Intuitionistic modalities, classical logic and lax logic.

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0. Plan of the talk.

1. Outlook: general ideas and problems.

2. On the categorical semantics of LK.

3. Polarized bi-intuitionistic logic.

4. Philosophical interlude.

6. Bi-intuitionistic modalities, classical logic.

6. Bi-intuitionistic modalities, lax logic.

7. Temporary conclusions.

1.1. Grand projects.

Project I - Double negation

Gentzen's Sequent Calculus LK *with*
Categorical model of it ? (a)
$\|$
$\sim\sim$ transl. $+$ Prawitz $\perp_C$ (b)
$\Downarrow$
$\lambda\mu$-calc *with*
Control Categories (c)

(a) **Which one?**
- Bellin-Hyland-Robinson-Urban. Categorical Proof Theory of the Classical Prop. Calculus *TCS* [2005]
- Führmann-Pym. Order-enriched categorical models of the class. seq. calculus. *J.Pure App.Algebra* [2006]
- Lamarche-Strassburger. Naming Proofs in Classical Logic, *TLCA* [2005]
- Dominic Hughes. Classical Logic = Fibered MLL, LICS 2005.
- Kosta Došen, Zoran Petrić *Proof-Net Categories*, Belgrade, 2007.

(b) **Who knows?**

(c) Peter Selinger. Control Category and Duality: on the categorical semantics of the lambda-mu calculus, *MSCS* [2001]

Project II - S4 translation

Natural Deduction NJ *with*
Cartesian Closed Categories

$\|$

S4 transl (a)

$\Downarrow$

Sequent calculus S4 *with*
Categorical model of it ?? (b)

(a) **Which translation?**

(b) **Extension of which classical model??**

1.2. Elementary remarks.

- Glivenko's and Gödel's double negation translation can be defined **functorially**:

  - **on formulas**:
  $(p)^* = \sim\sim p$, $(A \wedge B)^* = A^* \cap B^*$,
  $(A \to B)^* = A^* \supset B^*$, $(A \vee B)^* = \sim (\sim A^* \cap \sim B^*)$;

  - **on proofs**:
  by a map $(\ )^* : \mathbf{LK} \to \mathbf{LJ}$ such that

$$(A \vdash^{LK} A)^* \quad = \quad A^* \vdash^{LJ} A^* \quad \text{and}$$

$$\left(\frac{d_1 \qquad d_2}{S}\, {\scriptstyle cut} \right)^* = \frac{d_1^* \qquad d_2^*}{S^M}\, {\scriptstyle cut}$$

$\boxed{\textbf{Prawitz NK does not fit in here!}}$

- Gödel's McKinsey and Tarski's modal translation can be defined **functorially**:

  - **on formulas**:
  $(p)^M = \Box p$, $(A \cap B)^M = A^M \wedge B^M$,
  $(A \supset B)^M = \Box(A^M \to B^M)$, $(A \cup B)^M = A^M \vee B^M)$;

  - **on proofs**:
  by a map $(\ )^M : \mathbf{LJ} \to \mathbf{LK} - \mathbf{S4}$ such that

$$(A \vdash^{LJ} A)^M \quad = \quad A^M \vdash^{S4} A^M \quad \text{and}$$

$$\left(\frac{d_1 \qquad d_2}{S}\, {\scriptstyle cut} \right)^M = \frac{d_1^M \qquad d_2^M}{S^M}\, {\scriptstyle cut}$$

1.3. So what is Prawitz NK?

- **Natural Deduction:**
  classical NK = intuitionistic NJ + $\perp_C$

$$\text{where} \quad \dfrac{\begin{matrix}[\sim A]\\ \vdots \\ \perp\end{matrix}}{A}\perp_C \quad \equiv \quad \dfrac{\sim\sim A}{A}$$

- **but wait:** it uses the $\sim\sim$ translation *on formulas*
  letting $(\mathbf{p})^* = \mathbf{p}$, and then $(A \wedge B)^* = A^* \cap B^*$, $(A \to B)^* = A^* \supset B^*$, $(A \vee B)^* = \sim(\sim A^* \cap \sim B^*)$; and a proof d is
  - **intuitionistic and classical** if it makes no use of $\perp_C$
  - **classical** otherwise.

- We may think of a translation $\mathbf{LK} \to \mathbf{NK}$, but the symmetries between $\wedge$ and $\vee$ in **LK** are completely lost. Essentially, we extend intuitionistic **NJ**.

- Look for representations of classical logic within intuitionism (other than $\sim\sim$-trans)!

1.4. Hopeful projects.

Project I - Double negation

Gentzen's Sequent Calculus LK *with*
Categorical model of it.

$$\Downarrow$$

functorially? $\lambda\mu$ transl.?

$$\Downarrow$$

Polarized bi-intuitionistic logic + modality
with **Categorical model of it?**

$$\Downarrow$$

S4 transl

$$\Downarrow$$

Sequent calculus S4 *with*
Categorical model of it ??

2.1. Polycategories and proof-net categories. Hyland *et al.* 2005.

Definition. (Szabo 1975) A symmetric polycategory $\mathcal{P}$ consists of

• A collection ob$\mathcal{P}$ of *objects* and

- for every pair of finite sequences Γ and Δ of objects, a collection $\mathcal{P}(\Gamma; \Delta)$ of *polymaps* from Γ to Δ.

• For each re-ordering of the sequence Γ to produce a sequence Γ', an *isomorphism* from $\mathcal{P}(\Gamma; \Delta)$ to $\mathcal{P}(\Gamma'; \Delta)$, functorial in its action, and dually for Δ (*exchange*).

