An Introduction to Auslander-Reiten Theory

Lecture Notes
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Contents

Preliminaries

Aim of these lectures is to provide a brief introduction to the notion of an almost split sequence and its use in the representation theory of finite dimensional algebras. For a comprehensive treatment we refer to [16].
We will only assume some basic knowledge in module theory and category theory. In particular, we will use the following notions that can be found in any textbook on modules or algebras, e.g. in [2], or [19]: free module, maximal submodule, simple module, module of finite length, direct sum, direct summand, local ring, artinian ring, equivalence, duality, tensor product.

Further background material is collected in the first chapter.

1 SOME HOMOLOGICAL ALGEBRA

Throughout this chapter, let R be a ring, and denote by $R\,\mathrm{Mod}$ the category of all left R-modules.

1.1 Projective and injective modules

Definitions.
(1) A sequence of R-homomorphisms

$$\ldots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \ldots$$

is said to be *exact* in case $\mathrm{Im} f_n = \mathrm{Ker} f_{n+1}$ for each n.

(2) An exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a *short exact sequence*.

(3) A *split exact sequence* is a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ where g is a *split epimorphism*, i.e. there is a homomorphism $g' : C \to B$ such that the composition $g\, g' = \mathrm{id}_C$. This is equivalent to the condition that f is a *split monomorphism*, i.e. there is a homomorphism $f' : B \to A$ such that the composition $f'\, f = \mathrm{id}_A$.
(4) Let S be a ring, and let $F : R\ \mathrm{Mod} \to S\,\mathrm{Mod}$ be a covariant functor. If for every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ the sequence $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact, then F is said to be *left exact*. Similarly, if the sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact, then F is said to be *right exact*.
If F is a contravariant functor, then it is *left exact* provided for every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ the sequence $0 \to F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$ is exact, and right exactness of F is defined correspondingly.

(5) A functor that is both left and right exact is said to be *exact*. Note that the image of any exact sequence under an exact functor is exact.

An example of left exact functors is given by the Hom-functors.

Proposition 1.1.1. *If ${}_RM$ is a module, then the covariant functor* $\mathrm{Hom}_R(M,-) : R\ \mathrm{Mod} \to \mathbb{Z}\ \mathit{Mod}$ *and the contravariant functor* $\mathrm{Hom}_R(-,M) : R\ \mathrm{Mod} \to \mathbb{Z}\ \mathit{Mod}$ *are left exact.*

We are now going to discuss the modules for which the Hom functor is even exact.

Definition. A module ${}_RP$ is said to be *projective* if the functor $\mathrm{Hom}_R(P,-) : R\ \mathrm{Mod} \to \mathbb{Z}\,\mathrm{Mod}$ is exact. Dually, a module ${}_RI$ is said to be *injective* if the functor $\mathrm{Hom}_R(-,I) : R\ \mathrm{Mod} \to \mathbb{Z}\,\mathrm{Mod}$ is exact.

Theorem 1.1.2. [2, 17.2] *The following statements are equivalent for a module ${}_RP$.*

(1) P is projective

(2) P is isomorphic to a direct summand of a free left R-module.

(3) Every epimorphism $g : M \to P$ in R Mod *is a split epimorphism.*

We see that projective modules are closely related to free modules. In fact, they can also be characterized as follows.

Lemma 1.1.3. [2, Exercise 17.11] *A module ${}_RP$ is projective if and only if it has a* dual basis*, that is, a pair $(\,(x_i)_{i\in I},\, (\varphi_i)_{i\in I}\,)$ consisting of elements $(x_i)_{i\in I}$ in P and homomorphisms $(\varphi_i)_{i\in I}$ in $P^* = \mathrm{Hom}_R(P,R)$ such that every element $x \in P$ can be written as $x = \sum_{i\in I} \varphi_i(x)\, x_i$ with $\varphi_i(x) = 0$ for almost all $i \in I$.*

As a consequence, we obtain the following properties of the contravariant functor ${}^* = \mathrm{Hom}(-,R) : R\mathrm{Mod} \longrightarrow \mathrm{Mod}\,R$.

Proposition 1.1.4. *Let P be a finitely generated projective left R-module. Then P^* is a finitely generated projective right R-module, and $P^{**} \cong P$. Moreover, for all $M \in RMod$*

$$\mathrm{Hom}_R(P,M) \cong P^* \otimes_R M$$

Proof: We only sketch the arguments. First of all, note that the evaluation map $c : P \to P^{**}$ defined by $c(x)(\varphi) = \varphi(x)$ on $x \in P$ and $\varphi \in P^*$ is a monomorphism. Further, if $(\,(x_i)_{1\le i\le n},\, (\varphi_i)_{1\le i\le n}\,)$ is a dual basis of P, then it is easy to see that $(\,(\varphi_i)_{1\le i\le n},\, (c(x_i))_{1\le i\le n}\,)$ is a dual basis of P^*. This shows that P^* is finitely generated projective. The isomorphism $P^{**} \cong P$ is proved by showing that the assignment $P^{**} \ni f \mapsto \sum_{i=1}^n f(\varphi_i)\, x_i \in P$ is inverse to c.
Finally, for the last statement, one verifies that the map $\alpha : \mathrm{Hom}_R(P,M) \to P^* \otimes_R M$, $h \mapsto \sum_{i=1}^n \varphi_i \otimes h(x_i)$, and the map $\beta : P^* \otimes_R M \to \mathrm{Hom}_R(P,M)$ given by $\beta(\varphi \otimes m) : P \to M,\ x \mapsto \varphi(x)\, m,$ are well-defined and mutually inverse. $\square$

1.2 Finitely presented modules

Let ${}_RM$ be a module. Of course, M is epimorphic image of a free module. Moreover, M is *finitely generated* if and only if it is epimorphic image of a finitely generated free module, that is, there is an exact sequence $R^n \to M \to 0$ for some $n \in \mathbb{N}$. Finally, M is said to be *finitely presented* if there is an exact sequence $R^m \to R^n \to M \to 0$ for some $n, m \in \mathbb{N}$.
We will denote by Rmod the category of finitely presented modules.

1.3 Injective envelopes, projective covers

We have seen above that every module is epimorphic image of a projective module. Dually, every module can be embedded in an injective module. Moreover, this embedding can be chosen in a minimal way.

Definition. A homomorphism $f\colon A \to B$ is called *left minimal* if each $t \in \operatorname{End} B$ with $tf = f$ is an isomorphism. *Right minimal* homomorphisms are defined dually.

Theorem 1.3.1 (Eckmann-Schopf 1953 [2, 18.10]**).** *For every module ${}_RM$ there exist an injective module $E(M)$ and a left minimal monomorphism $e : M \to E(M)$. The module $E(M)$ is uniquely determined up to isomorphism and is called the* injective envelope *of M.*

Remark 1.3.2. Dually, a right minimal epimorphism $p : P(M) \to M$ where $P(M)$ is projective is called a *projective cover* of M. In general, however, projective covers need not exist. A ring R is said to be *semiperfect* if every finitely generated left R-module (or equivalently, every finitely generated right R-module) has a projective cover. For example, every finite dimensional algebra, or more generally, every Artin algebra, is semiperfect, cf. Section 3.
Over a semiperfect ring, a finitely presented module M always has a *minimal projective presentation* $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$, i. e. an exact sequence where P_0, P_1 are finitely generated projective, P_0 is a projective cover of M, and P_1 is a projective cover of $\operatorname{Ker} p_0$.

The original definition of injective envelope and projective cover uses the following notions.

Definition. Let ${}_RM$ be a module, and let U be a submodule of M.
(1) U is said to be an *essential submodule* in case for every submodule V of M the condition $U \cap V = 0$ implies $V = 0$.
(2) U is said to be an *superfluous submodule* in case for every submodule V of M the condition $U + V = M$ implies $V = M$.

Proposition 1.3.3. [33, 1.2.11 and 1.2.12] *Let ${}_RM$ be a module.*
(1) A monomomorphism $f : M \to I$ where I is injective is an injective envelope if and only if $\operatorname{Im} f$ is an essential submodule of I.
(2) An epimomorphism $g : P \to M$ where P is projective is a projective cover if and only if $\operatorname{Ker} g$ is a superfluous submodule of P.

We will later need the following observation which is an immediate consequence of 1.3.3(1).

