A pragmatic framework for intuitionistic modalities: Classical logic and Lax logic.

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Summary. We reconsider Dalla Pozza and Garola *pragmatic interpretation of intuitionistic logic* [13] where sentences and proofs formalize *assertions* and their justifications and revise it so that the costruction is done within an intuitionistic metatheory. We reconsider also the extension of Dalla Pozza and Garola'a approach to co-intuitionistic logic, seen as a logic of *hypotheses* [5, 9, 4] and the duality between *assertions* and *hypotheses* represented by two negations, the assertive and the hypothetical ones. By adding illocutionary forces of *conjecture*, defined as a *hypothesis that an assertion is justified* and of *expectation*, an *assertion that a hypothesis is justified* we obtain pragmatic counterparts of the modalities of classical **S4**, but also a framework for different interpretations of *intuitionistic* modalities *necessity* and *possibility*. We consider two applications: one is typing Parigot's $\lambda\mu$ calculus in a bi-intuitionistic logic of *expectations*. The second is an interpretation of Fairtlough and Mendler's *Propositional Lax Logic* as an extension of intuitionistic logic with a co-intuitionistic operator of *empirical possibility*.

1 Preface: intuitionistic pragmatics and its extensions.

Conceptually, this work is about a logico-philosophical framework called *Logic for Pragmatics*, initiated by Carlo Dalla Pozza and Claudio Garola [13], to represent intuitionistic Logic (**IL**) as a *logic of assertions and justifiability* and also to specify the relations of intuitionistic logic with Classical Logic (**CL**) as a logic of *propositions and truth* and with classical modal **S4**. We develop here an extension of such a logic for pragmatics to polarized bi-intuitionistic logic as a *logic of assertions and hypotheses* (**AHL**) [5, 4, 9]; here "polarized" means that the intuitionistic and co-intuitionistic parts are kept separate and are related only through two negations. Technically, our work is about applications of bi-intuitionistic logic; one of them uses a particular form of polarization to give an account of *possibility modulo constraints* as in Fairtlough and Mendler's version of *propositional lax logic* (**PLL**) [15].

Comparison of **IL** with classical **S4** is based on Gödel [18], McKinsey and Tarski [23] *modal* **S4** *translation*[1], which allows us to obtain Kripke's *possible world semantics* for **IL** from the possible world semantics for classical **S4** [21]. However our basic understanding of intuitionistic logic is based not on (a variant of) Tarski's semantics, but on the *Brouwer-Heyting-Kolmogorov interpretation* (BHK) which takes the informal notion of proof and of *mathematical method* as fundamental: an elementary formula is interpreted as the type of its informal proofs, an implication $A \supset B$ as the type of methods transforming a proof of A into a proof of B, universal quantification $\forall x.A$ as a method associating to each element a of the intended domain a proof $d(a)$ of $A(a)$, and so on. Perhaps one can argue that mathematical intuitionism, from topological models to topos theory, from type theory to categorical logic, formally develops and refines this basic idea.

Recently intensive efforts of research have been made on the *proof theory of classical logic*, in particular from Michel Parigot's $\lambda\mu$ calculus [26, 27] which provides a computational interpretation of classical logic as in Dag Prawitz's Natural Deduction **NK** [29]: Parigot's calculus has **NK** as typing system for programs with continuations and gave motivations also to Selinger's *control categories* [36][2]. It ought to be clear that such trend of research is based on interpretations of classical logic into intuitionistic logic, started by Glivenko's [16] (1929) and Gödel's [17] (1933) *double negation interpretation*[3]. Our technical contribution here is to show that if classical logic **CL** is translated into polarized bi-intuitionistic logic **AHL**, rather than just **IL**, then it becomes

[1] Consider the map between formulas of the intuitionistic predicate calculus into those of first order **S4** given by

$$\begin{array}{rcl} (p)^M = \Box p, & & (A \supset B)^M = \Box(A^M \supset B^M), \\ (A \cap B)^M = A^M \wedge B^M, & & (A \cup B)^* = A^M \vee B^M, \\ (\forall x.A)^M = \Box \forall x.A^M, & & (\exists x.A)^M = \exists x.A^M \end{array}$$

and the theorem, stated for Hilbert-style axiom systems for first order intuitionistic and modal **S4** logic: *A formula A is provable in intuitionistic logic* **IL** *if and only if* A^M *is provable in* **S4**.

[2] If a proof theory of classical logic has to provide a representation of classical proofs, with a reasonable notion of *identity of proofs*, modular with respect to cut-elimination and suitable for categorical treatment, and moreover if the comparison between the proof theory of **CL** and of **IL** has to yield a functorial translation into cartesian closed categories, then it ought to be clear that such a treatment is not available yet. Indeed for classical logic we would need to consider Gentzen's sequent calculus **LK**, but a suitable notion of identity of proofs for **LK** is still a topic of research (see, e.g., [8]).

[3] Consider the map between formulas of the classical predicate calculus into those of the intuitionistic predicate calculus given by Gödel in the form

$$\begin{array}{rcl} (p)^* = \sim\sim p, & & (A \wedge B)^* = A^* \cap B^*, \\ (A \to B)^* = A^* \supset B^*, & & (A \vee B)^* = \sim(\sim A^* \cap \sim B^*), \\ (\forall x.A)^* = \forall x.A^* & & (\exists x.A)^* = \sim \forall x. \sim A^* \end{array}$$

possible to give a treatment of the typing rules of the $\lambda\mu$ calculus according to the pattern *introduction-elimination.* Composing this translation with the modal **S4** translation, we obtain a variant of the $\Box\Diamond$ modal translation of propositional classical logic *without disjunction* [38], which retains the intuitionistic interpretations of the connectives and proves the double negation rule, but not the law of excluded middle.

Our goal here is not just to give some improvement on mathematical models of known logical systems, but rather to find *conceptually clarifying* ones; i.e, we hope to contribute to the goal of defining *canonical interpretations* of these systems in the framework of a bi-intuitionistic logic for pragmatics[4]. For this reason the style and intentions of our work here are also philosophical.

1.1 Plan of the paper.

In this paper we outline the clarifications and modifications to Dalla Pozza and Garola's Logic for pragmatics [13] that in our view are needed for *an intuitionistically acceptable* presentation of intuitionistic logic within this framework. We shall then proceed in the same way concerning the extension by Bellin, Biasi and Aschieri [5, 9, 4] to co-intuitionistic and bi-intuitionistic logic as logics of *assertions and hypotheses.* Next we give our pragmatic interpretations of classical logic as a *logic of expectations*, where *expecting p to be true* means to assert that the hypothesis that p is true can always be made. Finally, we show that a decomposition of Fairtlough and Mendler's modal operator $\bigcirc A$ of propositional lax logic **PLL** [15] into a *co-intuitionistic hypothetical modality* $\blacklozenge$ and an *intuitionistic assertive modality* allows us to account for "strange" model theoretic and proof theoretic features of their modality *"true modulo constraints"*.

2 A philosophical overview.

Carlo Dalla Pozza and Claudio Garola's *pragmatic interpretation* [13] presents intuitionistic logic as a *logic of assertions*, following suggestions by Michael Dummett; however, *intuitionistic pragmatics* is given in a two-layers formal system where *classical semantics* is also represented; broadly speaking, their goal is to show how classical logic can be reconciled with justificationist theories of meaning. The *elementary formulas* of Dalla Pozza and Garola's formal language $\mathcal{L}^P$ have the form $\vdash\alpha$ where Frege's symbol "$\vdash$" represents an (impersonal) *illocutionary force of assertion* and α is a propositional formula

and the following theorem, which can be stated, e.g., in Hilbert-style axiom systems and for classical and intuitionistic first order logic: *A formula A is provable in* **CL** *if and only if* A^* *is provable in* **IL**.

[4] See Dummett [14] for a convincing explanation of the difference between mathematical and philosophical interpretations.

interpreted in classical truth-functional semantics. Thus in accordance with Frege propositions are classically true or false; illocutionary acts of assertions can only be *justified* or *unjustified*. If we restrict ourselves to elementary expressions of the form $\vdash p$, where p is atomic (i.e., where the internal structure of p is not taken into account in the justification of $\vdash p$) and we apply the Brouwer-Heyting-Kolmogorov (BHK) interpretation to the expressions built up with intuitionistic connectives from such elementary expressions, then we have indeed an interpretation of *intuitionistic logic* in the "pragmatic layer" of this logic[5]. Here Gödel, McKinsey and Tarski's translation into classical **S4** is regarded as a *projection* of the intuitionistic pragmatic layer into the classical semantic layer, *extended with* **S4** *modalities*.

