\bullet})^{\vdash_k}$$

relativized obligation:
$$([\vdash \alpha]^{\bullet})^{\mathcal{O}\!-} \Rightarrow ([\vdash \alpha]^{\bullet})^{\mathcal{O}\!-_N}$$

TABLE 8. Substitution rules - II

Given a fully functorial translation $[\ \ ]^{\bullet}$:

$$\frac{([\vdash \mathbf{A}]^{\bullet})^{\circ\!-}, ([\mathbf{Z}]^{\bullet})^{\vdash} \Rightarrow ([\vdash \alpha]^{\bullet})^{\circ\!-},}{\circ\!-\, \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \circ\!-\, \alpha} \qquad \frac{([\vdash \mathbf{A}]^{\bullet})^{\circ\!-_N}, ([\mathbf{Z}]^{\bullet})^{\vdash} \Rightarrow ([\vdash \alpha]^{\bullet})^{\circ\!-_N},}{\circ\!-\, \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \circ\!-\, \alpha}$$

$$\frac{([\Gamma]^{\bullet})^{\vdash} \Rightarrow ([\delta]^{\bullet})^{\vdash}}{;\ (\Gamma)^{\vdash} \Rightarrow (\delta)^{\vdash}} \qquad \frac{([\Gamma]^{\bullet})^{\circ\!-} \Rightarrow ([\delta]^{\bullet})^{\circ\!-}}{;\ (\Gamma)^{\circ\!-} \Rightarrow (\delta)^{\circ\!-}}$$

$$\frac{([\Gamma]^{\bullet})^{\vdash_k} \Rightarrow ([\delta]^{\bullet})^{\vdash_k}}{;\ (\Gamma)^{\vdash_k} \Rightarrow (\delta)^{\vdash_k}} \qquad \frac{([\Gamma]^{\bullet})^{\circ\!-_N} \Rightarrow ([\delta]^{\bullet})^{\circ\!-_N}}{;\ (\Gamma)^{\circ\!-_N} \Rightarrow (\delta)^{\circ\!-_N}}$$

where the upper sequents are derivable in **ILLP**.

TABLE 9. Bridging rules.

THEOREM 3. *Let δ be a formula in the language of $\mathcal{L}^P$. The sequent* $;\ \Rightarrow \delta$ *is provable in* **ILP** *if and only if it is provable in* $\mathbf{ILP}^{+}$.

This result is an obvious consequence of the functoriality of the given translation $[\ \ ]^{\bullet} : \mathbf{IL} \to \mathbf{ILL}$. The *"only if"* part of Theorem 3 is obvious. To prove the *"if"* part notice that by Proposition 2 and the following remark we may assume that every subderivation d^{ℓ} ending with a bridging rule is cut-free and all the substitution rules occur at the end. If the last substitution rule is $\vdash$-*Subst Right*, then the conclusion of the bridging rule has the form $(\Gamma)^{\vdash} \Rightarrow (\delta)^{\vdash}$ and by the functoriality of the translation $[\ \ ]^{\bullet}$ we can transform d^{ℓ} into a derivation d' of $(\Gamma)^{\vdash} \Rightarrow (\delta)^{\vdash}$ in $\mathbf{ILP}^{-}$. Similarly if the last susbstitution is $\circ\!-$-*Subst Right*, $\vdash_k$-*Subst Right* and $\circ\!-_N$-*Subst Right*.

If the last substitution rule of d^{ℓ} is a *relevant implication* / $\circ\!-$ rule, then d^{ℓ} ends as follows:

$$\cfrac{\cfrac{[\vdash \mathbf{A}]^{\bullet}, [\mathbf{Z}]^{\bullet} \Rightarrow [\vdash \beta]^{\bullet}}{([\vdash \mathbf{A}]^{\bullet})^{\circ\!-}, [\mathbf{Z}]^{\bullet\,\vdash} \Rightarrow ([\vdash \beta]^{\bullet})^{\circ\!-}}\ \textit{rel. impl./}\ \circ\!-\ \textit{rule}}{\circ\!-\, \mathbf{A}, \mathbf{Z}\ ;\ \Rightarrow\ \circ\!-\, \beta}\ \textit{bridging rule}$$

and by the functoriality of the translation we can transform d^{ℓ} into a suitable derivation d':

$$\dfrac{\begin{array}{c} d' \\ \vdash \mathbf{A}, \mathbf{Z} \; ; \; \Rightarrow \; \vdash \beta \end{array}}{\circ\!\!- \mathbf{A}, \mathbf{Z} \; ; \; \Rightarrow \; \circ\!\!- \beta} \; \supset_{c} / \circ\!\!-$$