Remark 1.3.4. Let ${}_RS$ be a simple module and ${}_RI = E(S)$ its injective envelope.
(1) Every nonzero submodule ${}_RV$ of ${}_RI$ contains S.
(2) The endomorphism ring $\operatorname{End}_R I$ of ${}_RI$ is a local ring.

1.4 Cogenerators

Definition. A module ${}_RC$ is said to be a *cogenerator* of RMod if every left R-module M can be embedded in a product of copies of C, that is, for every ${}_RM$ there are a set I and a monomorphism $M \hookrightarrow C^I$.

Proposition 1.4.1. [2, 18.16] *Assume that $S_1, \ldots, S_n$ are representatives of the isomorphism classes of the simple left R-modules. Then $C = \bigoplus_{i=1}^n E(S_i)$ is an injective cogenerator of RMod. Moreover, C is a* minimal *cogenerator, that is, a left R-module ${}_RC'$ is a cogenerator of RMod if and only if it has a direct summand isomorphic to C.*

Cogenerators can be employed to detect monomorphisms. In particular, we have the following property of injective cogenerators.

Remark 1.4.2. [2, 18.14] Let C be an injective cogenerator of RMod. Then a sequence $M' \to M \to M''$ is exact if and only if the induced sequence $\operatorname{Hom}_R(M'', C) \to \operatorname{Hom}_R(M, C) \to \operatorname{Hom}_R(M', C)$ is exact.

1.5 Radical and Socle

Essential and superfluous submodules are closely related to the following notions. Given a module ${}_RM$, the *radical* of M is defined as

$$\operatorname{Rad} M = \bigcap \{U \mid U \text{ is a maximal submodule of } M\}$$

and the *socle* of M is defined as

$$\operatorname{Soc} M = \sum \{S \mid S \text{ is a simple submodule of } M\}$$

Proposition 1.5.1. [2, 9.13 and 9.7] *Let ${}_RM$ be a module.*
(1) $\operatorname{Rad} M = \sum \{V \mid V$ *is a superfluous submodule of* $M\}$.
(2) $\operatorname{Soc} M = \bigcap \{U \mid U$ *is an essential submodule of* $M\}$.

The radical of the left module ${}_RR$ coincides with the radical of the right module R_R, see [2, 15.14], and is called the *Jacobson radical* of R. We denote it by $J = J(R)$.

Remark 1.5.2. [2, 15.15] Let R be a local ring. Then J consists of all non-invertible elements of R, and it is the unique maximal (left or right) ideal of R. Moreover, R/J is the unique simple (left or right) R-module up to isomorphism, and $E(R/J)$ is a minimal injective cogenerator.

1.6 Push-out and Pull-back

Proposition 1.6.1. [30, pp. 41] *Consider a pair of homomorphisms in RMod*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

There is a module $_RL$ together with homomorphisms $\sigma : C \to L$ and $\tau : B \to L$ such that (i) the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \tau \\ C & \xrightarrow{\sigma} & L \end{array}$$

commutes; and
(ii) given any other module $_RL'$ together with homomorphisms $\sigma' : C \to L'$ and $\tau' : B \to L'$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \tau' \\ C & \xrightarrow{\sigma'} & L' \end{array}$$

commute, there exists a unique homomorphism $\gamma : L \to L'$ such that $\gamma\sigma = \sigma'$ and $\gamma\tau = \tau'$.

The module L together with σ, τ is unique up to isomorphism and is called push-out *of f and g.*

Proof: We just sketch the construction. The module L is defined as the quotient $L = B \oplus C \,/\, \{\, (f(a), -g(a)) \mid a \in A\}$, and the homomorphisms are given as $\sigma : C \to L,\ c \mapsto \overline{(0, c)}$, and $\tau : B \to L,\ b \mapsto \overline{(b, 0)}$. □

Remark 1.6.2. *If f is a monomorphism, also σ is a monomorphism, and* $\operatorname{Coker} \sigma \cong \operatorname{Coker} f$.

Dually, one defines the *pull-back* of a pair of homomorphisms

$$\begin{array}{ccc} & & B \\ & & \downarrow f \\ C & \xrightarrow{g} & A \end{array}$$

1.7 A short survey on Ext^1

Aim of this section is to give a brief introduction to the functor Ext^1, as needed in the sequel. For a comprehensive treatment we refer to textbooks in homological algebra, e.g. [30].

Definition. Let A, B be two R-modules. We define a relation on short exact sequences of the form $\mathfrak{E} : 0 \to B \to M \to A \to 0$ by setting

$$\mathfrak{E}_1 : 0 \to B \to E_1 \to A \to 0 \ \sim \ \mathfrak{E}_2 : 0 \to B \to E_2 \to A \to 0$$

if there is $f \in \mathrm{Hom}_R(E_1, E_2)$ making the following diagram commute

$$\begin{array}{cccccccccc}
\mathfrak{E}_1 : & 0 & \to & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \| & & \downarrow f & & \| & & \\
\mathfrak{E}_2 : & 0 & \to & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

It is easy to see that $\sim$ is an equivalence relation, and we denote by $\mathrm{Ext}^1_R(A, B)$ the set of all equivalence classes.

Next, we want to define a group structure on $\mathrm{Ext}^1_R(A, B)$. Let $[\mathfrak{E}]$ be the equivalence class of the short exact sequence $\mathfrak{E} : 0 \to B \to E \to A \to 0$. First of all, for $\beta \in \mathrm{Hom}_R(B, B')$ we can consider the short exact sequence $\beta\,\mathfrak{E}$ given by the push-out diagram

$$\begin{array}{cccccccccc}
\mathfrak{E} : & 0 & \to & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \downarrow \beta & & \downarrow & & \| & & \\
\beta\,\mathfrak{E} : & 0 & \to & B' & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

In this way, we can define a map

$$\mathrm{Ext}^1_R(A, \beta) : \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A, B'),\ [\mathfrak{E}] \mapsto [\beta\,\mathfrak{E}]$$

For $\beta_1 \in \mathrm{Hom}_R(B', B'')$ and $\beta_2 \in \mathrm{Hom}_R(B, B')$ one verifies

$$\mathrm{Ext}^1_R(A, \beta_1)\,\mathrm{Ext}^1_R(A, \beta_2) = \mathrm{Ext}^1_R(A, \beta_1\beta_2)$$

Dually, for $\alpha \in \mathrm{Hom}_R(A', A)$, we use the pull-back diagram

$$\begin{array}{cccccccccc}
\mathfrak{E}\,\alpha : & 0 & \to & B & \longrightarrow & E' & \longrightarrow & A' & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow \alpha & & \\
\mathfrak{E} : & 0 & \to & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

to define a map

$$\mathrm{Ext}^1_R(\alpha, B) : \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A', B),\ [\mathfrak{E}] \mapsto [\mathfrak{E}\,\alpha]$$

Since

$$\mathrm{Ext}^1_R(\alpha, B')\,\mathrm{Ext}^1_R(A, \beta)[\mathfrak{E}] = \mathrm{Ext}^1_R(A', \beta)\,\mathrm{Ext}^1_R(\alpha, B)[\mathfrak{E}]$$

the composition of the maps above yields a map

$$\mathrm{Ext}^1_R(\alpha, \beta) : \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A', B')$$

Now we are ready to define an addition on $\mathrm{Ext}^1_R(A,B)$, called *Baer sum.* Given two sequences $\mathfrak{E}_1 : 0 \to B \to E_1 \to A \to 0$ and $\mathfrak{E}_2 : 0 \to B \to E_2 \to A \to 0$, we consider the direct sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2 : 0 \to B \oplus B \to E_1 \oplus E_2 \to A \oplus A \to 0$ together with the diagonal map $\Delta_A : A \to A \oplus A$, $a \mapsto (a,a)$, and the summation map $\nabla_B : B \oplus B \to B$, $(b_1,b_2) \mapsto b_1 + b_2$. We then set

$$[\mathfrak{E}_1] + [\mathfrak{E}_2] = \mathrm{Ext}^1_R(\Delta_A, \nabla_B)([\mathfrak{E}_1 \oplus \mathfrak{E}_2]) \ \in \mathrm{Ext}^1_R(A,B)$$

In this way, $\mathrm{Ext}^1_R(A,B)$ becomes an abelian group. Its zero element is the equivalence class of all split exact sequences. The inverse element of the class $[\mathfrak{E}]$ given by the sequence $\mathfrak{E} : 0 \to B \xrightarrow{f} E \xrightarrow{g} A \to 0$ is the equivalence class of the sequence $0 \to B \xrightarrow{f} E \xrightarrow{-g} A \to 0$.