• An *identity* $id_A \in \mathcal{P}(A; A)$ for each object A; and a *composition* (*cut*)

$$\mathcal{P}(\Gamma; \Delta, A) \times \mathcal{P}(A, \Pi; \Sigma) \to \mathcal{P}(\Gamma, \Pi; \Delta, \Sigma)$$

for each $\Gamma, \Delta, A, \Pi, \Sigma$ coherent with reordering; these data satisfy the familiar identity and associativity laws.

Definition. A *symmetric $*$-polycategory* $\mathcal{P}$ consists of a polycategory $\mathcal{P}$ equipped with an involutory negation $(-)^*$ on object together with, for each Γ, Δ, A, an isomorphism $\mathcal{P}(\Gamma; \Delta, A) \cong \mathcal{P}(A^*, \Gamma; \Delta)$ coherent with reordering and composition.

We need:

1. *operations* for classical connectives $\wedge$, $\top$ corresponding to

$$\frac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge \text{ L} \qquad \frac{\Gamma \vdash \Delta, A \quad \Pi \vdash \Lambda, B}{\Gamma, \Pi \vdash \Delta, \Lambda, A \wedge B} \wedge\text{-R}$$

$$\frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta} \top \text{ L} \qquad \frac{}{\vdash \top} \top\text{-R}$$

2. negation is defined implicitly: given an involutory negation on atoms,

$$\begin{array}{cc} \top^* = \bot & \bot^* = \top, \\ (A \wedge B)^* = B^* \vee A^* & (A \vee B)^* = B^* \wedge A^*; \end{array}$$

   thus operations for $\vee$ and $\bot$ follow by duality;

3. *equalities on polymaps* from *naturality*, *commutative conversions* and *meaning-preserving* reductions;

4. implementation of *weakening* and *contraction*.

Proofs can be represented as proof-nets;
- also in the single-sided representation;
- it gives a simple algebraic notation for proofs

$$id_A, \quad f;g, \quad f\cdot g, \quad \overline{f}, \quad \star, \quad f^+$$

- two-sided nets are better for categories.

$$id_A : \frac{\textit{axiom}}{A^* \quad A} \qquad f;g \left\{ \frac{\begin{matrix} f & g \\ A & A^* \end{matrix}}{\textit{cut}} \right.$$

$$\overline{f} \left\{ \frac{\begin{matrix} f \\ A \quad B \end{matrix}}{A \vee B} \right. \qquad f\cdot g \left\{ \frac{\begin{matrix} f & g \\ A & B \end{matrix}}{A \wedge B} \right.$$

$$f^+ \left\{ \frac{\begin{matrix} f \\ A \end{matrix}}{\bot \quad A} \right. \qquad \star : \frac{\textit{axiom}}{\top}$$

$$\text{weak: } \left\{ \frac{\Gamma}{A, \Gamma} \right. \qquad \text{contr: } \left\{ \frac{A \quad A}{A} \right.$$

However proof-nets suggest unwanted identifications of proofs! (see below)
We work with polycategories then translate into cats.

Notes: (a) *Cut elimination is* **highly non-deterministic**.
(b) *Cut-elimination does not preserve* **identity of proofs**.
(c) *Equal proofs have the* **same set of normal forms**
(d) *but not necessarily the converse!*
(e) We indicate explicitly equations of proofs and *meaning preserving reductions* (also in proof-net notation).

2.2. Reductions, commutations.

- For simplicity, omit contexts Γ, Δ in $f : \Gamma, A \Rightarrow C, \Delta$.

- for $f : A \Rightarrow C$, $g : C \Rightarrow D$, $u : A' \Rightarrow A$, $v : B' \Rightarrow B$ write

$\{u, v\}; (f \cdot d)$ for

$$\text{cuts}\,\dfrac{\overset{u}{A' \Rightarrow A} \quad \overset{v}{B' \Rightarrow B} \quad \dfrac{\overset{f}{A \Rightarrow C} \quad \overset{g}{B \Rightarrow D}}{A, B \Rightarrow C \wedge D}}{A', B' \Rightarrow C \wedge D}$$

Logical cuts are meaning preserving.

$\wedge$ **reduction:**

for all $f : A \Rightarrow C$, $g : B \Rightarrow D$, $k : C, D \Rightarrow E$

$(\wedge)$ $$(f \cdot g); \overline{k} = \{f, g\}; k$$

where

$$\text{cut}\,\dfrac{\dfrac{\overset{f}{A \Rightarrow C} \quad \overset{g}{B \Rightarrow D}}{A, B \Rightarrow C \wedge D} \quad \dfrac{\overset{k}{C, D \Rightarrow E}}{C \wedge D \Rightarrow E}}{A, B \Rightarrow E}$$

reduces to

$$\text{cuts}\,\dfrac{\overset{f}{A \Rightarrow C} \quad \overset{g}{B \Rightarrow D} \quad \overset{k}{C, D \Rightarrow E}}{A, B \Rightarrow E}$$

$\top$ **reduction:**

$$(\top) \qquad \star; f^{+} = f, \qquad \text{for all } f : \Gamma \Rightarrow \Delta$$

where

$$cut\,\dfrac{\dfrac{\star}{\Rightarrow \top} \qquad \dfrac{\overset{f}{\Gamma \Rightarrow \Delta}}{\top, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta} \quad \text{red. to} \quad \overset{f}{\Gamma \Rightarrow \Delta}$$

Commutation equations.