However, if the logic for pragmatics has to serve the purpose of characterizing the conflictual difference between classical and intuitionistic logic, then we must make sure that the logical representation of intuitionistic logic within it does satisfy the requirements of an intuitionistic philosophy. Unfortunately this is not clear: the metatheory of Dalla Pozza and Garolas's logic for pragmatics makes essential use of classical logic. Thus the doubt remains that the logic for pragmatics may be just an attempt to represents those features of constructivism that can be arranged within classical logic, unless we show that the basic notions of Dalla Pozza and Garola's logic can be explained also in an intuitionistic metatheory and are thus "philosophically unbiased". It turns out that for this purpose only a few changes are needed to the framework in the papers by Dalla Pozza and Garola [13].

2.1 Expressive and descriptive uses.

An important distinction in Dalla Pozza and Garola's theory of pragmatics is between *expressive* and *descriptive* uses of illocutionary operators: we assert, command, promise, etc., and we report assertions, commands, promises. It may be useful to have different symbols "$\vdash$" and "A" for *expressing* and for *describing* the illocutionary force in acts of assertion.

In an elementary expression $\vdash p$ of Dalla Pozza and Garola's language $\mathcal{L}^P$ an *impersonal illocutionary force* is expressed: no subject uttering the assertion $\vdash p$ is indicated, but an intention is expressed that the hearer or reader may believe that the proposition p is true since *conclusive evidence* for it is available. Such an intention is not expressed, but only described when descriptive illocutionary operators are used.

Considering molecular expressions, e.g., the assertive $A \supset B$, the following questions arise:

[5] In this framework the BHK interpretation is as follows: $\vdash p$ is the type of informal justifications of the truth of p; $A \supset B$ is the type of methods transforming a justification of the assertion A into a justification of assertion B; $A \cap B$ is the type of pairs of justifications, the first of A and the second of B; $A_0 \cup A_1$ is the type of pairs $\langle p, i \rangle$ where $i = 0, 1$ and p is a justification of A_i; and so on.

(a) *Do molecular expressions of the language $\mathcal{L}^P$ have illocutionary force and does the distinction between expressive and descriptive uses apply to them?*
(b) *Do the proper components of a molecular expression describe their illocutionary force or do they also express it?.*

For us the only plausible answer to (b) is that the proper components of a molecular expressions are used descriptively. If we say *"whenever it rains, the basement is flooded"*, then we are only evoking circumstances justifying the assertion *"it rains"* in order to conclude that if those obtain, then also circumstances will occur that justify the assertion *"the basement is flooded"*. *Descriptive use* seems the right way of characterizing such uses of assertions.

However, there is a different sense of the distinction between "expressive" and "descriptive" that does not apply here: when John says *"it rains"* he *expresses* his belief, while when we say *"John believes that it rains"*, we *describe* his belief. Here the justification conditions for the assertion *"it rains"* are irrelevant to the justification conditions of *"John believes that it rains"*. On the contrary, the justification conditions of the assertion *"it rains"* are precisely the same both if it appears as an elementary expression or as an element of a molecular expression *"if it rains, then the basement is flooded"*. It is this second sense of the distinction that interests us here.

Similarly we answer to (a). An implication $A \supset B$ can be presented, albeit impersonally, with the intention of generating the belief that conclusive evidence of its truth is at hand, namely, that a method to transform a justification of A into a justification of B can be exhibited. Alternatively, such an intention may only be described and circumstances justifying the utterance $A \supset B$ may only be evoked, and no such method is presented ready for use. Thus we can say that in the first case the linguistic expression $A \supset B$ carries the illocutionary force of implication while in the second case such a force is only described. It is convenient to have distinct notations for expressive and descriptive uses also in the case of molecular expressions: we write $\vdash (A \supset B)$ and $\mathtt{A}\,(A \supset B)$ and similarly $\mathcal{H}\,(C \smallsetminus D)$ and $\mathtt{H}\,(C \smallsetminus D)$.

Our answers to (a) and (b) disagree with Dalla Pozza and Garola, who hold that illocutionary forces apply only to classical propositions, but are in agreement with Frege's principle that (expressive) signs of force can occur only as the outermost symbol in a logical expression.

2.2 Justifications conditions and semantic values.

Every expression of Dalla Pozza and Garola's formal language $\mathcal{L}^P$ of assertions has *justification conditions* in accordance with the BHK interpretation. An expression meeting such conditions is *justified*; otherwise it is *unjustified*. Moreover in any *elementary expression* $\vdash \alpha$ the *radical part* α is a proposition (in the sense of Frege); in any given semantic context α has a truth value *true* or *false* in the sense of classical logic, and such a semantic value contributes to determine whether or not the *justification conditions* for $\vdash \alpha$ are met. On the other hand *molecular expressions* A have a semantic value only through a semantic projection A^M in classical modal logic **S4**. However, such a semantic value is not a property of the expression A but of its **S4** translation A^M; in the case of an elementary expression $\mathsf{A}p$ both the radical p and its modal **S4** translation $\Box p$ have semantic values, but in different classical semantics.

In Dalla Pozza and Garola's representation of intuitionistic logic elementary expressions $\vdash \alpha$ must have an *atomic radical part* α, rightly so, since otherwise there would be a justification of an assertion $\vdash (p \vee \neg p)$ based on the fact that in this case α is a classical tautology. Does this restriction suffice to guarantee that intuitionistic logic is adequately represented in this logic for pragmatics? Actually, in this view it may very well happen that a proposition p is true without there being any evidence of this fact; in fact, there may be *pragmatically undecidable sentences* whose *assertion* is never justified and nevertheless do have a truth value in a semantic context.

In Michael Dummett's *justificationist* philosophy [14], *pragmatically undecidable* statements have no definite truth value; in fact, if we cannot specify conditions in which we would be able to obtain evidence of their truth, they must be considered meaningless. However Dummett does not propose to eliminate the property of statements of *being true* in favour of *being justifiably assertable*: the notion of truth carries a connotation of *objectivity* that must be retained. Thus Dummett advocates a notion of *intuitionistic truth* which may be applied only to assertive statements having definite justification conditions.

Now one may think that in Dalla Pozza and Garola's logic Dummett's objections against pragmatically undecidable statements can be met by retaining all such statements in the language $\mathcal{L}^P$ as equivalent tokens of type ⅄, a sentence which can never be justified. There are several kinds of such undecidable statements; we give three examples to be discussed below.

Example 1. (**a**) A statement by Jules-Henri Poincaré ([28], Book 2, chapter 1) is often quoted, that *last night all measures in the universe have increased one thousand times* - including the magnitudes we take as standards of measure. There can be no conclusive justification for asserting this statement nor for asserting its negation.

(**b**) No conclusive evidence can be given, thus no justification can be given for asserting the statement *"this iron bar has length of exactly one meter"* (in a given position on the earth and given temperature), where *"exactly one"* rules out any experimental error.

(**c**) No conclusive evidence can be given for the statement *the result of tossing this (fair) coin will be heads.*

We can sum up our discussion so far as follows. *Sentences in the "descriptive mode" are types of their justifications, and may be regarded as* intuitionistic propositions*; they have intuitionistic justification conditions and no semantic* *value in a classical semantics. The modal translation A^M of an intuitionistic formula A may have semantic value but has no justification conditions.*

2.3 Pragmatic interpretation of co-intuitionistic logic

Dalla Pozza's approach has also been extended to *co-intuitionistic* and *bi-intuitionistic* logic by Bellin, Biasi and Aschieri [5, 9], revised in [4]. *Co-intuitionistic logic* is interpreted as a logic of *hypothetical reasoning*, which is thought of as *dual* of assertive reasoning; such a duality is to be made precise both mathematically and conceptually.

Bi-Heyting algebras.