Moreover, the maps $\mathrm{Ext}^1_R(A,\beta), \mathrm{Ext}^1_R(\alpha,B)$ are group homomorphisms, and we have a covariant functor $\mathrm{Ext}^1_R(A,-) : R\ \mathrm{Mod} \to \mathbb{Z}\,\mathrm{Mod}$ and a contravariant functor $\mathrm{Ext}^1_R(-,B) : R\ \mathrm{Mod} \to \mathbb{Z}\,\mathrm{Mod}$.

The Ext^1-functors “repair” the non-exactness of the Hom-functors as follows.

Lemma 1.7.1. *Let* $\mathfrak{E} :\ 0 \to B \xrightarrow{\beta} B' \xrightarrow{\beta'} B'' \to 0$ *be a short exact sequence, and let* A *be an* R*-module. Then there is natural homomorphism* $\delta = \delta(A,\mathfrak{E})$ *such that the sequence* $0 \to \mathrm{Hom}_R(A,B) \xrightarrow{\mathrm{Hom}_R(A,\beta)} \mathrm{Hom}_R(A,B') \xrightarrow{\mathrm{Hom}_R(A,\beta')} \mathrm{Hom}_R(A,B'') \xrightarrow{\delta} \mathrm{Ext}^1_R(A,B) \xrightarrow{\mathrm{Ext}^1_R(A,\beta)} Ext^1_R(A,B') \xrightarrow{\mathrm{Ext}^1_R(A,\beta')} Ext^1_R(A,B'')$ *is exact. Here* δ *is defined by* $\delta(f) = [\mathfrak{E}\, f]$.

The dual statement for the contravariant functors $\mathrm{Hom}(-,B)$*,* $\mathrm{Ext}^1_R(-,B)$ *also holds true.*

Note that, since every short exact sequence starting at an injective module is split exact, we have $\mathrm{Ext}^1_R(A,I) = 0$ for all injective modules I and all modules A. As a consequence, we obtain the following description of Ext^1.

Proposition 1.7.2. *Let* A, B *be left* R*-modules.*
If $0 \to B \to I \xrightarrow{\pi} C \to 0$ *is a short exact sequence where* I *is injective, then*

$$\mathrm{Ext}^1_R(A,B) \cong \mathrm{Coker}\, \mathrm{Hom}_R(A,\pi)$$

Similarly, if $0 \to K \xrightarrow{\iota} P \to A \to 0$ *is a short exact sequence where* P *is projective, then*

$$\mathrm{Ext}^1_R(A,B) \cong \mathrm{Coker}\, \mathrm{Hom}_R(\iota,B)$$

2 ALMOST SPLIT SEQUENCES

2.1 Almost split maps

Let R be a ring, and let $\mathcal{M}$ be the category R Mod or R mod.

Definition.

(1) A homomorphism $g\colon B \to C$ in $\mathcal{M}$ is called *right almost split in* $\mathcal{M}$ if

(a) g is not a split epimorphism, and

(b) if $h\colon X \to C$ is a morphism in $\mathcal{M}$ that is not a split epimorphism, then h factors through g.

$$\begin{tikzcd} B \arrow[rr, "g"] & & C \\ & X \arrow[ul, dotted] \arrow[ur, "h"'] & \end{tikzcd}$$

(2) $g\colon B \to C$ is called *minimal right almost split in* $\mathcal{M}$ if it is right minimal and right almost split in $\mathcal{M}$.

The definition of a *(minimal) left almost split* map is dual.

Remark 2.1.1. Let $g\colon B \to C$ be right almost split in $\mathcal{M}$. Then $\operatorname{End} C$ is a local ring and $J(\operatorname{End} C) = g \circ \operatorname{Hom}_R(C, B)$. If C is not projective, then g is an epimorphism.

Proposition 2.1.2. *The following statements are equivalent for an exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* $\mathcal{M}$*.*

(1) f *is left almost split and* g *is right almost split in* $\mathcal{M}$*.*

(2) $\operatorname{End}_R C$ *is local and* f *is left almost split in* $\mathcal{M}$*.*

(3) $\operatorname{End}_R A$ *is local and* g *is right almost split in* $\mathcal{M}$*.*

(4) f *is minimal left almost split in* $\mathcal{M}$*.*

(5) g *is minimal right almost split in* $\mathcal{M}$*.*

Definition. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{M}$ is called *almost split* (*Auslander-Reiten sequence*) in $\mathcal{M}$ if it satisfies one of the equivalent conditions above.

Remark 2.1.3. [16, V.2, 1.16] Almost split sequences starting (or ending) at a given module are uniquely determined up to isomorphism. More precisely, if $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ are almost split sequences, then $A \cong A'$ if and only if $C \cong C'$ if and only if there are isomorphisms a, b, c making the following diagram commute

$$\begin{array}{ccccccccc}
0 & \to & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \to & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

We will now show the abundance of almost split sequences. We first need some preliminaries.

2.2 The Auslander-Bridger transpose

We have seen in 1.1.4 that the functor $^* = \mathrm{Hom}(-, R) : R\mathrm{Mod} \longrightarrow \mathrm{Mod}\, R$ induces a duality between the full subcategories of finitely generated projective modules in R Mod and Mod$R$. The following construction from [13] can be viewed as a way to extend this duality to all finitely presented modules.

We denote by $R\mathrm{mod}_{\mathcal{P}}$ the full subcategory of Rmod consisting of the modules without non-zero projective summands.

From now on, we will assume that the ring R is semiperfect, cf. Remark 2.4.2.

Let $M \in R\ \mathrm{mod}_{\mathcal{P}}$ and let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M. Applying the functor $^* = \mathrm{Hom}_R(-, R)$ on it, we obtain a minimal projective presentation

$$P_0^* \xrightarrow{p_1^*} P_1^* \to \mathrm{Coker}\, p_1^* \to 0\,.$$

Set $\mathrm{Tr}\, M = \mathrm{Coker}\, p_1^*$. Then $\mathrm{Tr}\, M \in \mathrm{mod}\, \mathrm{R}_{\mathcal{P}}$. Moreover, the following hold true.

(1) The isomorphism class of $\mathrm{Tr}\, M$ does not depend on the choice of $P_1 \to P_0 \to M \to 0$.

(2) There is a natural isomorphism $\mathrm{Tr}^2(M) \cong M$.

Let us now consider a homomorphism $f \in \mathrm{Hom}_R(M, N)$ with $M, N \in R$ mod. It induces a commutative diagram

$$\begin{array}{ccccccc}
P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & M & \longrightarrow & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
Q_1 & \xrightarrow{q_1} & Q_0 & \xrightarrow{q_0} & N & \longrightarrow & 0
\end{array}$$

Applying $^* = \mathrm{Hom}(-, R)$, we can construct $\widetilde{f} \in \mathrm{Hom}(\mathrm{Tr}\, N, \mathrm{Tr}\, M)$ as follows:

$$\begin{array}{ccccccc}
P_0^* & \xrightarrow{p_1^*} & P_1^* & \longrightarrow & \mathrm{Tr}\, M & \longrightarrow & 0 \\
\uparrow f_0^* & & \uparrow f_1^* & & \uparrow \widetilde{f} & & \\
Q_0^* & \xrightarrow{q_1^*} & Q_1^* & \longrightarrow & \mathrm{Tr}\, N & \longrightarrow & 0
\end{array}$$

Note that this construction is not unique since $\widetilde{f}$ depends on the choice of f_0, f_1. However, if we choose another factorization of f, say by maps g_0 and g_1, and conctruct $\widetilde{g}$ correspondingly, then the difference $f_0 - g_0 \in \mathrm{Ker}\, q_0 = \mathrm{Im}\, q_1$ factors through Q_1, and so $\widetilde{f} - \widetilde{g}$ factors through P_1^*, as illustrated below:

$$\begin{tikzcd} P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] \arrow[d, dotted, bend right, "g_1"'] & P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] \arrow[d, dotted, bend right, "g_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\ Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, "q_0"'] & N \arrow[r] & 0 \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} P_0^* \arrow[r, "p_1^*"] & P_1^* \arrow[r] & \operatorname{Tr} M \arrow[r] & 0 \\ Q_0^* \arrow[r, "q_1^*"'] \arrow[u, "f_0^*"] \arrow[u, dotted, bend right, "g_0^*"'] & Q_1^* \arrow[r] \arrow[u, "f_1^*"] \arrow[u, dotted, bend right, "g_1^*"'] & \operatorname{Tr} N \arrow[r] \arrow[u, "\widetilde{f}"] \arrow[u, dotted, bend right, "\widetilde{g}"'] \arrow[ul] & 0 \end{tikzcd}$$

In other words, if we consider the subgroups
$P(M,N) = \{f \in \operatorname{Hom}(M,N) \mid f \text{ factors through a projective module}\} \leq \operatorname{Hom}_R(M,N)$, then $\widetilde{f}$ is uniquely determined modulo $P(\operatorname{Tr} N, \operatorname{Tr} M)$.