For all $f :\Rightarrow A, B$, $g :\Rightarrow C$, $h \Rightarrow D$,

$$(f \cdot g) \cdot h = f \cdot (g \cdot h) :\Rightarrow A \wedge C, B \wedge D$$

For all $f : A, B \Rightarrow C$, $g :\Rightarrow D$,

$$\overline{f} \cdot g = \overline{f \cdot g} : A \wedge B \Rightarrow C \wedge D$$

For all $f : A, B \Rightarrow C, D$, $\overline{\overline{f}} = \overline{\overline{f}}$
where

$$\dfrac{\dfrac{\overset{f}{A, B \Rightarrow C, D}}{A \wedge B \Rightarrow C, D}}{A \wedge B \Rightarrow C \vee D} = \dfrac{\dfrac{\overset{f}{A, B \Rightarrow C, D}}{A, B \Rightarrow C \vee D}}{A \wedge B \Rightarrow C \vee D}$$

$$f^{+} \cdot g = (f \cdot g)^{+}, \quad \overline{f}^{+} = \overline{f^{+}}, \quad f^{++} = f^{++}$$

and all variants by duality.

2.3. Naturality equations.

(i) for $\wedge$ L: for $h : A, B \Rightarrow E$ and $w : E \Rightarrow E'$

$$\boxed{\overline{h}; w = \overline{h; w}}$$

i.e.,

$$\textit{cut}\,\frac{\dfrac{\overset{h}{A, B \Rightarrow E}}{A \wedge B \Rightarrow E} \qquad \overset{w}{E \Rightarrow E'}}{A \wedge B \Rightarrow E'} \quad = \quad \frac{\textit{cut}\,\dfrac{\overset{h}{A, B \Rightarrow E} \qquad \overset{w}{E \Rightarrow E'}}{A, B \Rightarrow E'}}{A \wedge B \Rightarrow E'}$$

(ii) for T L: for $h : A \Rightarrow B$, $v : B \Rightarrow B'$

$$\boxed{h^{+}; w = (h; w)^{+}}$$

(iii) for $\wedge$ R: **for arbitrary** u, v, f, g,

$$\boxed{(\S) \quad \{u, v\}; (f \cdot g) \neq (u; f) \cdot (v; g)}$$

Namely:

$$
cuts\,\frac{\overset{u}{A' \Rightarrow A} \quad \overset{v}{B' \Rightarrow B} \quad \dfrac{\overset{f}{A \Rightarrow C} \quad \overset{g}{B \Rightarrow D}}{A, B \Rightarrow C \wedge D}}{A', B' \Rightarrow C \wedge D}
$$

(§) **is not equal to**

$$
\frac{cut\,\dfrac{\overset{u}{A' \Rightarrow A} \quad \overset{f}{A \Rightarrow C}}{A' \Rightarrow C} \quad cut\,\dfrac{\overset{v}{B' \Rightarrow B} \quad \overset{g}{B \Rightarrow D}}{B' \Rightarrow D}}{A', B' \Rightarrow C \wedge D}
$$

Definition. (a) We call maps u, v for which

$$u; (f \cdot g) = (u; f) \cdot g, \quad v; (f \cdot g) = f \cdot (v; g)$$

(and so also (§), and their dual) hold **linear**.
(b) *Axioms* (id and $\star$) are linear; linear maps are *closed under the logical operations* $(- \cdot -), \overline{(-)}, (-)^{+}$ (and their duals).

2.4. Nonlinear maps.

We implement *weakening* and *contraction* by cut with *generic instances* of them:

$$t_A : \left\{ \frac{\Rightarrow \top}{A \Rightarrow \top} \text{W L} \qquad d_A : \left\{ \frac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A \wedge A}}{A \Rightarrow A \wedge A} \text{C L} \right.\right.$$

Dually, have $u_A : \bot \Rightarrow A$ and $m_A : A \vee A \Rightarrow A$. To *weaken* f with A we reduce the cut $t_A; f^+$ where $f^+ : \top, \Gamma \Rightarrow \Delta$;
to *contract* A in g we reduce the cut $d_A; \overline{g}$ where $\overline{g} : A \wedge A, \Gamma \Rightarrow \Delta$ (and dually!).

Remember that *logical cuts preserve meaning!*.

There are obvious naturality and commuting conversions (e.g., for C L analogue to those of $\wedge$ R) and other equalities (omitted).

Claim: we must have

$$(\S\S) \qquad m_A; (id_A \cdot id_B) \;\neq\; (m_A; id_A) \cdot id_B$$

The l.h.s. of (§§) $m_A;(id_A \cdot id_B)$:

$$\mathit{cut}\,\dfrac{\mathrm{C\ R}\,\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A \vee A \Rightarrow A, A}}{A \vee A \Rightarrow A} \qquad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A \wedge B}}{A \vee A, B \Rightarrow A \wedge B}$$

First reduction. "m_A up" yields $m_A \cdot id_B$

$$\text{NF1:} \qquad \dfrac{\mathrm{C\ R}\,\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A \vee A \Rightarrow A, A}}{A \vee A \Rightarrow A} \qquad B \Rightarrow B}{A \vee A, B \Rightarrow A \wedge B}$$

the r.h.s. of (§§) $(m_A; id_A) \cdot id_B$ red. to NF1.

Second reduction: "$(id_A \cdot id_B)$ up" yields $d_b;(id_A \cdot id_B \cdot id_A \cdot id_B); m_{A \wedge B}$

$$\text{NF2:} \qquad \begin{array}{r} \vee\ \mathrm{L} \\ \mathrm{C\ R,\ CR} \end{array} \begin{array}{c} \dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{AB \Rightarrow A \wedge B} \quad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A \wedge B}}{A \vee A, B, B \Rightarrow A \wedge B, A \wedge B} \\ \hline\hline A \vee A, B \Rightarrow A \wedge B \end{array}$$

No reduction from r.h.s. (§§) to NF2.

So the l.h.s. and the r.h.s. of (§§) have different sets of normal forms, qed.

2.5. Non-functoriality of $\wedge$.

Given $f : A \Rightarrow B$, $g : C \Rightarrow D$, write $f \wedge g = \overline{f \cdot g} : A \wedge C \Rightarrow B \wedge D$.