The mathematical theory of co-intuitionistic logic, initiated by Cecylia Rauszer [34, 35] and promoted among others by William Lawvere [22], begins by dualizing the language and the known semantics of intuitionistic logic. A Heyting algebra is a structure $\mathcal{L} = (L, \wedge, \vee, 0, 1, \rightarrow)$ where $(L, \wedge, \vee, 0, 1)$ is a lattice [6] and $\rightarrow$ is a binary operation on L satisfying the adjunction (1). A co-Heyting algebra is a lattice $\mathcal{L}$ such that its dual $\mathcal{L}^{op}$ obtained by reversing the order is a Heyting algebra. Thus a co-Heyting algebra has a binary operation $\smallsetminus$ satisfying (1). A bi-Heyting algebra is a lattice with the structure of a Heyting and a co-Heyting algebra.

$$\begin{array}{cc} \textit{Heyting adjunction} & \textit{co-Heyting adjunction} \\ \dfrac{a \wedge b \leq c}{b \leq a \rightarrow c} & \dfrac{c \leq b \vee a}{c \smallsetminus a \leq b} \end{array} \tag{1}$$

We are concerned with *polarized* structures, where the dual algebras are distinct and related by two negations representing the duality. We work mostly with the closed semilattices $(A, \wedge, 1, \rightarrow)$ and $(C, \vee, 0, \smallsetminus)$[7].

In logical terms we define propositional intuitionistic logic **IL** on connectives $\cap$, $\supset$, $\cup$, $\sim$ and co-intuitionistic logics **co-IL** on connectives $\curlyvee$, $\smallsetminus$, $\curlywedge$, $\frown$ (cfr. Definition 1). Also let $\Gamma = A_1, \ldots, A_n$ and $\Delta = C_1, \ldots, C_n$; in **IL** we have the consequence relation $\Gamma \Rightarrow A$ (*the conjunction* $A_1 \cap \ldots \cap A_n$ *entails* A)

[6] Remember that a lattice is a partially ordered set A in which every finite subset has both join $\vee$ and meet $\wedge$. Equivalently, a lattice is a structure $(A, \wedge, \vee, 0, 1)$ such that both $(A, \vee, 0)$ and $(A, \wedge, 1)$ are semilattices (commutative monoid where every element is an idempotent) and the induced orders are opposite: $a \leq b$ iff $b = a \vee b$ iff $a = b \wedge a$.

[7] In order to generalize these notions to a categorical semantics for bi-intuitionistic logic, the result by Tristan Crolard [11] that co-exponents are trivial in the category **Sets** is a main motivation for polarization. It also implicitly suggests that in the proof-theory of co-intuitionistic logic disjunction ought to be defined *multiplicatively*, rather than *additively*, as indeed it is in **Sets**. We shall follow this principle in our proof-theoretic presentation here, unlike in [4].

and in **co-IL** the consequence relation $C \Rightarrow \Delta$ (*C entails the disjunction* $C_1 \curlyvee \ldots \curlyvee C_n$).

Proposition 1. *There is a duality* $(\ )^\perp$ *between* **IL** *and* **co-IL** *with the following property. Let* $C = A^\perp$ *and* $\Delta = C_1^\perp, \ldots, C_n^\perp = \Gamma^\perp$. *For any of the algebraic, topological or Kripke semantics,* $\Gamma \Rightarrow A$ *is valid if and only if* $C \Rightarrow \Delta$ *is valid. In particular* A *is* valid [contradictory] *iff* C *is* contradictory [valid].

See, e.g., [11]. Significant corollaries are the following:

1. Dual to assertive intuitionistic *contradiction* $A \cap \sim A$ is hypothetical excluded middle $C \curlyvee \frown C$, which is *valid in co-intuitionistic logic*;
2. Dual to assertive intuitionistic *excluded middle* $A \cup \sim A$ is hypothetical contradiction $C \curlywedge \frown C$, which is *consistent in co-intuitionistic logic.*

Co-intuitionistic pragmatic language.

What could an *intended interpretation* be in common sense reasoning of co-intuitionistic logic, based on a logic of illocutionary forces? We try to develop a *logic of hypotheses.* Here *hypothetical* elementary formulas have the form $\mathcal{H}\ p$, where $\mathcal{H}$ expresses the illocutionary force of hypothesis, with its descriptive counterpart $\mathtt{H}\ p$. From them we build molecular expressions with co-intuitionistic connectives, in particular *subtraction* $C \smallsetminus D$, the dual of implication, whose intended meaning is the hypothesis that we may retain the hypothesis C while dismissing the hypothesis D. In general we may have many variants of hypothetical co-intuitionistic connectives in our logic for pragmatics and also *mixed connectives* yielding assertive or hypothetical expressions out of assertive and hypothetical expressions [5]. However, interesting results can be obtained already with the pragmatic interpretation of a small fragment of *polarized bi-intuitionistic* logic, whose language $\mathcal{L}^{AH}$ has an intuitionistic part built from assertive elementary formulas through the assertive connective *conjuncton* ($\cap$) and *implication* ($\supset$) assertive logic, and its co-intuitionistic dual, built from hypothetical elementary formulas with hypothetical connectives *disjunction* ($\curlyvee$) and *subtraction* ($\smallsetminus$); the two fragments are then related through intuitionistic *assertive negation* ($\sim$) and co-intuitionistic hypothetical *doubt* ($\frown$).

The distinction about descriptive and expressive uses of illocutionary forces and the relevant discussion apply here to, so we write "$\mathcal{H}$" and "$\mathtt{H}$" for the expressive and descriptive uses of the operator of hypothesis. Here is our official definition of the extended pragmatic language $\mathcal{L}^{AH}$ of assertions and hypotheses.

Definition 1. *(assertions, hypotheses)*

Descriptive types:

- *assertive:*

$A, B \; := \text{A}\, p \mid \text{A}\,(A \supset B) \mid \text{A}\,(A \cap B) \mid \text{A}\,(A \cup B) \mid \text{A}\,(\sim A) \mid \text{A}\,(\sim C)$

- *hypothetical:*

$C, D \; := \text{H}\, p \mid \text{H}\,(C \smallsetminus D) \mid \text{H}\,(C \curlyvee D) \mid \text{H}\,(C \curlywedge D) \mid \text{H}\,(\frown C) \mid \text{H}\,(\frown A)$

Expressive formulas:

- *elementary:* $\vdash p, \;\; \mathcal{H}\, p.$
- *molecular:* $\vdash A, \;\; \mathcal{H}\, C$ *where* $\text{A}\, A$ *and* $\text{H}\, C$ *are descriptive types.*

To simplify notations, we omit signs of illocutionary force except in elementary expressions.

2.4 Justification conditions and semantic values for hypotheses.

How are justification conditions and semantic values assigned to *hypothetical* expressions? Solving this task would provide interesting insight on justification conditions in general. We want to extend the BHK interpretation to a co-intuitionistic logic of hypotheses. In the case of intuitionistic assertive A, a justification for A is *conclusive evidence* for A; here what counts as conclusive evidence is relative to the standards of scientific disciplines and styles of discourse, in particular, in mathematics we need a proof if A. In the co-intuitionistic case, we may use legal terminology and say that a justification for $\mathcal{H}\, p$ is a *"scintilla of evidence"* that p may be true, where again "scintilla of evidence" is defined relatively to a scientific or conversational context. Explaining the intended meaning of this notion is a main task of a philosophical account of a logic of hypothetical reasoning.

There are at least two approaches: one is to say that if it is compatible with the context of our knowledge to make the hypothesis C, then the simple awareness of this *generic compatibility* must count as evidence for C. Another approach is to say that, on the contrary, some circumstances giving *positive confirmation* for the truth of C are required. A choice between these two approaches may have major consequences. Let us consider in particular the examples in Section 2.2.

(**a**) Poincaré's statement is certainly compatible with our present knowledge, although it is beyond our means to put it to test; but exactly the same can be said of its negation. Hence the only "scintilla of evidence " for it and for its negation is a mere realization of compatibility. In general any proposition p a classical logician may consider as meaningful and capable of a semantic value can also be tested for compatibility with our present knowledge; if the result of this test is positive, then it constitutes evidence for $\mathcal{H}\, p$. On the other hand, there is no doubt that for Dummett the statement ought to be regarded as meaningless. Also for Poincaré the statement makes sense only assuming a notion of *absolute space* which in his view is meaningless ([28], Book 2, chapter 1). Thus in the approach *"hypothetical evidence by compatibility"* the dispute between an intuitionistic and a classical theory of meaning becomes one about the borderline between what can be asserted and what can be merely object of a hypothesis. On the other hand, in the approach *"hypothetical evidence as confirmation"* statements beyond any possible confirmation ought to be removed from serious consideration either as assertions or as hypotheses in scientific discourse, although a classical philosopher may not want to regard them as meaningless. We do not have a strong argument against any one of the two approaches, but our feeling is that "scintilla of evidence" for a hypothesis cannot be just an abstract test of compatibility[8]. In the rest of this paper we assume that *every hypothetical statement C under consideration belongs to a "context of justification" in which there is at least a "scintilla of evidence" for C or against it, i.e, in favour of $\frown C$.*

(**b**) A different insight comes from the example about experimental error. One can say that in principle an iron bar might have *any length*, under a suitable general notion of possibility. However, the hypothesis that the bar is one meter long *modulo experimental error* results from a sophisticated procedure, where the degree of confirmation is increased by repeated successful measurements. Indeed in the end we *assert* that the iron bar is one meter long modulo experimental error. This notion seems related to *truth under constraints* (Fairtlough and Mendler [15], see below).