We set $\underline{\operatorname{Hom}}_R(M,N) = \operatorname{Hom}_R(M,N)/P(M,N)$, and let $\mathrm{R}\,\underline{\mathrm{mod}}$ be the category with the same objects as R mod and morphisms $\underline{\operatorname{Hom}}_R(M,N)$. It is called the *stable category* of R mod modulo projectives. We obtain the following.

Proposition 2.2.1.

(1) There is a group isomorphism $\underline{\operatorname{Hom}}(M,N) \to \underline{\operatorname{Hom}}(\operatorname{Tr} N, \operatorname{Tr} M)$, $\underline{f} \mapsto \underline{\widetilde{f}}$.

(2) $\operatorname{End}_R M$ *is local if and only if* $\operatorname{End} \operatorname{Tr} M_R$ *is local.*

(3) Tr *induces a duality* $\mathrm{R}\,\underline{\mathrm{mod}} \to \underline{\mathrm{mod}}\,\mathrm{R}$.

As we will see in Section 5.1, over hereditary rings the transpose is isomorphic to the functor $\operatorname{Ext}^1_R(-,R)$. In general, we have the following result.

Lemma 2.2.2. *Let* $\mathfrak{E} : 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be a short exact sequence, and let* $A \in R\mathit{mod}_{\mathcal{P}}$. *Then there is a natural homomorphism* $\delta = \delta(A, \mathfrak{E})$ *such that the sequence* $0 \to \operatorname{Hom}_R(A,X) \to \operatorname{Hom}_R(A,Y) \to \operatorname{Hom}_R(A,Z) \xrightarrow{\delta} \operatorname{Tr} A \otimes_R X \to \operatorname{Tr} A \otimes_R Y \to \operatorname{Tr} A \otimes_R Z \to 0$ *is exact.*

Proof: Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \to 0$ be a minimal projective presentation of A. Since the P_i, $i = 0,1$, are finitely generated projective, we know from 1.1.4 that $\operatorname{Hom}_R(P_i, M) \cong P_i^* \otimes_R M$ for any $M \in R\mathrm{Mod}$. So the cokernel of $\operatorname{Hom}(p_1, M) : \operatorname{Hom}_R(P_0, M) \longrightarrow \operatorname{Hom}_R(P_1, M)$ is isomorphic to $\operatorname{Tr} A \otimes_R M$. Hence we have a commutative diagram with exact rows

$$\begin{array}{cccccccc} & 0 & & 0 & & 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \operatorname{Hom}_R(A,X) & \longrightarrow & \operatorname{Hom}_R(A,Y) & \longrightarrow & \operatorname{Hom}_R(A,Z) & & \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \operatorname{Hom}_R(P_0,X) & \longrightarrow & \operatorname{Hom}_R(P_0,Y) & \longrightarrow & \operatorname{Hom}_R(P_0,Z) & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \operatorname{Hom}_R(P_1,X) & \longrightarrow & \operatorname{Hom}_R(P_1,Y) & \longrightarrow & \operatorname{Hom}_R(P_1,Z) & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ & \operatorname{Tr} A \otimes X & \longrightarrow & \operatorname{Tr} A \otimes Y & \longrightarrow & \operatorname{Tr} A \otimes Z & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ & 0 & & 0 & & 0 & & \end{array}$$

and by the snake-lemma [30, 6.5] we obtain the claim. $\square$

2.3 The local dual

Lemma 2.3.1. *Let M_R be a finitely presented right R-module with local endomorphism ring. Let $S = \operatorname{End} M_R$ and ${}_S V$ be an injective envelope of ${}_S S/J(S)$. Set*

$$M^+ = {}_R \operatorname{Hom}_S(M, V)$$

Then M^+ is a left R-module with local endomorphism ring $\operatorname{End}_R M^+ \cong \operatorname{End}_S V$.

Proof: M^+ is a left R-module with respect to $r \cdot f : M \to V$, $m \mapsto f(mr)$.
We have $\operatorname{End}_R M^+ = \operatorname{Hom}_R ({}_R \operatorname{Hom}_S(M, V),\, {}_R \operatorname{Hom}_S(M, V)) \cong$
$\cong \operatorname{Hom}_S (M_R \otimes \operatorname{Hom}_S(M, V),\, {}_S V) \cong \operatorname{Hom}_S (\operatorname{Hom}_S (\operatorname{Hom}_R (M, M), {}_S V), {}_S V) \cong \operatorname{End}_S V$ where the first isomorphism follows by Hom-$\otimes$-adjointness, and the second holds true because M_R is finitely presented and ${}_S V$ is injective, see e.g. [20, 3.2.11]. Finally, $\operatorname{End}_S V$ is local by 1.3.4. □

In general, the local dual M^+ of a finitely presented module M need not be finitely presented.

Example 2.3.2. [31, p.1936] Take the upper triangular matrix ring $R = \begin{pmatrix} G & {}_G G_F \\ 0 & F \end{pmatrix}$ where $F \subset G$ is a field extension with $\dim G_F < \infty$ and $\dim_F G = \infty$. The ring R is artinian hereditary, see [16, III,2.1]. Set $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $P_1 = e_1 R$ is an indecomposable projective right R-module with $\operatorname{End} P_1 \cong G$, and $P_1^+ = {}_R \operatorname{Hom}_G(P_1, G)$ is an indecomposable injective left R-module which is not finitely generated. Moreover, using results of W. Zimmermann [34], one can show that there is no almost split sequence in mod$R$ starting at P_1.

2.4 Existence of almost split sequences

Theorem 2.4.1 (Auslander 1978). *Let ${}_R C$ be a finitely presented non-projective module with local endomorphism ring. Then there is an almost split sequence $0 \to A \to B \to C \to 0$ in R Mod, and $A = (\operatorname{Tr} C)^+$.*

This result was first proven in [10]. We here present a proof from [34].

Proof: First of all, since $\operatorname{End} C$ local, also $T = \operatorname{End} \operatorname{Tr} C$ is local by Proposition 2.2.1. Let ${}_T U$ be an injective envelope of ${}_T T/J(T)$, and ${}_R A = \operatorname{Hom}_T(\operatorname{Tr} C, {}_T U) = (\operatorname{Tr} C)^+$. Then by Lemma 2.3.1 also $S = \operatorname{End}_R A \cong \operatorname{End}_T U$ is local.
Let $0 \to K \xrightarrow{i} P \xrightarrow{p} C \to 0$ be a projective cover. The strategy of the proof now consists

in constructing a map $a\colon K \to A$, such that the push-out

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "a"'] & P \arrow[r, "p"] \arrow[d, dotted] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, dotted, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

yields an almost split sequence.

1. Step: Let us first establish the required properties of the map a.
If $a \in \mathrm{Hom}_R(K, A)$ satisfies

(i) a does not factor through i, but

(ii) the composition sa factors through i for any $s \in J(S)$,

then the sequence $0 \to A \underset{f}{\to} B \underset{g}{\to} C \to 0$ constructed as above is almost split in RMod. In fact, f is not a split monomorphism by condition (i). Moreover, we claim that every $h : A \longrightarrow X$ which is not a split monomorphism factors through f. To this end, we consider the push-out diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, dotted] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, dotted, "f'"'] & B' \arrow[r, "g'"'] & C \arrow[r] & 0 \end{tikzcd}$$

Now, if $l \in \mathrm{Hom}_R(X, A)$, then $lh \in J(S)$, and lha factors through i by condition (ii), so by the push-out property l factors through f'. This shows that the sequence $0 \to \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(B', A) \xrightarrow{(f',A)} \mathrm{Hom}_R(X, A) \to 0$ is exact. Since ${}_TU$ is an injective cogenerator, using 1.4.2 together with Hom-$\otimes$-adjointness, we obtain that also $0 \to \mathrm{Tr}\, C \otimes_R X \to \mathrm{Tr}\, C \otimes_R B' \to \mathrm{Tr}\, C \otimes_R C \to 0$ is exact.
Then from the long exact sequence of Lemma 2.2.2

$$0 \to \mathrm{Hom}_R(C, X) \to \mathrm{Hom}_R(C, B') \xrightarrow[\mathrm{Hom}_R(C,g')]{} \mathrm{End}_R\, C \xrightarrow{\delta} \mathrm{Tr}\, C \otimes_R X \hookrightarrow \mathrm{Tr}\, C \otimes_R B'$$

we infer that $\mathrm{Hom}_R(C, g')$ is an epimorphism, hence g' a split epimorphism. But then the bottom row in the second diagram above is split exact, and h factors through f.