Functoriality of $\wedge$ fails:

$$\boxed{\begin{array}{rll} (u \wedge v); (f \wedge g) & = \overline{\{u, v\}; (f \cdot g)} & (\wedge \text{ red}) \\ & \neq \overline{(u; f) \cdot (v; g)} & (\S\S) \\ & = (u; f) \wedge (v; g) & (\wedge \text{ red}) \end{array}}$$

Note: the operation $\overline{(-)}$ is injective.

Definition. *Linear idempotents* are linear maps e s.t. $e; e = e$.
Fact: *identities* are linear idempotents; $e_\top = \star^+$ is a linear idempotent;
if $e_A : A \Rightarrow A$ and $e_B : B \Rightarrow B$ are linear idempotents then so are

$$e_A \wedge e_B = \overline{e_A \cdot e_B} \quad \text{and dually defined} e_A \vee_B .$$

Notice that $e_{A \wedge B} \neq e_A \wedge e_B$, i.e., $id_{A \wedge B} \neq \overline{id_A \cdot id_B}$.

Definition. (**guarded categories**) A *guarded category* is a category with a class of linear idempotents.

2.6. Categorical models compared.

Definition. (1) (**guarded functors**) A *guarded functor* $F : \mathcal{C} \to \mathcal{D}$ between guarded categories consists of the usual data for a functor such that F maps linear idempotents to linear idempotents and whenever e and e' are linear idempotents then

$$F(e); F(f); F(g); F(e') = F(e); F(f; g); F(e')$$

(2) (**guarded transformations**) Let $F, G : \mathcal{C} \to \mathcal{D}$ be guarded functors. A *guarded transformation* consists of data $\alpha_A : F(A) \to G(a)$ satisfying

$$F(id_A); F(u); \alpha_B = \alpha_A; G(u); G(id_B)$$

for all $u : A \to B$ in $\mathcal{C}$.

We define a categorical model for classical logic letting logical operations be *guarded functorial*. Details are omitted.

Theorem. *Let $\mathcal{C}$ b a categorical model for classical proof in the above sense. Then the following are equivalent.*

(1) *The identity conditions $id_A \wedge id_B = id_{A \wedge B}$ and $e_\top = id_\top$.*

(2) *Full functoriality of $\wedge$ and $\top$.*

(3) *Representability of polymaps by $\wedge$, $\top$ and $\vee$, $\bot$.*

Conditions (1) - (3) are satisfied in Fühman-Pym's model and Lamarche-Strassburger's models accept an even larger set of equalities of proofs.

3. Polarized bi-intuitionism.

Language $\mathcal{L}^{AH}$:

$$A, B \quad := \quad \vdash p \mid \top \mid \sim A \mid A \supset B \mid A \cap B \mid \sim C$$

$$C, D \quad := \quad \mathcal{H} p \mid \bot \mid \frown C \mid C \smallsetminus D \mid C \curlyvee D \mid \frown A$$

$\vdash p$: the type of assertions that prop. p is true.
$\vdash$denotes the illocutionary force of *assertion*.

$\mathcal{H} p$: the type of hypotheses that p may be true.
$\mathcal{H}$ the illocutionary force of *hypothesis*.

- $\vdash p$, $A \supset B, A \cap B, \top$: *assertive types*, (implication, conjunction, validity);
- $\mathcal{H}\ p$, $C \smallsetminus D, C \curlyvee D, \bot$: *hypothetical types*, (subtraction, disjunction, invalidity);

($C \smallsetminus D$ *perhaps* C *and not* D).

- *definitely not*: $\sim A =_{df} A \supset \mathbf{inv}$, **inv** invalid assertive sentence;
- *perhaps not*: $\frown C =_{df} \mathbf{val} \smallsetminus C$, **val** valid conjectural sentence;
- $\sim C$, $\frown A$ *dualities*.

Extended KHB interpretation for $\mathcal{L}^{AH}$.

- $\vdash p$ is justified by *conclusive evidence* that p is true;
- $\mathcal{H} p$ is justified by a *scintilla of evidence* that p is true;
- $A \supset B$ is justified by a *method* transforming a justification of A into a justification of B
- $C \smallsetminus D$ is justified by a *scintilla of evidence* that there is a justification of C and no justification of D; etc.

3.1. Interpretation in S4.

$$
\begin{array}{cc}
(\top)^M =_{df} \mathbf{true} & (\bot)^M =_{df} \ \mathbf{false} \\
(\vdash p)^M =_{df} \Box p & (\mathcal{H} p)^M =_{df} \Diamond p \\
(A \supset B)^M =_{df} \Box(A^M \to B^M) & (C \smallsetminus D)^M =_{df} \Diamond(C^M \wedge \neg D^M) \\
(A_1 \cap A_2)^M =_{df} A_1^M \wedge A_2^M & (C_1 \curlyvee C_2)^M =_{df} C_1^M \vee C_2^M \\
(\sim A)^M = \Box\neg A^M & (\frown C)^M = \Diamond\neg C^M \\
(\sim C)^M = \neg C^M & (\frown A)^M = \neg A^M
\end{array}
$$

- *Epistemic interpretation* of **S4**: models $(W, R, \Vdash)$ with R a pre-order representing the evolution of states of knowledge $w_i \in W$.

- **AH**: the set of all expressions in $\mathcal{L}^{AH}$ that are valid in the **S4** modal translation.

- **AH-G1**: a sequent calculus for **AH** *sound and complete for* **S4** where sequents are of the form

$$\Theta \ ; \ \Rightarrow \ A \ ; \ \Upsilon$$

$$\Theta \ ; \ C \ \Rightarrow \ ; \ \Upsilon$$

- Θ is a sequence of *assertive* formulas $A_1, \ldots, A_m$;

- Υ a sequence of *hypothetical* formulas $C_1, \ldots, C_n$.