(**c**) The example from probability also presents a contrast between the generic possibility that tossing a coin may yield tails and the more refined notion of possibility that *it will be heads with a 50% probability.* This requires a probabilistic setup, and constitutes evidence for the *hypothesis that it will be heads* with a 50 % probability, a hypothesis we may *assert modulo probabilistic contraints.* In both cases we have a generic notion of possible situations, which refutes an assertion as unwarranted, and a more restrictive notion of possibility and truth under constraints.

2.5 Hypothetical contexts and connectives.

Giving an intuitionistically acceptable explanation of hypothetical co-intuitionistic logic in Dummett and Prawitz's *meaning as use* approach [14, 32], requires explaning not only elementary formulas, as done in section (2.4), but also the co-intuitionistic consequence relation

$$C \Rightarrow D_1, \ldots, D_n \tag{2}$$

and then giving an account of how the justification condition for molecular expressions according to the meaning of their main connective. Here the meaning of a connective may be explained according to (a) an introduction rule in Natural Deduction [a right rule in the sequent calculus] or (b) according to an elimination rule [a left rule] [32]. In case (a) one shows how the *justification conditions* of a molecular formula are determined from the justification

[8] We follow here a view expressed by Carlo Dalla Pozza in private conversation.

conditions of its component formulas; in case (b) one shows how the *immediate consequences* of a molecular formula are determined from the immediate consequences of its component formulas. Then one justifies an introduction-elimination pair [a right-left pair] by the *inversion principle* [29], i.e., by showing that *nothing is derivable before a normalization step* [*a symmetric cut elimination step*] *that is not derivable afterwards.*

To explain the consequence relation in (2), three approaches are available. One is to explain how given refutations for $D_1, \ldots, D_n$ one constructs a refutation of C; this works, since the *refutability* conditions for C are the same as the *provability* conditions for $C^\perp$ and we do have a *meaning-as-use* explanation of intuitionistic logic. However, although this may allow us to formalize the medieval practice of *refutations* [2], it does not give any positive information about making hypotheses. The second, which we sketch here, explains how given evidence x for the hypothesis C we obtain *"parcels of hypothetical evidence"* $t_1, \ldots, t_n$ for $D_1, \ldots, D_n$. The third, which is related to David Nelson's *constructible falsity* [25] and also to game semantic notions, provides both provability and refutability conditions for each formula; we do not consider this approach here.

We give an informal account of the second approach, which can be formalized as a *"calculus of coroutines"* as done by G. Bellin [4], adapting an idea by Crolard [12]. Thus we have a *computational context of justification*

$$x : C \Rightarrow \mathcal{S}_x : \Delta \tag{3}$$

which is *distributed* since $\mathcal{S}_x = t_1, \ldots, t_n$ consists of separate "parcels of evidence" for (the hypothetical disjunction of) the hypotheses $D_1, \ldots, D_n = \Delta$. Essential to the *co-intuitionistic* nature of the calculus is that C is the only open assumption in the context, thus that x is the one and only *free variable* occurring in $\mathcal{S}_x$. In our distributed system the variable x also points to a *location*; moving the variable to a new location means moving the entire computational context to it. Formally, we have an operation transforming a free variable y into a term $\mathtt{y}(t)$, whose informal meaning is approximately y `away from` t, with the effect that the computational context of y is now moved to a location away from that of t and thus cannot interact with terms in the computational context of t.

We need to give a *meaning as use* interpretation of co-intuitionistic connectives. We focus on *subtraction*, as the treatment of the other connectives is more familiar; thus we need an interpretation according to the sequent calculus rules

$$\frac{H \Rightarrow \Gamma, C \qquad D \Rightarrow \Delta}{H \Rightarrow \Gamma, C \smallsetminus D, \Delta} \smallsetminus \mathrm{R} \qquad\qquad \frac{C \Rightarrow D, \Upsilon}{C \smallsetminus D \Rightarrow \Upsilon} \smallsetminus \mathrm{L}$$

and with reduction rule

$$\dfrac{\dfrac{H \Rightarrow \Gamma, C \qquad D \Rightarrow \Delta}{H \Rightarrow \Gamma, C \setminus D, \Delta}\setminus \mathrm{R} \qquad \dfrac{C \Rightarrow D, \Upsilon}{C \setminus D \Rightarrow \Upsilon}\setminus \mathrm{L}}{H \Rightarrow \Gamma, \Delta, \Upsilon}\,cut$$

reduces to

$$\dfrac{\dfrac{H \Rightarrow \Gamma, C \qquad C \Rightarrow D, \Upsilon}{H \Rightarrow \Gamma, D, \Upsilon}\,cut_1 \qquad D \Rightarrow \Delta}{H \Rightarrow \Gamma, \Delta, \Upsilon}\,cut_2$$

Suppose we have a context $z : C \Rightarrow \ell : D, \mathcal{S}_z : \Upsilon$. Then we define the *immediate consequences* of $C \setminus D$ as follows: to have a justification v of $C \setminus D$ means to be able to extract from it a justification for C and also to *set aside* the justification ℓ of D from the justification z of C (we use a term $\texttt{postpone}(z \mapsto \ell, v)$ for this purpose); *"setting aside"* ℓ is the *operational meaning* of the inconsistency between $C \setminus D$ and the evidence ℓ: for this reason the `postopone` term is assignd to $\bullet$ which is not a type and thus cannot occur as a subformula of other formulas. Formally, we have a $\setminus$ *left* rule of the following form:

$$\dfrac{z : C \Rightarrow \ell : D,\ \mathcal{S}_z : \Upsilon}{v : C \setminus D \Rightarrow \texttt{postpone}(z \mapsto \ell, v) : \bullet,\ \mathcal{S}_{\mathtt{z}(v)} : \Upsilon}\setminus \mathrm{L} \tag{4}$$

In the dual introduction rule we have an operation `make-coroutine` connecting two computational contexts: one depends on $x : H$ and yields parcels of justifications $\mathcal{S}_x \cup \{t\}$ for Γ, C, the other depends on $y : D$ and yields justifications $\mathcal{S}_y : \Delta$. The point is that we can merge the two contexts into one, still depending on $x : H$, *provided that* $\mathcal{S}_y$ *is moved to a location "away from"* t, i.e., away from x. Formally, we have an assignment to the $C \setminus D$ *right* rule:

$$\dfrac{x : H \Rightarrow \mathcal{S}_x : \Gamma,\ t : C \qquad y : D \Rightarrow \mathcal{S}_y : \Delta}{x : H \Rightarrow \mathcal{S}_x : \Gamma,\ \texttt{make-coroutine}(t, \mathtt{y}) : C \setminus D,\ \mathcal{S}_{\mathtt{y}(t)} : \Upsilon}\setminus \mathrm{R} \tag{5}$$

Notice that we have specified the meaning of `make-coroutine` in terms of its action on the *consequences* of H, as required by a justification from an *elimination rule* [32].

Now consider a *cut* where the left and right cut-formula $C \setminus D$ has been introduced by the right and left rules, as in (5) and (4), respectively. Then a justification for $C \setminus D$ of the form `make-coroutine` is substituted for y in $\texttt{postpone}(x \mapsto \ell, y)$. The upshot is that when eliminating the *cut*, the *"postponed"* computation ℓ can be used to *"fill the jump"* from C to D in (5).

$$\dfrac{\dfrac{x : H \Rightarrow \mathcal{S}_x : \Gamma,\ t : C \qquad z : C \Rightarrow \ell : D,\ \mathcal{S}_z : \Delta}{x : H \Rightarrow \mathcal{S}_x : \Gamma, \ell' : D,\ \mathcal{S}_t : \Delta} \qquad y : D \Rightarrow \mathcal{S}_y : \Upsilon}{x : H \Rightarrow \mathcal{S}_x : \Gamma, \mathcal{S}_t : \Delta, \mathcal{S}_{\ell'} : \Upsilon} \tag{6}$$

where $\mathcal{S}_t =_{df} \mathcal{S}_x\{x := t\}$, $\ell' =_{df} \ell\{x := t\}$ and $\mathcal{S}_{\ell'}$ is a list of contexts resulting from repeatedly substituting each term $j \in \ell'$ for z in $\mathcal{S}_z$.