2 Step: Existence of a.
In the long exact sequence of Lemma 2.2.2

$$0 \to \mathrm{Hom}_R(C, K) \to \mathrm{Hom}_R(C, P) \xrightarrow[\mathrm{Hom}_R(C,p)]{} \mathrm{End}_R\, C \xrightarrow{\delta} \mathrm{Tr}\, C \otimes_R K \xrightarrow{\mathrm{Tr}\, C \otimes i} \mathrm{Tr}\, C \otimes_R P$$

we have $\mathrm{Ker}\, \delta = \mathrm{Im}\, \mathrm{Hom}_R(C, p) = P(C, C)$. So, by Proposition 2.2.1 (1) we have an exact sequence of T-modules

$$\begin{tikzcd} 0 \arrow[r] & T/P(\mathrm{Tr}\, C, \mathrm{Tr}\, C) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Tr}\, C \otimes_R K \arrow[r, "\mathrm{Tr}\, C \otimes i"] & \mathrm{Tr}\, C \otimes_R P \\ & \mathrm{End}_R\, C/P(C, C) \arrow[ur, "\overline{\delta}"'] & & \end{tikzcd}$$

Now applying $\mathrm{Hom}_T(-, {}_TU)$, and using Hom-$\otimes$-adjointness together with the isomorphism $S \cong \mathrm{End}_T\, U$, we obtain an exact sequence of right S-modules

$$\mathrm{Hom}_R(P, A) \xrightarrow{\mathrm{Hom}_R(i,A)} \mathrm{Hom}_R(K, A) \xrightarrow{\widetilde{\delta}} \mathrm{Hom}_T(T/P(\mathrm{Tr}\, C, \mathrm{Tr}\, C), {}_TU) \to 0$$

Note that $T/P(\mathrm{Tr}\, C, \mathrm{Tr}\, C) \neq 0$ as $\mathrm{Tr}\, C$ is not projective. Hence the left T-module $\mathrm{Hom}_T(T/P(\mathrm{Tr}\, C, \mathrm{Tr}\, C), {}_TU)$ is a non-zero submodule of $\mathrm{Hom}_T(T, U) \cong {}_TU$, and by Remark 1.3.4 it contains a simple module ${}_TM$ isomorphic to ${}_TT/J(T)$. Let us choose a generator γ of ${}_TM$ and let $a \in \mathrm{Hom}_R(K, A)$ be a preimage of γ under $\widetilde{\delta}$. Then a satisfies condition (i) since $\widetilde{\delta}(a) \neq 0$. Moreover, using that $\mathrm{Soc}_T\, U \cdot J(\mathrm{End}_T\, U) = 0$, it is not hard to infer that a also satisfies (ii). $\square$

Remark 2.4.2. Let R be an arbitrary ring. Then the construction of $\mathrm{Tr}\, M$ depends on the initial choice of the projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$. For details, we refer to [10, pp.13]. In particular, if $\mathrm{Tr}_1\, M$ and $\mathrm{Tr}_2\, M$ are obtained by using different projective presentations, then we can only say that there are projective modules P, Q such that $\mathrm{Tr}_1\, M \oplus P \cong \mathrm{Tr}_2\, M \oplus Q$. This shows that 2.2.1 (2) is no longer valid.

For the proof of Theorem 2.4.1 one then proceeds as follows. Let ${}_RC$ be a finitely presented non-projective module with local endomorphism ring. Fix a projective presentation of C, construct $\mathrm{Tr}\, C$, and set $T = \mathrm{End}\,\mathrm{Tr}\, C$. Since $P(C, C) \subset J(\mathrm{End}_R\, C)$, by 2.2.1 (1) there is a surjective ring homomorphism $T \to \mathrm{End}_R\, C/J(\mathrm{End}_R\, C)$. So, $\mathrm{End}_R\, C/J(\mathrm{End}_R\, C)$ can be viewed as simple left T-module. Choose an injective envelope ${}_TU$ of ${}_T\,\mathrm{End}_R\, C/J(\mathrm{End}_R\, C)$ and set ${}_RA = \mathrm{Hom}_T(\mathrm{Tr}\, C, {}_TU)$. Then one checks as in 2.3.1 that $S = \mathrm{End}_R\, A \cong \mathrm{End}\, {}_TU$ is local. Moreover, by [10, I, 3.4] there is an isomorphism

$$\mathrm{Hom}_T\left(\underline{\mathrm{Hom}}_R\,(C, C),\ {}_TU\right) \cong \mathrm{Ext}^1_R\,(C, A)$$

This isomorphism is used to construct the almost split sequence $0 \to A \to B \to C \to 0$ in R Mod, see [10, I, 3.4 and II, 5.1].

Further existence results for almost split sequences over rings can be found in [8, 9, 34, 36, 37, 3, 31, 4, 24].

3 ARTIN ALGEBRAS

Definition. An *Artin algebra* is an algebra over a commutative artinian ring k which, in addition, is a finitely generated k-module.

Throughout this chapter, we fix a commutative artinian ring k and an Artin algebra Λ over k. We start out by collecting some well-known properties of Artin algebras.

3.1 First properties of Artin Algebras

(1) Λ is an artinian ring.

(2) [16, I.1 and I.3.2] All finitely generated Λ-modules have finite length.

(3) Every finitely generated Λ-module M has an *indecomposable decomposition* $M = \bigoplus_{i=1}^{n} M_i$ with $\operatorname{End}_\Lambda M_i$ local for all $1 \leq i \leq n$ (Theorem of Krull-Schmidt).

(4) If M, N are finitely generated Λ-modules, then $\operatorname{Hom}_\Lambda(M, N)$ is a finitely generated k-module via the multiplication

$$\alpha \cdot f \colon m \mapsto \alpha f(m) \quad \text{for} \quad \alpha \in k,\ f \in \operatorname{Hom}_\Lambda(M, N)$$

In particular, $\operatorname{End}_\Lambda N$ and $(\operatorname{End}_\Lambda N)^{\mathrm{op}}$ are again Artin k-algebras, and N is a Λ-$(\operatorname{End}_\Lambda N)^{\mathrm{op}}$-bimodule via the multiplication

$$n \cdot s := s(n) \quad \text{for} \quad n \in N,\ s \in \operatorname{End}_\Lambda N$$

Moreover, $\operatorname{Hom}_\Lambda(M, N)$ is an $\operatorname{End} N$-$\operatorname{End} M$-bimodule which has finite length on both sides.

(5) [2, 15.16, 15.19, 27.6] The Jacobson radical $J = J(\Lambda)$ is nilpotent, and Λ/J is semisimple. In particular, Λ is semiperfect, and there are orthogonal idempotents $e_1, \ldots, e_n \in \Lambda$ such that $1 = \sum_{i=1}^{n} e_i$, and $e_i \Lambda e_i$ is a local ring for every $1 \leq i \leq n$. Note that ${}_\Lambda\Lambda = \bigoplus_{i=1}^{n} \Lambda e_i$ and ${}_\Lambda\Lambda/J \cong \bigoplus_{i=1}^{n} \Lambda e_i/Je_i$ are indecomposable decompositions.

(6) Λ is Morita equivalent to a *basic* Artin algebra, that is, to an Artin algebra S with the property that ${}_S S$ is a direct sum of pairwise nonisomorphic projectives, or equivalently, $S/J(S)$ is a product of division rings, see [2, p. 309] or [16, II.2].