Basic laws of AH:

$A \equiv \sim\frown A$ as $(\sim\frown A)^M = \Box A^M = A^M$.

$C \equiv \frown\sim C$ as $(\frown\sim C)^M = \Diamond C^M = C^M$.

Sequent calculus AH-G1
rules for implication, subtraction

right $\supset$*:*

$$\frac{\Theta, A\ ;\ \Rightarrow\ B\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ A \supset B\ ;\ \Upsilon}$$

left $\smallsetminus$*:*

$$\frac{\Theta\ ;\ C\ \Rightarrow\ ;\ \Upsilon, D}{\Theta\ ;\ C \smallsetminus D\ \Rightarrow\ ;\ \Upsilon}$$

left $\supset$*:*

$$\frac{\Theta\ ;\ \Rightarrow\ A\ ;\ \Upsilon \qquad B, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{A \supset B,\ \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

right $\smallsetminus$*:*

$$\frac{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, C \qquad \Theta\ ;\ D\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, C \smallsetminus D}$$

intuitionistic modalities:

$\Box_I C = \sim\frown C$ (*assertive necessity*)

- where $(\sim\frown C)^M = \Box C^M$

$\Diamond_I C = \frown\sim A$ (*hypothetical possibility*)

- where $(\frown\sim A)^M = \Diamond A^M$.

Write $\mathcal{E}p =_{df} \Box_I\ \mathcal{H}p$ (**expectation** that p)

- the type of assertions that p has to be a hypothesis.

Write $\mathcal{C}p =_{df} \Diamond_I\ \vdash p$ (**conjecture** that p)

- the type of hypotheses that p could be asserted.

3.2. Bi-intuitionistic illocutions.

$$
\begin{tikzcd}
& & \Diamond p & & & \\
& & & & \Diamond\Box\Diamond p \arrow[ull] & \\
p \arrow[uurr] & & & \Diamond\Box p \arrow[ur] & & \Box\Diamond p \arrow[ul] \\
& & & & \Box\Diamond\Box p \arrow[ul] \arrow[ur] & \\
& & \Box p \arrow[uull] \arrow[urr] & & &
\end{tikzcd}
$$

The modalities of **S4**

$$
\begin{tikzcd}
& & \mathcal{H}p & & & \\
& & & & \frown\frown \mathcal{H}p \arrow[ull] & \\
p \arrow[uurr] & & & \mathcal{C}p \arrow[ur] & & \mathcal{E}p \arrow[ul] \\
& & & & \sim\sim \vdash p \arrow[ul] \arrow[ur] & \\
& & \vdash p \arrow[uull] \arrow[urr] & & &
\end{tikzcd}
$$

Assertions, hypotheses, conjectures, expectations

3.3. Bimodal Interpretation.

$$
\begin{array}{rclcrcl}
(\vdash p)^M &=& \Box p & \qquad & (\mathcal{H} p)^M &=& \rlap{-}\Diamond p \\
(A \supset B)^M &=& \Box(A^M \to B^M) & & (C \smallsetminus D)^N &=& \rlap{-}\Diamond(C^M \wedge \neg D^M) \\
(A \cap B)^M &=& A^M \wedge B^M & & (C \curlyvee D)^M &=& C^M \vee D^M \\
(A \cup B)^M &=& A^M \vee B^M & & (C \curlywedge D)^M &=& C^M \wedge D^M \\
(\sim C)^M &=& \Box \neg C^M & & (\frown A)^M &=& \rlap{-}\Diamond \neg A^M
\end{array}
$$

- Bimodal **S4** frames have the form (W, R, S) where both R and S are preorders over the set W.
- Kripke models for *bimodal* **S4** have the form $\mathcal{M} = (W, R, S, \mathcal{V})$, with $\mathcal{V}$ a valuation of the atoms over W and the forcing conditions are

1. $w \Vdash \Box X$ if and only if $\forall w'$, wRw' implies $w' \Vdash X$;

2. $w \Vdash \rlap{-}\Diamond X$ if and only if $\exists w'$ such that wSw' and $w' \Vdash X$.

• **Consider bimodal frames where** $S \subseteq R$.

- In such a model it is easier to falsify $\rlap{-}\Diamond X$ than $\Diamond X$.

• Let **ASH** be the set of $\mathcal{L}^{AH}$ formulas valid in all such frames.

• Let **ASH-G1** be the sequent calculus *sound and complete* for the semantics of **ASH**.

Sequent calculus ASH-G1
specific rules

right $\supset$*:*

$$\frac{\Theta, A \; ; \; \Rightarrow \; B \; ;}{\Theta \; ; \; \Rightarrow \; A \supset B \; ; \; \Upsilon}$$

left $\smallsetminus$*:*

$$\frac{\Theta \; ; \; C \; \Rightarrow \; ; \; \Upsilon, D}{\Theta \; ; \; C \smallsetminus D \; \Rightarrow \; ; \; \Upsilon}$$

3.4. Categorical semantics?

The categorical semantics of subtraction is given by *coexponents*.
- Given two objects $A, B \in \mathcal{C}$, a *coexponent* for A, B is an object B_A, together with an arrow $\ni_{A,B}$: $B \to B_A \sqcup C$ in $\mathcal{C}$, satisfying the following property:
For any object C and any arrow $f : B \to C \sqcup A$, there is a unique $h : B_A \to C$ (written $f_\star$) such that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "\ni_{A,B}"'] & C \sqcup A \\ & B_A \sqcup A \arrow[u, "h \sqcup 1_A"'] \end{tikzcd} \qquad \begin{tikzcd} C \\ B_A \arrow[u, "h"'] \end{tikzcd}$$

Lemma. (T. Crolard) *In the category of sets, the coexponent B_A of two sets A and B is defined if and only if $A = \emptyset$ or $B = \emptyset$.*

Proof. In **Sets**, the coproduct $\sqcup$ is the disjoint union; thus if $A \neq \emptyset \neq B$ then the functions f and $\ni_{A,B}$ for every $b \in B$ must *choose a side*, left or right, of the coproduct in their target and moreover $f_\star \sqcup 1_A$ leaves the side unchanged. Hence, if we take a nonempty set C and f with the property that for some b different sides are chosen by f and $\ni_{A,B}$, then the diagram does not commute. QED.