The *meaning as use* interpretation, in the style of Dummett and Prawitz, of the connective of *subtraction* is given precisely by the $\setminus$-*left* rule and by the `postpone` operation, namely, by the act of *removing from the current* *space of justification the computations that derive a justification for D from a justification for C*: indeed we make it impossible for D to be an immediate consequence of $C \smallsetminus D$, as we regard D to be incompatible with it. Also the connective *subtraction* is justified in a *meaning as use* interpretation by the evident *harmony* between left and right rules for it.

2.6 AHL and strictness: Kripke semantics and sequent calculi

In formalizing polarized bi-intuitionistic logic it is convenient to project the pragmatic language of assertions and hypotheses $\mathcal{L}^{AH}$ over *bimodal* **S4**, letting

$$\begin{array}{rlrl} (\vdash p)^M &= \Box p & (\mathcal{H} p)^M &= \Diamond p \\ (A \supset B)^M &= \Box(A^M \to B^M) & (C \smallsetminus D)^N &= \Diamond (C^M \wedge \neg D^M) \\ (A \cap B)^M &= A^M \wedge B^M & (C \curlyvee D)^M &= C^M \vee D^M \\ (A \cup B)^M &= A^M \vee B^M & (C \curlywedge D)^M &= C^M \wedge D^M \\ (\sim C)^M &= \Box \neg C^M & (\frown A)^M &= \Diamond \neg A^M \end{array} \tag{7}$$

Bimodal **S4** frames have the form (W, R, S) where both R and S are preorders over the set W. Kripke models for *bimodal* **S4** have the form $\mathcal{M} = (W, R, S, \mathcal{V})$, with $\mathcal{V}$ a valuation of the atoms over W and the forcing conditions are

1. $w \Vdash \Box X$ if and only if $\forall w'$, wRw' implies $w' \Vdash X$;
2. $w \Vdash \Diamond X$ if and only if $\exists w'$ such that wSw' and $w' \Vdash X$.

Then one can study different bi-intuitionistic systems, depending on the relations between R and S.

Definition 2. (Strictness) *Let $\mathcal{L}^{AH}$ be the pragmatic language for polarized bi-intuitionistic logic.*
(i) Assertive Hypothetical Logic **AHL** *is the set of all $\mathcal{L}^{AH}$ formulas which are valid under semantic projection over all bimodal frames with $R = S$;*
(iii) Assertive and Strictly Hypothetical Logic **ASHL** *is the set of all $\mathcal{L}^{AH}$ formulas which are valid over all bimodal frames with $S \subseteq R$;*
(ii) Strict Assertive and Hypothetical Logic **SAHL** *is the set of all $\mathcal{L}^{AH}$ formulas which are valid over all bimodal frames with $R \neq S$.*

Here *assertive and strictly hypothetical logic* **ASHL** formalizes logics where the justification conditions for hypothetical statements are *stronger* than those for assertive statement: an example is our treatment of Fairlough and Mendler's *propositional lax logic* below.

Lemma 1. *Let $\mathcal{F} = (W, R, S)$ be a multimodal frame, where R and S are preorders.*

(i) *The following are valid in $\mathcal{F}$*

$$\Box \boxminus \Box \alpha \to \Box \alpha \qquad \textit{and} \qquad \boxminus \Box \boxminus \alpha \to \boxminus \alpha$$

$(ii)(a)$ *The following are equivalent:*

1.a: $S \subseteq R$;
2.a: the following scheme is valid in $\mathcal{F}$*:*
(Ax.a) $\Box\alpha \to \Box \boxminus \Box\alpha$;

$(ii)(b)$ *The following are equivalent*

1.b: $R \subseteq S$;
2.b: the following scheme is valid in $\mathcal{F}$*:*
(Ax.b) $\boxminus\ \alpha \to \boxminus\ \Box \boxminus\ \alpha$

Proof of $(ii)(a)$. (1.a $\Rightarrow$ 2.a) is obvious. (2.a $\Rightarrow$ 1.a): If S is not a subset of R, then given wSv and not wRv define a model on $\mathcal{F}$ where $w' \Vdash p$ for all w' such that wRw' but $v \not\Vdash p$; thus $\Box p \to \Box \boxminus \Box p$ is false at w.

A versatile tool for studying variants of bi-intuitionistic logic is given by sequent calculi with sequents of the form

$$\Theta\ ;\ \epsilon \Rightarrow \epsilon'\ ;\ \Upsilon \tag{8}$$

where Θ and ϵ' consist of *assertive formulas*, ϵ and Υ of *hypothetical* ones, and at most one of ϵ and ϵ' is non-empty and contains one formula. A sequent calculus **AH-G3** for our basic system **AHL** is given in Section 5, as in [5, 4][9].

To formalize the logics with strictness conditions we restrict the rules of **AH-G3** as indicated below; the restrictions apply only to the *"bi-intuitionistically sensitive"* rules $\supset$*-right*, $\sim$*-right*, $\smallsetminus$*-left* and $\frown$*-left*. Thus we obtain sequent calculi **SAH-G3** for **SAHL** and **ASH-G3** for **ASHL**.

$$\supset\text{-R}\ \frac{\Theta, A_1\ ;\ \Rightarrow\ A_2\ ;\ \Upsilon^{**}}{\Theta\ ;\ \Rightarrow\ A_1 \supset A_2\ ;\ \Upsilon}\ (\P\P) \qquad \smallsetminus\text{-L}\ \frac{\Theta^*\ ;\ C_1\ \Rightarrow\ ; C_2, \Upsilon}{\Theta\ ;\ C_1 \smallsetminus C_2 \Rightarrow\ ;\ \Upsilon}\ (\P)$$

$$\sim\text{-R}\ \frac{\Theta\ ;\ C\ \Rightarrow\ ;\ \Upsilon^{**}}{\Theta\ ;\ \Rightarrow \sim C\ ;\ \Upsilon}\ (\P\P) \qquad \frown\text{-L}\ \frac{\Theta^*\ ;\ \Rightarrow\ A\ ;\ \Upsilon}{\Theta\ ;\ \frown A\ \Rightarrow\ ;\ \Upsilon}\ (\P)$$

Υ^{**} not allowed in **SAH-G3**, **ASH-G3** | Θ^* not allowed in **SAH-G3**

Using methods of [5] we can prove the following result.

Theorem 1. *The sequent calculi* **AH-G3** [**SAH-G3**, **ASH-G3**] without the cut rules *are sound and complete with respect to Kripke semantics over bimodal preordered frames, induced by the semantic projection of* **AHL** [**SAHL**, **ASHL**, respectively] *in bimodal* **S4**.

To see why the restrictions are needed and suffice, notice that by the valid scheme $(Ax.a)$ of Lemma 1.$(ii)(a)$

[9] In the classification of Troelstra and Schwichtenberg [39], **G3** sequent calculi have implicit structural rules *exchange, weakening* and *contraction* and are suitable for proving completeness results with a tableaux-like methods.

$$A \Rightarrow \sim\frown A \quad \text{is valid in the semantics of } \mathbf{AHL} \text{ and of } \mathbf{ASHL} \tag{9}$$

i.e., $\Box A^M \to \Box\neg\Diamond\neg\Box A^M$ is valid in a bimodal frame (W, R, S) iff $S \subseteq R$. Moreover notice that $A \Rightarrow \sim\frown A$ is derivable in **AH-G3** and in **ASH-G3** since the restriction (¶) does not apply. Finally, the unrestricted rule $\frown$-*left* of **AH-G3** and **ASH-G3** becomes derivable in **SAH-G3** using *cut* with the scheme (1) taken as axiom:

$$\text{cut}\ \dfrac{\overset{(1)}{B\,;\ \Rightarrow\ \sim\frown B\,;} \qquad \sim\text{-L}\ \dfrac{\frown\text{-L}\ \dfrac{\frown\text{-R}\ \dfrac{B\,;\ \Rightarrow\ A\,;\ \Upsilon}{;\ \Rightarrow\ A\,;\ \Upsilon, \frown B}}{;\ \frown A\ \Rightarrow\ ;\ \Upsilon, \frown B}\ (\P)}{\sim\frown B\,;\ \frown A\ \Rightarrow\ ;\ \Upsilon}}{B\,;\ \frown A\ \Rightarrow\ ;\ \Upsilon}$$

Thus absence of the restriction (¶) on sequents is equivalent to the requirement $S \subseteq R$ on bimodal frames in the interpretation of **AHL** and **ASHL**. Similarly, using the fact that

$$\frown\sim C \Rightarrow C \quad \text{is valid in the semantics of } \mathbf{AHL} \tag{10}$$

and is derivable in **AH-G3** since the restriction (¶¶) does not apply there, we show that absence of the restriction (¶¶) is equivalent to the requirement $R \subseteq S$ in the interpretation of **AHL** .