From now on, we will assume that Λ is basic. Then

$$\Lambda e_1, \ldots, \Lambda e_n$$

are representatives of the isomorphism classes of the indecomposable projectives in $\Lambda \operatorname{Mod}$, and

$$\Lambda e_1/Je_1, \ldots, \Lambda e_n/Je_n$$

are representatives of the isomorphism classes of the simples in $\Lambda \operatorname{Mod}$. In order to describe the indecomposable injectives, we have to introduce the following construction.

3.2 The duality

Let ${}_kI$ be an injective envelope of $k/J(k)$. Then ${}_kI$ is an injective cogenerator of $\operatorname{Mod} k$.

Proposition 3.2.1. [16, II.3]

(1) ${}_kI$ *is finitely generated.*

(2) If ${}_kM$ *is a finitely generated module, then* $\operatorname{Hom}_k(M, I)$ *is a finitely generated* k*-module of the same length.*

(3) If $M \in \Lambda \operatorname{mod}$*, then* $\operatorname{Hom}_k(M, I)$ *is a finitely generated right* Λ*-module via the multiplication*

$$f \cdot r : m \mapsto f(rm) \quad \textit{for} \quad f \in \operatorname{Hom}_k(M, I), r \in \Lambda$$

(4) The evaluation map $c : M \to \operatorname{Hom}_k(\operatorname{Hom}_k(M, I), I)$*, given by* $c(m)(f) = f(m)$ *is a functorial isomorphism of* Λ*-modules.*

Proof: We sketch the argument for (4): Of course, c is a monomorphism of k-modules, and it is an isomorphism since the lengths of M and $\operatorname{Hom}_k(\operatorname{Hom}_k(M, I), I)$ coincide by (2). Further, c is easily seen to be Λ-linear. □

So, the functor $D : \Lambda \operatorname{mod} \longrightarrow \operatorname{mod} \Lambda$, $M \mapsto \operatorname{Hom}_k(M, I)$ is a duality. We are now going to see that

$$D(e_1\Lambda), \ldots, D(e_n\Lambda)$$

are representatives of the isomorphism classes of the indecomposable injectives in $\Lambda \operatorname{Mod}$.

Lemma 3.2.2. *Let* $1 \le i \le n$*. Then*

(1) $D(\Lambda e_i/Je_i) \cong e_i\Lambda/e_iJ$.

(2) ${}_\Lambda D(e_i\Lambda)$ *is an injective envelope of* $\Lambda e_i/Je_i$*.*

(3) ${}_\Lambda D(\Lambda_\Lambda)$ *is an injective envelope of* ${}_\Lambda\Lambda/J$ *and an injective cogenerator of* ΛMod*.*

Proof: We only sketch the arguments and refer to [16, II.4] for more details.
(1) Note that $D(\Lambda e_i/Je_i) \cdot J = 0$ because for $f \in D(\Lambda e_i/Je_i)$, $r \in J$, $x \in \Lambda$, we have $(f \cdot r)(\overline{xe_i}) = f(r \cdot \overline{xe_i}) = f(\overline{rxe_i}) = 0$. Hence $D(\Lambda e_i/Je_i)$ is a Λ/J-module, thus it is semisimple. As $\operatorname{End} D(\Lambda e_i/Je_i) \cong (\operatorname{End} \Lambda e_i/Je_i)^{\mathrm{op}}$ by means of the duality D, and $\operatorname{End} \Lambda e_i/Je_i$ is local, we conclude that $D(\Lambda e_i/Je_i)$ is simple. It remains to prove that $D(\Lambda e_i/Je_i) \cong e_i\Lambda/e_iJ$. For $0 \neq \overline{e_i} \in \Lambda e_i/Je_i$ there exists $\alpha \in \operatorname{Hom}_k(\Lambda e_i/Je_i, I) = D(\Lambda e_i/Je_i)$ such that $\alpha(\overline{e_i}) \neq 0$. Then $0 \neq \alpha e_i \in D(\Lambda e_i/Je_i)_\Lambda$ since $\alpha e_i(\overline{e_i}) = \alpha(e_i\overline{e_i}) = \alpha(\overline{e_i})$. So, we can define a non-zero Λ-linear map $f \colon e_i\Lambda \to D(\Lambda e_i/Je_i)$, $e_ir \mapsto \alpha e_ir$. Then f must be a Λ-epimorphism with $\operatorname{Ker} f = e_iJ$, and the claim is proven.
(2) The onto map $f : e_i\Lambda \twoheadrightarrow D(\Lambda e_i/Je_i)$ is a projective cover. Hence $\Lambda e_i/Je_i \cong D^2(\Lambda e_i/Je_i) \hookrightarrow D(e_i\Lambda)$ is an injective envelope.
(3) is an immediate consequence of (2) and 1.4.1. □

As a further consequence, we see that D and the local duality coincide for Artin algebras.

Corollary 3.2.3. *Let $M \in \operatorname{mod}\Lambda$ be indecomposable. Then $M^+ \cong D(M)$.*

Proof: Since Λ is an Artin algebra, also the endomorphism ring S of M_Λ is an Artin algebra by 3.1 (4), thus $D(S_S)$ is an injective envelope of ${}_SS/J(S)$, see Lemma 3.2.2. So, we choose ${}_SV = D(S_S)$ and $M^+ = \operatorname{Hom}_S(M, D(S_S)) \cong \operatorname{Hom}_k(S \otimes_S M, I) \cong D(M)$. $\square$

3.3 The Auslander-Reiten translation

We now combine the transpose with the duality D.

Denote by $\Lambda\mathrm{mod}_{\mathcal{I}}$ the full subcategory of Λmod consisting of the modules without non-zero injective summands. For $M, N \in \Lambda\operatorname{mod}$ consider further the subgroup
$I(M,N) = \{f \in \operatorname{Hom}_\Lambda(M,N) \mid f \text{ factors through an injective module}\} \leq \operatorname{Hom}_\Lambda(M,N)$, set $\overline{\operatorname{Hom}}_\Lambda(M,N) = \operatorname{Hom}_\Lambda(M,N)/I(M,N)$, and let $\Lambda\,\overline{\operatorname{mod}}$ be the category with the same objects as $\Lambda\operatorname{mod}$ and morphisms $\overline{\operatorname{Hom}}_\Lambda(M,N)$.

Proposition 3.3.1. *(1) The duality D induces a duality $\Lambda\,\underline{\operatorname{mod}} \to \overline{\operatorname{mod}}\,\Lambda$.*

(2) The composition $\tau = D\operatorname{Tr}\colon \Lambda\,\underline{\operatorname{mod}} \to \Lambda\,\overline{\operatorname{mod}}$ is an equivalence with inverse $\tau^- = \operatorname{Tr} D\colon \Lambda\,\overline{\operatorname{mod}} \to \Lambda\,\underline{\operatorname{mod}}$.

Example: Let $\Lambda = kA_3$ be the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$. We compute $\tau\, S_2$. The indecomposable projectives are $P_1,\ P_2 = JP_1,\ P_3 = S_3 = JP_2$, and the indecomposable injectives are $I_1 = S_1 = I_2/S_2,\ I_2 = I_3/S_3,\ I_3 = P_1$. We compute τS_2. Applying the functor * on $0 \to P_3 \to P_2 \to S_2 \to 0$ we obtain $0 \to e_2\Lambda \to e_3\Lambda \to \operatorname{Tr} S_2 \to 0$, and applying D we get $0 \to \tau S_2 \to I_3 \to I_2 \to 0$, showing that $\tau S_2 = S_3$.

The functor τ is called *Auslander-Reiten translation* and plays a fundamental role in the computation of almost split sequences. In fact, as a direct consequence of Theorem 2.4.1, we obtain the following result.

Theorem 3.3.2 (Auslander-Reiten 1975). *(1) For every finitely generated indecomposable non-projective module M there is an almost split sequence $0 \to \tau M \to B \to M \to 0$ in Λ Mod with finitely generated modules.*

(2) For every finitely generated indecomposable non-injective module M there is an almost split sequence $0 \to M \to E \to \tau^- M \to 0$ in Λ Mod with finitely generated modules.