Thus co-intuitionistic logic with disjunction and subtraction has only a degenerate model in **Sets**. Look for models in linear logic and *monoidal categories*.

• *Additive* intro rules for disjunction involve a *choice* between the disjunct. Thus we must have *multiplicative* rules for disjunction.
- "Commas" in the right hypothetical area are Girard's *par*.

Sequent calculus AH-G1
rules for disjunction

right $\curlyvee$*:*

$$\frac{\Theta \;;\; \epsilon \;\Rightarrow\; \epsilon' \;;\; C_0, C_1, \Upsilon}{\Theta \;;\; \epsilon \Rightarrow\; \epsilon' \;;\; , C_0 \curlyvee C_1, \Upsilon}$$

left $\curlyvee$*:*

$$\frac{\Theta \;;\; C_0 \;\Rightarrow\; ;\; \Upsilon \quad \Theta' \;;\; C_1 \;\Rightarrow\; ;\; \Upsilon'}{\Theta, \Theta' \;;\; C_0 \curlyvee C_1 \;\Rightarrow\; ;\; \Upsilon}$$

4.1. Normalization in Prawitz 1965.

- **Technically, $\bot_C$ is an anomaly.**
  Natural Deduction inferences consist of *introduction* - *elimination* rules;
  - Intuitionistic *"ex falso"* $\dfrac{\bot}{A}\,\bot I$ is an intro.
  - **classic $\bot_C$ is neither an intro nor an elim.**

- A derivation is a *redex* if it ends with an *elim* whose main premise is conclusion of an *intro*.
  A *reduction* eliminates the *intro-elim* pair.

$$\dfrac{\dfrac{\begin{matrix}[A]\\ d_1\\ B\end{matrix}}{A \supset B}\supset I \qquad \begin{matrix}d_0\\ A\end{matrix}}{\mathrm{B}}\supset E \qquad \text{reduces to} \qquad \begin{matrix}d_0\\ [A]\\ d_1\\ B\end{matrix}$$

  Under the $\sim\sim$ translation the rule $\bot_C$ is needed only with atomic conclusions ("minimize the disturbance to the intro-elim pattern").

- **Not possible with disjunction!**

4.2. Prawitz 1977: "Meaning as use".

- Principle of *harmony*:
  - *Intro* rules give the *operational meaning* to a connective (Gentzen).

$$\cap\ \mathrm{I}\,\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \cap B}$$

  - Elim rules are *justified* by the meaning given by an intro rule.

$$\cap_1 E\,\frac{\Gamma \vdash A \cap B}{\Gamma \vdash A} \qquad \cap_2 E\,\frac{\Gamma \vdash A \cap B}{\Gamma \vdash B}$$

  Then an *intro* followed by an *elim* should yield the same information than was already in (one of) the premises (*inversion principle*, Prawitz 1965).

$$\cap_1 E\,\dfrac{\cap I\,\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \cap B}}{\Gamma \vdash A} \quad \text{reduces to} \quad \Gamma \vdash A$$

  A $\cap^*$ I rule *not* in harmony with $\cap_1$ E and $\cap_2$ E:

$$\cap^* I\,\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \cap B}$$

4.3. Prawitz: revision of use.

• The double negation rule neither gives operational meaning nor is not justified by it. **Revision of use is needed** when classical logic conflicts with intuitionism in mathematics and in ordinary language.

Dummett's justificationism.

Intuitionism is the **logic of assertions** and of their justifications.

Some assertions about the past, the future, some applications of the notion of classical continuum to physics, Laplace's determinism, etc. are in principle unjustifiable.

In this case Dummett holds that not only these *assertions*, but also their *propositional content* ought to be regarded as meaningless.

• Dummett refuses to apply a *correspondence theory of truth* to abstract mathematical constructions; in particular, unlike Prawitz and Martin-Löf, does not accept that proofs have an atemporal existence, independently of our knowledge of them.

• Dummett appears to accept a *correspondence theory of truth* w.r.t. the *objects of perception* (e.g., in *Thought and Reality*).

4.4. Prawitz on proofs and justifications.

"Is there a general concept of proof?"

• The conceptual problem: how and why a proof succeeds in giving knowledge.

- why does a proof justify the last assertion?
- it gives **conclusive grounds** for that assertion.

• Why an inference succeeds in justifying the conclusion given the justification of the premisses?

• What constitutes a justification of an assertion?

- *Direct, canonical* means to justify an assertion (introduction rule)
- *Indirect, non-canonical* means must be reduced to canonical ones.

• Heyting: *A proof is the realization of the intention expressed by the proposition.*

• Prawitz: To know the meaning of a sentence A is to know what forms a canonical ground for A has and what conditions the parts of A satisfy.

- Inference acts may now be seen as the act of operating on grounds from the premises.
- The conjecture on identity of proofs is no longer valid. It becomes a definition about the identity of grounds. (My notes from CLMPS 2011, Nancy.)

4.5. Comments by GB, 2012

• Whether or not the notion of truth can be separated form the notion of justification, there are *pragmatic components of meaning*.
• A **pragmatic interpretation of classical logic** is the specification of domains of discourse where the use of classical logic yields correct inferences.