3 Intuitionistic modalities.

Consider *polarized bi-intuitionistic* logic (**AHL**) and its semantic projection in **S4**. Write X for an arbitrary (assertive or hypothetical) formula in $\mathcal{L}^{AH}$. Here pairs of dual negations yield *interior* and *closure* operators in the topology associated with the classical **S4** modalities:

$$(\sim\frown X)^M = \Box\neg\Diamond\neg X^M = \Box X^M \qquad (\frown\sim X)^M = \Diamond\neg\Box\neg X^M = \Diamond X^M \tag{11}$$

Thus if $X = A$ (*assertive*) then $(\sim\frown A)^M \equiv A^M$ and if $X = C$ (*hypothetical*) then $(\frown\sim C)^M \equiv C^M$. But

$$(\sim\frown C)^M = \Box C^M \qquad (\frown\sim A)^M = \Diamond A^M. \tag{12}$$

Hence in **AHL** pairs of dual negations behave like *polarity changing modalities*. In general we have modalities

$$\begin{array}{lll} \textit{descriptive modalities} & \mathtt{A}C =_{df} {\scriptstyle\mathtt{A}} \sim\frown C; & \mathtt{H}A =_{df} {\scriptstyle\mathtt{H}} \frown\sim A \\ \textit{expressive modalities} & \mathcal{A}C =_{df} {\scriptstyle\vdash} \sim\frown C; & \mathcal{H}A =_{df} {\scriptstyle\mathcal{H}} \frown\sim A \end{array} \tag{13}$$

Proposition 2. (substitution) *Let X be an arbitrary formula and let A be assertive formulas and let C be hypothetical. If X is valid or contradictory in the Kripke semantics for* **AHL** *then so are* $X\big[A/\,{\scriptstyle\mathtt{A}}\,p\big]$ *and* $X\big[\mathtt{A}C/\,{\scriptstyle\mathtt{A}}\,p\big]$, *where* $X\left[A/\mathtt{A}p\right]$ *and* $X\left[\mathtt{A}C/\mathtt{A}p\right]$ *are the result of substituting* A *and* $\mathtt{A}C$ *for all occurrences of the elementary formula* $\mathtt{A}p$ *in* X*. Similarly* $X\left[C/\mathtt{H}p\right]$ *and* $X\left[\mathtt{H}A/\mathtt{H}p\right]$ *are valid or contradictory if* X *is.*

An important use of intuitionistic modalities is the representation of two additional illocutionary operators within **AHL**:

- a descriptive *conjecture* $\mathtt{C}p =_{df} \mathtt{H}\,\mathtt{A}\,p$, with its expressive counterpart $\mathcal{C}p =_{df} \mathcal{H}\,\mathtt{A}\,p$ is the *hypothesis that an assertion is justified*;
- a descriptive *expectation* $\mathtt{E}p =_{df} \mathtt{A}\,\mathtt{H}\,p$, with its expressive counterpart $\mathcal{E}p =_{df} \mathcal{A}\,\mathtt{H}\,p$ is the *assertion that a hypothesis is justified.*

We notice that expectations and conjectures allow us to find an intuitionistic pragmatic counterpart to all the modalities of classical **S4**, as in Table 1.

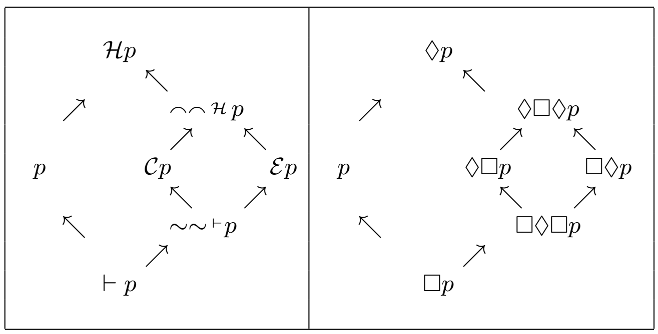

Table 1. Assertions, conjectures, expectations and hypotheses

3.1 Expectations: A Pragmatic Interpretation of Classical Logic.

Our first application is a representation of the *conjunctive implicational fragment of classical logic*, formalized as in Prawitz's Natural Deduction **NK** [29] into an extension of intuitionistic Natural Deduction with *axioms for hypotheses* and *introduction and elimination rules* for *expectations*, which allow us to derive the double negation law.

Definition 3. (Expectations) *The language of the logic of* expectations **AEL** *extends the language* $\mathcal{L}^A$ *of assertive intuitionistic logic with an infinite sequences of elementary* hypothetical *expressions* $\{\mathtt{H}\,p_i, i \geq 0\}$ *and the corresponding expressions of* expectations $\mathtt{E}\,p_i =_{df} \mathtt{A}\ \mathtt{H}\,p_i$*. The bi-intuitionistic sequent calculus for* expectations **AE-G3** *has sequents of the forms*

$$\Gamma\ ;\Rightarrow\ A\ ;\ \mathtt{H}p_1, \ldots \mathtt{H}p_n \qquad \textit{and} \qquad \Gamma\ ;\ \mathtt{H}p \Rightarrow\ ;\ \mathtt{H}p, \mathtt{H}p_1, \ldots \mathtt{H}p_n \tag{14}$$

In addition to the intuitionistic rules for conjunction and implication, right and left rules for expectations. The semantic projection is on classical **S4**.

$$\frac{\Theta\ ;\ \Rightarrow \bot\ ;\ \mathtt{H}p, \Upsilon}{\Theta\ ;\ \Rightarrow \mathtt{E}p\ ;\ \Upsilon}\ exp\ R \qquad \frac{\Theta\ ;\ \mathtt{H}p \Rightarrow\ ;\ \Upsilon}{\Theta, \mathtt{E}p\ ;\ \Rightarrow \bot\ ;\ \Upsilon}\ exp\ L$$

Here $\bot$ is an assertive expression which is always unjustified. The double negation rule for *expectations* is proved as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{;\ \mathtt{H}p \Rightarrow\ ;\ \mathtt{H}p}{\mathtt{E}p\ ;\ \Rightarrow \bot\ ;\ \mathtt{H}p}\ \text{exp L}}{;\ \Rightarrow \sim\mathtt{E}p\ ;\ \mathtt{H}p}\ \supset \text{R}}{\sim\sim\mathtt{E}p\ ;\ \Rightarrow\ \bot\ ;\ \mathtt{H}p}\ \supset \text{L}}{\sim\sim\mathtt{E}p\ ;\ \Rightarrow\ \mathtt{E}p\ ;}\ \text{exp L}$$

Notice also that the language of our logic of expectations *cannot be extended with ordinary intuitionistic disjunction, since intuitionistically* $\sim\sim (E_0 \cup E_1) \not\Rightarrow (E_0 \cup E_1)$*, hence the resulting logic would not be closed under substitution*[10]. Adding the obvious atomic cut elimination steps for *expectations* is unproblematic for any cut-elimination algorithm.

Recent type-theoretic and categorical developments, initiated by Michel Parigot $\lambda\mu$ calculus [26, 27], have identified computational properties characterizing proofs with classical principles with respect to intuitionistic proofs. The rules of the μ operator, in addition to those of the simply typed λ calculus, are typed in a multiple conclusion Natural Deduction setting by introduction and elimination rules for the operator $\mathcal{E}$:

$$\mu\ \frac{\Gamma\ ;\Rightarrow t : \bot\ ;\ \Delta, \alpha : \mathcal{H}p}{\Gamma\ ;\Rightarrow \mu\alpha.t : \mathcal{E}p\ ;\ \Delta}\ \mathcal{E}\ \text{intro}$$

$$[\alpha]\ \frac{\Gamma\ ;\Rightarrow t : \mathcal{E}p\ ;\ \Delta \qquad ;\ \alpha : \mathcal{H}p \Rightarrow\ ;\ \alpha : \mathcal{H}p}{\Gamma\ ;\Rightarrow [\alpha]t : \bot\ ;\ \Delta, \alpha : \mathcal{H}p}\ \mathcal{E}\ \text{elim} \tag{15}$$

The familiar β reduction and η expansion operations on $\lambda\mu$ terms,

$$[\alpha]\mu\alpha.t\ \rightsquigarrow\ t \qquad t\ \rightsquigarrow\ \mu\alpha.[\alpha]t \tag{16}$$

have a natural proof-theoretic interpretation.