Proof: For (1), combine Theorem 2.4.1 with Lemma 3.2.3.
(2) By (1) there is an almost split sequence ending at $D(M)$. Applying D on it, we obtain an almost split sequence in Λmod with end-term $D(\tau\, D(M)) \cong \operatorname{Tr} D(M) = \tau^- M$. It is

even an almost split sequence in ΛMod. This follows for instance from [34, Proposition 3] by using that M is pure-injective. In fact, all finitely generated Λ-modules are endofinite, i. e. they have finite length over their endomorphism ring, see Property (4) in 3.1, and it is well known that endofinite modules are pure-injective (even Σ-pure-injective). □

The Theorem above was originally proved in [14] using the Auslander-Reiten formula that we are going to discuss next. Another proof, using functorial arguments, is given in [11].

3.4 The Auslander-Reiten formula

Theorem 3.4.1 (Auslander-Reiten 1975). *Let A, C be Λ-modules with $A \in \Lambda\, mod_{\mathcal{P}}$. Then there are natural k-isomorphisms*

$$\begin{array}{rl} \text{(I)} & \overline{\mathrm{Hom}}_\Lambda\,(C, \tau\, A) \cong D\,\mathrm{Ext}^1_\Lambda\,(A, C) \\ \text{(II)} & D\,\underline{\mathrm{Hom}}_\Lambda\,(A, C) \cong \mathrm{Ext}^1_\Lambda\,(C, \tau\, A) \end{array}$$

These formulae were first proven in [14], see also [26]. A more general version of (II), valid for arbitrary rings, is proven in [10, I, 3.4], cf. 2.4.2. We here sketch a proof from [23].

Following [16, IV.4], we associate to any short exact sequence $\mathfrak{E} : 0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \longrightarrow 0$ and any module M the *covariant defect* $\mathfrak{E}_*(M)$ and the *contravariant defect* $\mathfrak{E}^*(M)$ defined as the cokernels $\mathfrak{E}_*(M) = \mathrm{Coker}\,\mathrm{Hom}(\iota, M)$, and $\mathfrak{E}^*(M) = \mathrm{Coker}\,\mathrm{Hom}(M, \pi)$.

Lemma 3.4.2. *If $\mathfrak{E} : 0 \longrightarrow X \xrightarrow{i} Y \longrightarrow Z \longrightarrow 0$ is an exact sequence and $A \in \Lambda\, mod_{\mathcal{P}}$, then there is a k-isomorphism $\mathfrak{E}_*(\tau\, A) \cong D\,\mathfrak{E}^*(A)$.*

Proof: By Lemma 2.2.2 we have an exact sequence $0 \to \mathfrak{E}^*(A) \to \mathrm{Tr}\, A \otimes X \xrightarrow{\mathrm{Tr}\, A \otimes i} \mathrm{Tr}\, A \otimes Y$ of abelian groups, and it is easy to check that it is also an exact sequence of k-modules. Now applying the duality D and using Hom-$\otimes$-adjointness, we obtain a commutative diagram

$$\begin{tikzcd} D(\mathrm{Tr}\, A \otimes Y) \arrow[r] \arrow[d, "\text{adj. } \cong"'] & D(\mathrm{Tr}\, A \otimes X) \arrow[r] \arrow[d, "\text{adj. } \cong"'] & D\,\mathfrak{E}^*(A) \arrow[r] \arrow[d, "\cong"'] & 0 \\ \mathrm{Hom}_R(Y, \tau A) \arrow[r, "\mathrm{Hom}(i, \tau A)"] & \mathrm{Hom}_R(X, \tau A) \arrow[r] & \mathfrak{E}_*(\tau A) \arrow[r] & 0 \end{tikzcd}$$

inducing the desired isomorphism. □

Proof of Theorem 3.4.1: (I) Take an injective envelope $\mathfrak{E} : 0 \to C \xrightarrow{i} I \to Z \to 0$ of C and use 1.7.2. Then $\mathfrak{E}^*(A) \cong \mathrm{Ext}^1_\Lambda(A, C)$ and $\mathfrak{E}_*(\tau A) = \mathrm{Coker}\,\mathrm{Hom}(i, \tau\, A) = \overline{\mathrm{Hom}}_\Lambda(C, \tau\, A)$ because $I(C, \tau\, A) = \mathrm{Im}\,\mathrm{Hom}(i, \tau\, A)$. So Lemma 3.4.2 yields the claim.
(II) is proven dually.□

3.5 The Auslander-Reiten quiver

We now use almost split maps to study the category $\Lambda\,\mathrm{mod}$. First of all, we have to take care of the indecomposable projective and the indecomposable injective modules.

Proposition 3.5.1.

(1) If P indecomposable projective, then the embedding $g : \mathrm{Rad}\, P \hookrightarrow P$ is minimal right almost split in ΛMod.

(2) If I indecomposable injective, then the natural surjection $f : I \to I/\,\mathrm{Soc}\, I$ is minimal left almost split in ΛMod.

Proof: (1) Note that $\mathrm{Rad}\, P = JP$ and P/JP is simple [16, I,3.5 and 4.4], so $\mathrm{Rad}\, P$ is the unique maximal submodule of P. Thus, if $h\colon X \to P$ is not a split epimorphism, then it is not an epimorphism and therefore $\mathrm{Im} h$ is contained in $\mathrm{Rad}\, P$. Hence g is right almost split. Moreover, g is right minimal since every $t \in \mathrm{End}\,\mathrm{Rad}\, P$ with $gt = g$ has to be a monomorphism, hence an isomorphism.
(2) is proven with dual arguments. $\square$

Let now $M \in \Lambda\,\mathrm{mod}$ be indecomposable. From 3.5.1 and 3.3.2 we know that there is a map $g : B \to M$ with $B \in \Lambda\,\mathrm{mod}$ which is minimal right almost split, and there is a map $f : M \to N$ with $N \in \Lambda\,\mathrm{mod}$ which is minimal left almost split. Consider decompositions

$$B = \bigoplus_{i=1}^{n} B_i \quad \text{and} \quad N = \bigoplus_{k=1}^{m} N_k$$

into indecomposable modules B_i and N_k. The maps

$$g|B_i \quad \text{and} \quad \mathrm{pr}_{N_k} f$$

are characterized by the property of being irreducible in the following sense, see [16, V.5.3].

Definition. A homomorphism $h\colon M \to N$ between indecomposable modules M, N is said to be *irreducible* if h is not an isomorphism, and in any commutative diagram

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "\alpha"'] & & N \\ & Z \arrow[ur, "\beta"'] & \end{tikzcd}$$

either α is a split monomorphism or β is a split epimorphism.

In particular, if h is irreducible, then $h \neq 0$ is either a monomorphism or an epimorphism.

Irreducible morphisms can also be described in terms of the following notion, which is treated in detail in [16, V.7].

Definition. For two modules $M, N \in \Lambda\,\mathrm{mod}$, we define the *radical* of $\mathrm{Hom}_\Lambda(M, N)$ by

$$r(M,N) = \{\, f \in \mathrm{Hom}_\Lambda(M,N) \mid \text{ for each indecomposable module } Z \in \Lambda\,\mathrm{mod}, \text{ every composition of the form } Z \to M \xrightarrow{f} N \to Z \text{ is a non-isomorphism}\}$$

For $n \in \mathbb{N}$ set

$$r^n(M,N) = \{f \in \mathrm{Hom}_\Lambda(M,N) \mid f = gh \text{ with } h \in r(M,X), g \in r^{n-1}(X,N), X \in \Lambda\,\mathrm{mod}\}$$

Proposition 3.5.2. *If $M, N \in \Lambda$ mod are indecomposable modules, then*

(1) $r(M,N)$ consists of the non-isomorphisms in $\mathrm{Hom}_\Lambda(M,N)$, so $r(M,M) = J(\mathrm{End}_\Lambda M)$.

(2) $f \in \mathrm{Hom}_\Lambda(M,N)$ is irreducible if and only if $f \in r(M,N) \setminus r^2(M,N)$.

Since the irreducible morphisms arise as components of minimal right almost split maps and minimal left almost split maps, we obtain the following result.

Proposition 3.5.3. *Let M, N be indecomposable modules with an irreducible map $M \to N$. Let $g\colon B \to N$ be a minimal right almost split map, and $f\colon M \to B'$ a minimal left almost split map. Then there are integers $a, b > 0$ and modules $X, Y \in \Lambda\,\mathrm{mod}$ such that*

(1) $B \cong M^a \oplus X$ and M is not isomorphic to a direct summand of X,

(2) $B' \cong N^b \oplus Y$ and N is not isomorphic to a direct summand of Y.