3.3. Examples. The following derivation shows that the principle *"if p is obligatory and if the assertibility of p causally implies the assertibility of q, then q is obligatory"*, is provable in **ILP**:

$$\begin{array}{ll}
\vdash p \; ; \; \Rightarrow \; \vdash p \qquad \vdash q \; ; \; \Rightarrow \; \vdash q & \\ \hline
\vdash p, \; \vdash p \supset_{c} \vdash q \; ; \; \Rightarrow \; \vdash q & \supset_{c} \\ \hline
\circ\!\!- p, \; \vdash p \supset_{c} \vdash q \; ; \; \Rightarrow \; \circ\!\!- q & \supset_{c} - \circ\!\!- \\ \hline
; \; \circ\!\!- p, \; \vdash p \supset_{c} \vdash q \Rightarrow \circ\!\!- q & \textit{derelictions} \\ \hline
; \; \circ\!\!- p, \; \circ\!\!- p \cap (\vdash p \supset_{c} \vdash q) \Rightarrow \circ\!\!- q & \textit{left } \cap \\ \hline
; \; \circ\!\!- p \cap (\vdash p \supset_{c} \vdash q) \Rightarrow \circ\!\!- q & \textit{left } \cap, \textit{ contraction} \\ \hline
; \; \Rightarrow (\circ\!\!- p \cap (\vdash p \supset_{c} \vdash q)) \supset \circ\!\!- q & \textit{right } \supset
\end{array}$$

In a similar way we derive $(\circ\!\!-_N p \cap (\vdash p \supset_{c} \vdash q)) \supset \circ\!\!-_N q$.
By the cut-elimination theorem we cannot derive any of the following formulas:

$$\begin{array}{ll}
(\circ\!\!- p \cap (\vdash p \supset \vdash q)) \supset \circ\!\!- q & (\vdash p \cap (\circ\!\!- p \supset \circ\!\!- q)) \supset \circ\!\!- q \\
(\vdash p \cap (\circ\!\!- p \supset \circ\!\!- q)) \supset \vdash q & (\vdash_k p \cap (\circ\!\!- p \supset \circ\!\!- q)) \supset \vdash_k q \\
(\vdash p \cap (\vdash_k p \supset \vdash_k q)) \supset \vdash q & (\circ\!\!- p \cap (\vdash_k p \supset \vdash_k q)) \supset \circ\!\!- q \\
(\vdash_k p \cap (\vdash_{k'} p \supset \vdash_{k'} q)) \supset \vdash_{k'} q & (\circ\!\!- p \cap (\circ\!\!-_N p \supset \circ\!\!-_N q)) \supset \circ\!\!- q
\end{array}$$

§4. Concluding remark. From a mathematical point of view the extension from **ILP**$^-$ to **ILP**$^+$ through the linear system **ILLP** is an immediate corollary of the existence of fully functorial translations **IL** $\rightarrow$ **ILL** and does not seem to add much to the direct extension from **ILP**$^-$ to **ILP**. Its significance lies in the following facts. On one hand **ILP** formalizes reasoning with mixed contexts and with various notions of implications using a *mixed relevant and intuitionistic consequence relation*, where we can infer properties of *causal implication* by using the relevant area only and also derive standard intuitionistic formulas using the intuitionistic area of the consequence relation, but the relations between the two forms of reasoning are not fully clarified. Moreover, other uses of reasoning with mixed contexts may require other extensions of standard intuitionistic calculi: this could certainly be done on a case by case basis. On the other hand linear logic has been proposed as a general framework for such extensions, where the relations between these different forms of reasoning may be clarified. But the explanations so far given of the "intended interpretations" of linear logic and its place in the context of logical theory appear unsatisfactory to us. In our approach the language $\mathcal{L}_i^\ell$ and the calculus **ILLP** have a clear "intended interpretation" in the notion of a *free logic of pragmatic force*. It is not claimed that this is the only possible "natural reasoning interpretation" of linear logic. However, our interpretation shows that reasoning with the linear consequence relation is an *extension* of the ordinary logic of assertions and obligations rather than a *deviant logic*. The connectives and operators of intuitionistic linear logic also obtain in this way a *partial interpretation* in the pragmatic framework, although through the rather cumbersome machinery of translations and substitution operators.

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FACOLTÀ DI SCIENZE
UNIVERSITÀ DI VERONA
STRADA LE GRAZIE, CÀ VIGNAL 2
37134 VERONA, ITALY
and
DEPARTMENT OF COMPUTER SCIENCE
QUEEN MARY AND WESTFIELD COLLEGE
LONDON E1 4NS
E-mail: bellin@sci.univr.it

DIPARTIMENTO DI FILOSOFIA
UNIVERSITÀ DI LECCE
VIA STAMPICCHIA, PALAZZO PARLANGELI
73100 LECCE, ITALY