Technically, composing Glivenko's and Gödel's [16, 17] double negation translation with the modal **S4** translation for the fragment of classical logic without disjunction and existential quantification, we obtain a modal translation A^{*M} of classical logic with mappings $p \mapsto \Box\Diamond\Box p$ for the atoms, while for non-atomic formulas $A \mapsto \Box\Diamond\Box A^{*M}$ follows from the modal translation of intuitionistic logic. It is known [38] that a direct translation (A^m) of the full language of (propositional) classical logic in classical **S4** can be given mapping $p \mapsto \Box\Diamond p$; but here, crucially, we need the translation $(A \vee B)^m = \Box\Diamond(A^m \vee B^m)$. On the

[10] Our treatment here resembles work by Miglioli, Moscato, Ornaghi and Usberti [24] in a different context. We thank Piero Pagliani for pointing at this connection.

other hand, excluding disjunction from the classical (propositional) language we may still obtain the A^m modal translation from the modal translation of intuitionistic logic, starting from the map $p \mapsto \Box\Diamond p$ for the atoms[11].

Conceptually, we see no reason why our theory of expectations should not be acceptable from the viewpoint of an intuitionistic philosophy in the sense of Dummett and Prawitz: notice that in our framework the classical law of double negation is a property of the *pragmatic attitude of expectation*, not a universal principle of logic. Focussing on *expectations*, we notice that the justification of an expression $\mathtt{E}\,p$ presented in such a mood does not require conclusive evidence of the truth of p, but only evidence that it is inescapable to consider the truth of p in any state of our knowledge; this is the same as to say that *not* p can never be conjectured. The notion of a "verification by approximation" to p seems to be implicit in the justification conditions for such an illocutionary act. As descriptive illocutionary forces are *intuitionistic modalities*, the meaning of $\mathtt{E}\,p$ essentially depends on what it means to single out a hypothesis as *assertible* among several other hypotheses which are not distinguished in this way: this is what we do in an *expectation introduction* inference. Moreover, it is in harmony with the meaning given by such an operation that we may decide to relinquish the special status of an expectation $\mathcal{E}p$ and retain the relevant information only as a hypothesis $\mathcal{H}p$: this is what we do in an *expectation elimination* inference. Eliminating an expectation $\mathcal{E}p$ allows us to raise another hypothetical piece of information $\mathcal{H}\,q$ to the status of an expectation $\mathcal{E}q$. Admittedly, this may seem uninformative, but it is the kind of meaning one can obtain from the use of a modality. It is likely that a much more informative analysis of the meaning of expectations is to come from *game theoretic* proof procedures for classical logic, as presented in Aschieri's work [3]; but we cannot discuss them here.

3.2 Hypotheses "modulo constraints": pragmatics of Lax Logic.

Our second application is to classify *intuitionistic modal logics* and to distinguish logics where the axioms for possibility and falsity $\Diamond(A \vee B) \Rightarrow \Diamond A \vee \Diamond B$ and $\Rightarrow \neg\Diamond\bot$ hold from those where they don't. Among the former we have the intuitionistic modalities in Alex Simpson's thesis [37], which are modelled on *first order intuitionistic quantifiers* and which may be called *assertive modalities.* Among the latter we have *Propositional Lax Logic* **PLL** as in Matt Fairtlough and Michael Mendler [15] and *constructive* **CS4** by Natasha Alechina, Valeria de Paiva, Michael Mendler and Eike Ritter [1], which we regard as having a *hypothetical possibility operator* within intuitionistic logic. Here we focus on **PLL**, giving a sort of *pragmatic interpretation* to it.

Definition 4. (Hypothetical possibility with constraints) *The logic of* pragmatic **PLL** *is a fragment of the logic* **ASHL**, *section 2.6. Its language extends*

[11] We thank Grigori Mints for showing the connection between our work and the classical translation $\Box\Diamond$ [38].

the language $\mathcal{L}^A$ *of assertive intutionistic logic with a* hypothetical possibility *modality* $\blacklozenge$*, an* assertive necessity *modality* $\mathtt{A}$ *and constants* $\blacklozenge\curlywedge$ *and* $\mathtt{A}\curlywedge$*. Semantic projection is in* bimodal **S4** *on frames* (W, R, S) *with* $S \subseteq R$*; here* $\mathcal{L}^A$ *and "*$\mathtt{A}$*" are interpreted through the accessibility relation* R *and "*$\blacklozenge$*" through* S*. The bi-intuitionistic sequent calculus for* pragmatic **PLL** *has sequents of the forms*

$$\Gamma\ ;\ \Rightarrow A\ ; \qquad \Gamma\ ;\ \Rightarrow\ ;\ \blacklozenge A \quad \textit{and} \quad \Gamma\ ;\ \blacklozenge A\ \Rightarrow\ ;\ \blacklozenge B \tag{17}$$

and in addition to the assertive intuitionistic rules, the following axioms and rules for modalities.

hypotheses	*non-assertability*
$;\ \blacklozenge\curlywedge\ \Rightarrow\ ;\ \blacklozenge\curlywedge$	$\Gamma, \mathtt{A}\curlywedge;\ \Rightarrow\ B\ ;$
$\dfrac{\Theta\ ;\ \Rightarrow A\ ;}{\Theta\ ;\ \Rightarrow\ ;\ \blacklozenge A}\ \blacklozenge\ R$	$\dfrac{\Theta, A\ ;\ \Rightarrow\ ;\ \blacklozenge B}{\Theta\ ;\ \blacklozenge A;\ \Rightarrow\ ;\ \blacklozenge B}\ \blacklozenge\ L$
$\dfrac{\Theta\ ;\ \Rightarrow\ ;\ \blacklozenge A}{\Theta\ ;\ \Rightarrow \mathtt{A}\blacklozenge A\ ;}\ \mathtt{A}\ R$	$\dfrac{\Theta\ ;\ \blacklozenge A \Rightarrow\ ;\ \blacklozenge B}{\Theta,\ \mathtt{A}\blacklozenge A\ ;\ \Rightarrow\ ;\ \blacklozenge B}\ \mathtt{A}\ L$

In our "pragmatic interpretation" we decompose the modal operator of **PLL** as $\bigcirc B = \mathtt{A}\blacklozenge B$. Then the *right* and *left* rules for $\bigcirc$ in the sequent calculus for **PLL**

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \bigcirc A}\ \bigcirc\ \mathrm{R} \qquad \frac{\Gamma, A \Rightarrow \bigcirc B}{\Gamma, \bigcirc A \Rightarrow \bigcirc B}\ \bigcirc\ \mathrm{L} \tag{18}$$

are decomposed in our setting as follows:

$$\dfrac{\dfrac{\Gamma\ ;\Rightarrow A\ ;}{\Gamma\ ;\Rightarrow;\ \blacklozenge A}\ \blacklozenge\ \mathrm{R}}{\Gamma\ ;\Rightarrow \mathtt{A}\blacklozenge A;}\ \mathtt{A}\ \mathrm{R} \qquad \dfrac{\dfrac{\dfrac{\Gamma, A;\ \Rightarrow;\ \blacklozenge B}{\Gamma;\ \blacklozenge A \Rightarrow;\ \blacklozenge B}\ \blacklozenge\ \mathrm{L}}{\Gamma, \mathtt{A}\blacklozenge A\ ;\Rightarrow;\ \blacklozenge B}\ \blacklozenge\ \mathrm{R}}{\Gamma, \mathtt{A}\blacklozenge A\ ;\Rightarrow \mathtt{A}\blacklozenge B\ ;}\ \mathtt{A}\ \mathrm{L} \tag{19}$$