Classical logic may be not only about (Frege's bivalent notion of) truth, but also about properties of epistemic attitudes and illocutionary acts *different from assertions*; see below **expectations**.

• Classical reasoning may become "pragmatically justified" when formulas express different "intentions" and have different justifications from those required by the illocutionary act of *assertion.*
• Thus revision of use is about *fallacies* of scientific or common sense reasoning.

5. Classical logic in bi-intuitionism.

The language $\mathcal{L}^{\mathcal{E}}$ of the pragmatic interpretations of classical logic:

$$\boxed{\mathcal{L}^{\mathcal{E}} = \{E, F, \mathcal{H}p\} \text{ where } E, F \ := \ \mathcal{E}\, p \mid E \supset F \mid E \cap F \mid \ \sim E}$$

with interpretation into **S4**:

$$\left(\mathcal{L}^{\mathcal{E}}\right)^M = \{E^M, F^M, \Diamond p\} \text{ with } (\mathcal{E}\, p)^M = \Box\Diamond p;$$

$$(E \supset F)^M = \Box(E^M \to F^M), (E \cap F)^M = E^M \wedge F^M, (\sim E)^M = \Box\neg E^M$$

To the calculus **AH-G1** add the following rules:

$$\mathcal{E}\text{-R}\ \frac{\Theta \ ; \ \Rightarrow \ ; \ \ \mathcal{H}p, \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \ \mathcal{E}p \ ; \ \Upsilon} \qquad\qquad \mathcal{E}\text{-L}\ \frac{\Theta \ ; \ \ \mathcal{H}p \ \ \Rightarrow \ ; \ \ \Upsilon}{\Theta, \mathcal{E}p \ ; \ \ \Rightarrow \ ; \ \ \Upsilon}$$

Here is a derivation of $\sim\sim \mathcal{E}p :\Rightarrow; \mathcal{E}p$.

$$\boxed{\mathcal{E}\text{-R}\ \cfrac{\supset\text{-L}\ \cfrac{\supset\text{-R}\ \cfrac{\mathcal{E}\text{-L}\ \cfrac{; \ \mathcal{H}p \Rightarrow \ ; \ \mathcal{H}p}{\mathcal{E}p \ ; \ \ \Rightarrow \ ; \ \mathcal{H}p}}{; \ \Rightarrow \ \sim \mathcal{E}p \ ; \ \mathcal{H}p}}{\sim\sim \mathcal{E}p \ ; \ \ \Rightarrow \ ; \ \mathcal{H}p}}{\sim\sim \mathcal{E}p \ ; \ \ \Rightarrow \mathcal{E}p \ ;}}$$

Notice that $(E \cup \neg E)^M = \Box\Diamond E^M \vee \Box\neg E^M)$ is invalid;

- only $(E \cup \neg E)^{\Box\Diamond} = \Box\Diamond(E^M \vee \Box\neg E^M)$ is valid.

5.1. $\lambda\mu$ -calculus.

In (a variant of) Parigot's $\lambda\mu$-calculus, terms are defined by the grammar

$$t, u \quad := \quad x \mid \alpha \mid tu \mid \lambda x.t \mid [\alpha]t \mid \mu\alpha.t$$

and are typed by sequents of the form $\Gamma \vdash t : A \mid \Delta$, with contexts $\Gamma = x_1 : C_1, \ldots, x_m : C_m$ and $\Delta = \alpha_1 : D_1, \ldots, \alpha_n : D_n$, where the x_i are *variables* and the α_j are μ-*variables* and a term t is assigned to the formula A occurring in the *stoup*. In addition to the rules of the simply typed lambda calculus, there are *naming rules*

$$\frac{\Gamma \vdash t : A \mid \alpha : A, \Delta}{\Gamma \vdash [\alpha]t : \bot \mid \alpha : A, \Delta}[\alpha] \qquad \frac{\Gamma \vdash t : \bot \mid \alpha : A, \Delta}{\Gamma \vdash \mu\alpha.t : A \mid \Delta}\mu$$

5.4. Typing $\lambda\mu$ with expectations.

We type the $\lambda\mu$-calculus in a logic of expectations: namely, all formulas which an abstraction $[\alpha]$ or the μ-rule is applied to elementary expressions of the form $\mathcal{E}p$.

We consider a sequent-style Natural Deduction with **AHL-G1** sequents of the forms

$$\Theta\ ;\ \Rightarrow A\ ;\ \Upsilon \quad \text{or} \quad ;\ \mathcal{H}p\ \Rightarrow\ ;\ \Upsilon$$

where A and all formulas in Θ are assertive and Υ consists of expressions of the form $\mathcal{H}p_i$ only.

Then the $[\alpha]$ and μ rules become the *elimination* and *introduction* rules for *expectations*:

$$\mathcal{E}\ \mathrm{E}\frac{\Gamma \vdash t : \mathcal{E}p \mid \alpha : \mathcal{H}p, \Delta \qquad \alpha : \mathcal{H}p \vdash \alpha : \mathcal{H}p}{\Gamma \vdash [\alpha]t : \bot \mid \alpha : \mathcal{H}p, \Delta}[\alpha]$$

$$\mathcal{E}\ \mathrm{I}\frac{\Gamma \vdash t : \bot \mid \alpha : \mathcal{H}p, \Delta}{\Gamma \vdash \mu\alpha.t : \mathcal{E}p\ ;\ \Delta}\mu$$

The *renaming* and $\mu\eta$ operations on $\lambda\mu$ terms,

$$[\alpha]\mu\beta.t \rightsquigarrow t[\alpha/\beta] \qquad \mu\alpha.[\alpha]t \rightsquigarrow t \qquad (1)$$

are interpreted as β and η reductions for $\mathcal{E}$.

6. Lax Logic and "non-standard" bi-intuitionism.

Propositional Lax Logic **PLL** is intuitionistic logic extended with a modal operator $\bigcirc$, interpreted as "*true modulo constraints*".