For instance we derive

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, B;\ \Rightarrow A \cap B\ ;}{A, B;\ \Rightarrow\ ;\ \blacklozenge(A \cap B)}\ \blacklozenge\ \mathrm{R}}{A;\ \blacklozenge B \Rightarrow\ ;\ \blacklozenge(A \cap B)}\ \blacklozenge\ \mathrm{L}}{A, \bigcirc B;\ \Rightarrow\ ;\ \blacklozenge(A \cap B)}\ \mathtt{A}\ \mathrm{L}}{\bigcirc B;\ \blacklozenge A\ \Rightarrow\ ;\ \blacklozenge(A \cap B)}\ \blacklozenge\ \mathrm{L}}{\bigcirc A, \bigcirc B;\ \Rightarrow\ ;\ \blacklozenge(A \cap B)}\ \mathtt{A}\ \mathrm{L}}{\bigcirc A, \bigcirc B;\ \Rightarrow\ \bigcirc(A \cap B)\ ;}\ \mathtt{A}\ \mathrm{R}$$

Allowing intuitionistic disjunction $A \cup B$ in the language, the first characteristic properties of $\bigcirc$ obviously holds in our interpretation:

$$\mathtt{A}\blacklozenge(A \cup B)\ ; \not\Rightarrow\ (\mathtt{A}\blacklozenge A) \cup (\mathtt{A}\blacklozenge B)\ ; \tag{20}$$

To show that the second characteristic property of **PLL** also holds in our interpretation:

$$; \not\Rightarrow \sim \mathtt{A}\blacklozenge\curlywedge \; ; \qquad (21)$$

we must look for a sentence $\curlywedge$ which is never justifiably asserted, but that may become a possibly true hypothesis *under some constraint.* Here we think of $\blacklozenge$ as meaning *"possibly true within the margin of physical error"* in the sentence (**b**) *this iron bar has length of one meter up to experimental error* or *"possibly true with 50 % probability"* as in sentence (**c**) *the result of tossing this fair coin may be heads* in the examples of section 2.4[12].

What have we achieved by representing **PLL** into **ASHL**? Two remarks are essential: one is that the language of **PLL** is represented by *assertive* formulas and Gentzen's systems for it have sequents with only one formula in the succedent. The second is that even when a co-intuitionistic modality $\blacklozenge$ is introduced, the strictness conditions (¶¶) of **SAH-G3** in section 2.6 block any application of the intuitionistic right rules when a formula $\blacklozenge B$ occurs in the succedent of a sequent[13]; this guarantees that the representation is sound. It follows that the "necessity component" of the operator $\bigcirc$ of **PLL** is naturally explained as an application of the *assertive modality* $\mathtt{A}$, which is needed to bring back co-intuitionistic formulas $\blacklozenge B$ to the assertive side; this provides a rationale for the $\bigcirc$ right and left rule within our framework[14].

4 Conclusions.

One of the aims of this paper has been to develop Dummett's and Prawitz's ideas focussing on what it means to justify an elementary linguistic expression when its *illocutionary force* is taken into account. Dummett and Prawitz's analysis of proof-theoretic meaning yields an explanation of the logical connectives based on the assertability conditions and on the direct consequences of *molecular* expressions without taking into account the meaning of elementary expressions: these are invariably assumed to be assertions. As we identify four distinct modes of presenting elementary expressions as *assertions*, *hypotheses*, *conjectures* and *expectations* we do not simply remark that the justification conditions for such illocutionary acts are different, but also we show that the logics of assertive and hypothetical expressions are necessarily different: the

[12] In Fairtlough and Mendler's paper the operator $\bigcirc$ is interpreted as *"true modulo constraints"* and a motivating example is an electric circuit where the current may oscillate before reaching a stationary value in a given time interval. It would seem that many very diverse classes of constraints might be considered that would yield interesting models of **PLL**.

[13] Suitable restrictions on permutations of rules and on the cut-elimination procedure follow as well; we cannot pursue the topic here.

[14] There seems to be no difficulty to extend such an account to **CS4** along similar lines. We cannot pursue this line of research here.

law of excluded middle is valid for hypothetical disjunctions (being the dual of the law of no contradiction for assertive conjunctions) while it is the *law of no contradiction* for hypothetical conjunctions which is co-intuitionistically invalid (being the dual of the law of excluded middle for assertive disjunctions). Thus in this extended framework an analysis of *meaning as use* is inevitably incomplete if it does not take into account the justification conditions for elementary expressions.

In conclusion, it seems that as the representation of intuitionistic logic within classical logic requires an extension of classical logic with **S4** modalities, so the intuitionistic representation of classical proof theory through the double negation interpretation is best understood in an *extension* of intuitionistic logic with a fragment of polarized bi-intuitionistic logic, in which the type theoretic setting of Parigot's $\lambda\mu$ calculus fits best. A similar point can be made about the representation of the logic of *truth modulo constraints* **PLL** within an *asymmetric version* of polarized bi-intuitionistic logic. It seems interesting to us that the properties of the "mysterious" operator $\bigcirc$ follow at once in the framework of the polarized bi-intuitionistic logic of assertions and hypotheses, when strictness conditions are added. These result give more than a "scintilla of evidence" that a pragmatic interpretation may be of technical and conceptual significance in the treatment of intuitionistic modalities.

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5 APPENDIX. Sequent Calculus AH-G3.

identity rules

logical axiom: $A, \Theta \; ; \; \Rightarrow \; A \; ; \; \Upsilon$

logical axiom: $\Theta \; ; \; C \; \Rightarrow \; ; \; \Upsilon, C$

cut_1:

$$\frac{\Theta \; ; \; \Rightarrow \; A \; ; \; \Upsilon \qquad A, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

cut_2:

$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, C \qquad \Theta' \; ; \; C \; \Rightarrow \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

ASSERTIVE LOGICAL RULES

validity axiom:

$$\Theta \,;\; \Rightarrow\; \curlyvee \,;\, \Upsilon$$

right $\supset$*:*

$$\frac{\Theta, A_1 \,;\, \Rightarrow\, A_2 \,;\, \Upsilon}{\Theta \,;\, \Rightarrow\, A_1 \supset A_2 \,;\, \Upsilon}$$

left $\supset$*:*

$$\frac{A_1 \supset A_2, \Theta;\, \Rightarrow\, A_1 \,;\, \Upsilon \qquad A_2, \Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon}{A_1 \supset A_2, \Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon}$$

right $\cap$*:*

$$\frac{\Theta \,;\, \Rightarrow\, A_1 \,;\, \Upsilon \qquad \Theta \,;\, \Rightarrow\, A_2 \,;\, \Upsilon}{\Theta \,;\, \Rightarrow\, A_1 \cap A_2 \,;\, \Upsilon}$$

left $\cap$*:*

$$\frac{A_0, A_1, \Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon}{A_0 \cap A_1, \Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon}$$

CONJECTURAL RULES

absurdity axiom:

$$;\, \curlywedge \,\Rightarrow\, ;\, \Upsilon$$

right $\smallsetminus$*:*

$$\frac{\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, C_1 \qquad \Theta \,;\, C_2 \,\Rightarrow\, ;\, \Upsilon, C_1 \smallsetminus C_2}{\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, C_1 \smallsetminus C_2}$$

left $\smallsetminus$*:*

$$\frac{\Theta;\, C_1 \,\Rightarrow\, ;\, \Upsilon, C_2}{\Theta \,;\, C_1 \smallsetminus C_2 \,\Rightarrow\, ;\, \Upsilon}$$

right $\curlyvee$*:*

$$\frac{\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, C, \upsilon_0, C_1}{\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, C_0 \curlyvee C_1}$$

left $\curlyvee$*:*

$$\frac{\Theta \,;\, C_1 \,\Rightarrow\, ;\, \Upsilon \qquad \Theta \,;\, C_2 \,\Rightarrow\, ;\, \Upsilon}{\Theta \,;\, C_1 \curlyvee C_2 \,\Rightarrow\, ;\, \Upsilon}$$

MIXED-TYPE NEGATIONS:

right $\sim$:

$$\frac{\Theta \,;\, C \,\Rightarrow\, ;\, \Upsilon}{\Theta \,;\, \Rightarrow \sim C \,;\, \Upsilon}$$

left $\sim$:

$$\frac{\sim C, \Theta;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, C}{\sim C, \Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon}$$

right $\frown$:

$$\frac{\Theta, A \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \frown A}{\Theta \,;\, \epsilon \,\Rightarrow\, \epsilon' \,;\, \Upsilon, \frown A}$$

left $\frown$:

$$\frac{\Theta;\, \Rightarrow\, A \,;\, \Upsilon}{\Theta \,;\, \frown A \,\Rightarrow\, ;\, \Upsilon}$$