The sequent calculus rules for the modality **PLL**:

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \bigcirc A} \bigcirc \mathsf{R} \qquad\qquad \frac{\Gamma, A \Rightarrow \bigcirc B}{\Gamma, \bigcirc A \Rightarrow \bigcirc B} \bigcirc \mathsf{L} \qquad (2)$$

Characteristic of **PLL** is the invalidity of standard axioms for $\diamond$:

$$\diamond(A \vee B) \Rightarrow \diamond A \vee \diamond B \quad \text{and} \quad \Rightarrow \neg\diamond\bot$$

with $\bigcirc$ for $\diamond$.

- Fairtlough and Mendler. Propositional Lax Logic, *Information and Computation*, [1997]
- Alechina De Paiva, Mendler and Ritter. Categorical and Kripke Semantics for Constructive S4 Modal Logic, CSL [2001]

6.1. ASH decomposition.

Bi-intuitionistic modalities in **ASH** are
- *hypothetical possibility* modality $\blacklozenge$,
- *assertive necessity* modality $\Box_I$ and
- constants $\blacklozenge\curlywedge$ and $\Box_I\curlywedge$.

These are interpreted in *bimodal* **S4** on frames (W, R, S) with $S \subseteq R$;
- $\mathcal{L}^A$ and "$\Box_I$" are interpreted through R and
- "$\blacklozenge$" through S.

The bi-intuitionistic sequent calculus for *pragmatic* **PLL** has sequents of the forms

$$\Gamma\ ;\ \Rightarrow A\ ; \qquad \Gamma\ ;\ \Rightarrow\ ;\ \blacklozenge A \qquad \Gamma\ ;\ \blacklozenge A\ \Rightarrow\ ;\ \blacklozenge B$$

and in addition to the assertive intuitionistic rules, the following axioms and rules for modalities.

hypotheses

$$;\ \blacklozenge\curlywedge\ \Rightarrow\ ;\ \blacklozenge\curlywedge$$

non-assertability

$$\Gamma, \Box_I\curlywedge;\ \Rightarrow\ B\ ;$$

$$\frac{\ominus\ ;\ \Rightarrow A\ ;}{\ominus\ ;\ \Rightarrow\ ;\ \blacklozenge A}\,\blacklozenge\ \mathrm{R} \qquad \frac{\ominus, A\ ;\ \Rightarrow\ ;\ \blacklozenge B}{\ominus\ ;\ \blacklozenge A;\ \Rightarrow\ ;\ \blacklozenge B}\,\blacklozenge\ \mathrm{L}$$

$$\frac{\ominus\ ;\ \Rightarrow\ ;\ \blacklozenge A}{\ominus\ ;\ \Rightarrow \Box_I\blacklozenge A\ ;}\,\Box_I\ \mathrm{R} \qquad \frac{\ominus\ ;\ \blacklozenge A \Rightarrow\ ;\ \blacklozenge B}{\ominus,\ \Box_I\blacklozenge A\ ;\ \Rightarrow\ ;\ \blacklozenge B}\,\Box_I\ \mathrm{L}$$

Pragmatic interpretation: decompose

$$\bigcirc B = \Box_I \blacklozenge B.$$

Then the *right* and *left* rules for $\bigcirc$ in the sequent calculus for **PLL**

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \bigcirc A}\bigcirc\,\mathsf{R} \qquad\qquad \frac{\Gamma, A \Rightarrow \bigcirc B}{\Gamma, \bigcirc A \Rightarrow \bigcirc B}\bigcirc\,\mathsf{L} \tag{3}$$

are decomposed in our setting as follows:

$$\frac{\dfrac{\Gamma\ ; \Rightarrow A\ ;}{\Gamma\ ; \Rightarrow ;\ \blacklozenge A}\blacklozenge\,\mathsf{R}}{\Gamma\ ; \Rightarrow \Box_I \blacklozenge A;}\Box_I\,\mathsf{R} \qquad \frac{\dfrac{\dfrac{\Gamma, A;\ \Rightarrow ;\ \blacklozenge B}{\Gamma;\ \blacklozenge A \Rightarrow ;\ \blacklozenge B}\blacklozenge\,\mathsf{L}}{\Gamma, \Box_I \blacklozenge A\ ; \Rightarrow ;\ \blacklozenge B}\blacklozenge\,\mathsf{R}}{\Gamma, \Box_I \blacklozenge A\ ; \Rightarrow \Box_I \blacklozenge B\ ;}\Box_I\,\mathsf{L} \tag{4}$$

Allowing intuitionistic disjunction $A \cup B$ in the language, the first characteristic properties of $\bigcirc$ obviously holds in our interpretation:

$$\square_I \blacklozenge (A \cup B) \; ; \not\Rightarrow \; (\square_I \blacklozenge A) \cup (\square_I \blacklozenge B) \; ; \qquad (5)$$

To show that the second characteristic property of **PLL** also holds in our interpretation:

$$; \not\Rightarrow \; \sim \square_I \blacklozenge \curlywedge \; ; \qquad (6)$$

we must look for a sentence $\curlywedge$ which is never justifiably asserted, but that may become a possibly true hypothesis *under some constraint*.

7. Summary.

In this talk we have done the following;

- Argued for Hyland et al [2005] as a categorical proof theory of classical logic.

- Introduced polarised bi-intuitionistic logic as logics of assertions and hypotheses.

- Used the modality of "expectations" to interpret the double negation law and to type the $\lambda\mu$ calculus.

- Suggested that this pragmatic interpretation of classical logic is acceptable in M. Dummett's "justificationism".

- Given an interpretation of Lax Logic in a "non standard" system of bi-intuitionistic logic.

- **Thank you!**