Moreover,

$$a = \dim r(M,N)/r^2(M,N)_{\mathrm{End}\,M/J(\mathrm{End}\,M)}$$

$$b = \dim_{\mathrm{End}\,N/J(\mathrm{End}\,N)} r(M,N)/r^2(M,N)$$

Thus $a = b$ provided that k is an algebraically closed field.

Proof: The $\mathrm{End}\,N$-$\mathrm{End}\,M$-bimodule structure on $\mathrm{Hom}_\Lambda(M,N)$ from 3.1 (4) induces an $\mathrm{End}\,N/J(\mathrm{End}\,N)$-$\mathrm{End}\,M/J(\mathrm{End}\,M)$-bimodule structure on $r(M,N)/r^2(M,N)$. Now $\mathrm{End}\,N/J(\mathrm{End}\,N)$ and $\mathrm{End}\,M/J(\mathrm{End}\,M)$ are skew fields. Consider the minimal right almost split map $g : B \longrightarrow N$. If $g_1, \ldots, g_a : M \to N$ are the different components of $g\,|_{M^a}$, then $\overline{g_1}, \ldots, \overline{g_a}$ is the desired $\mathrm{End}\,M/J(\mathrm{End}\,M)$-basis. Dual considerations yield an $\mathrm{End}\,N/J(\mathrm{End}\,N)$-basis of $r(M,N)/r^2(M,N)$. For details, we refer to [16, VII.1].
Finally, since $\mathrm{End}\,N/J(\mathrm{End}\,N)$ and $\mathrm{End}\,M/J(\mathrm{End}\,M)$ are finite dimensional skew field extensions of k, we conclude that $a = b$ provided that k is an algebraically closed field.□

Definition. The *Auslander-Reiten quiver (AR-quiver)* $\Gamma = \Gamma(\Lambda)$ of Λ is constructed as follows. The set of vertices Γ_0 consists of the isomorphism classes $[M]$ of finitely generated indecomposable Λ-modules. The set of arrows Γ_1 is given by the following rule: set an arrow

$$[M] \xrightarrow{(a,b)} [N]$$

if there is an irreducible map $M \to N$ with (a,b) as above in Proposition 3.5.3.

Observe that Γ is a locally finite quiver (i.e. there exist only finitely many arrows starting or ending at each vertex) with the simple projectives as sources and the simple injectives as sinks. Moreover, if k is an algebraically closed field, we can drop the valuation by drawing multiple arrows.

Proposition 3.5.4. *Consider an arrow from* Γ

$$[M] \xrightarrow{(a,b)} [N]$$

(1) Translation of arrows*:*
If M, N *are indecomposable non-projective modules, then in* Γ *there is also an arrow*

$$[\tau M] \xrightarrow{(a,b)} [\tau N]$$

(2) Meshes*:*
If N *is an indecomposable non-projective module, then in* Γ *there is also an arrow*

$$[\tau N] \xrightarrow{(b,a)} [M]$$

Proof: (1) can be proven by exploiting the properties of the equivalence $\tau = D\,\mathrm{Tr}\colon \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ from 3.3.1. In fact, the following is shown in [15, 2.2]: If N is an indecomposable non-projective module with a minimal right almost split map $g : B \longrightarrow N$, and $B = P \oplus B'$ where P is projective and $B' \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$ has non non-zero projective summand, then there are an injective module $I \in \Lambda\,\mathrm{mod}$ and a minimal right almost split map $g' : I \oplus \tau B' \longrightarrow \tau N$ such that $\tau(\underline{g}) = \overline{g'}$. Now the claim follows easily.
(2) From the almost split sequence $0 \longrightarrow \tau N \longrightarrow M^a \oplus X \longrightarrow N \longrightarrow 0$ we immediately infer that there is an arrow $[\tau N] \xrightarrow{(b',a)} [M]$ in Γ. So we only have to check $b' = b$. We know from 3.5.3 that $b' = \dim r(\tau N, M)/r^2(\tau N, M)_{\mathrm{End}\,\tau N/J(\mathrm{End}\,\tau N)}$. Now, the equivalence $\tau = D\,\mathrm{Tr}\colon \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ from 3.3.1 defines an isomorphism $\underline{\mathrm{End}}_\Lambda N \cong \overline{\mathrm{End}}_\Lambda \tau N$, which in turn induces an isomorphism $\mathrm{End}\,N/J(\mathrm{End}\,N) \cong \mathrm{End}\,\tau N/J(\mathrm{End}\,\tau N)$. Moreover, using 3.5.3 and denoting by ℓ the length of a module over the ring k, it is not difficult to verify that $b' \cdot \ell(\mathrm{End}\,\tau N/J(\mathrm{End}\,\tau N)) = a \cdot \ell(\mathrm{End}\,M/J(\mathrm{End}\,M)) = \ell(r(M,N)/r^2(M,N)) = b \cdot \ell(\mathrm{End}\,N/J(\mathrm{End}\,N))$, which implies $b' = b$. $\square$

Example: Let $\Lambda = K\mathbb{A}_3$ be the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$.
Λ is a serial algebra. The module $I_3 \cong P_1$ has the composition series $P_1 \supset P_2 \supset P_3 \supset 0$. Furthermore, $I_3/\operatorname{Soc} I_3 \cong I_2$, and $I_2/\operatorname{Soc} I_2 \cong I_1$. So, there are only three almost split sequences, namely $0 \to P_3 \to P_2 \to S_2 \to 0$, and $0 \to P_2 \to S_2 \oplus P_1 \to I_2 \to 0$, and

$0 \to S_2 \to I_2 \to I_1 \to 0$. Hence $\Gamma(\Lambda)$ has the form

$$\begin{tikzcd}
 & & P_1 \cong I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & \cdots\cdots\cdots & I_2 \arrow[dr] & \\
P_3 \arrow[ur] & \cdots\cdots\cdots & S_2 \arrow[ur] & \cdots\cdots\cdots & I_1
\end{tikzcd}$$

4 ARTIN ALGEBRAS OF FINITE REPRESENTATION TYPE

Definition. An Artin algebra Λ is said to be *of finite representation type* if there are only finitely many isomorphism classes of finitely generated indecomposable left Λ-modules. This is equivalent to the fact that there are only finitely many isomorphism classes of finitely generated indecomposable right Λ-modules.

Artin algebras of finite representation type are completely described by their AR-quiver.

Theorem 4.0.5 (Auslander 1974, Ringel-Tachikawa 1973). *Let Λ be an Artin algebra of finite representation type. Then every module is a direct sum of finitely generated indecomposable modules. Moreover, every non-zero non-isomorphism $f\colon X \to Y$ between indecomposable modules X, Y is a sum of compositions of irreducible maps between indecomposable modules.*

Proof: The first statement was proven independently in [6] and [32, 29]. Let us briefly explain a further proof due to W. Zimmermann which uses the notion of a pure-exact sequence, for details see [22, Theorem 14]. We take a complete irredundant set $M_1, \dots, M_n$ of finitely generated indecomposable left modules and consider $M = \bigoplus_{i=1}^n M_i$. It is enough to show that every left module A is isomorphic to a direct summand of a direct sum of copies of M. First of all, A is a direct limit of finitely presented modules. Thus there are a set I and a pure-exact sequence

$$(*) \qquad\qquad 0 \to K \to M^{(I)} \to A \to 0$$

Since M is finitely presented, applying $\mathrm{Hom}(M, -)$ we obtain an exact sequence

$$(**) \qquad\qquad 0 \to \mathrm{Hom}(M, K) \to \mathrm{Hom}(M, M^{(I)}) \to \mathrm{Hom}(M, A) \to 0$$

of left modules over $S = (\mathrm{End}\, M)^{op}$. Moreover, $(**)$ is even pure-exact, as we can easily verify by using Hom-$\otimes$-adjointness. Note that the middle-term of $(**)$ is isomorphic to $S^{(I)}$, and the ring S is perfect. It follows that $(**)$ splits. But then also the sequence

$$0 \to M \otimes_S \mathrm{Hom}(M, K) \to M \otimes_S \mathrm{Hom}(M, M^{(I)}) \to M \otimes_S \mathrm{Hom}(M, A) \to 0$$

splits. Now remember that M is a generator of ΛMod, which implies $M \otimes_S \operatorname{Hom}(M, X) \cong X$ for any left module X. So, we conclude that our original sequence $(*)$ splits, and A has the required property.
We now sketch the proof of the second statement, and refer to [16, V.7] for details. Take a non-zero non-isomorphism $f\colon X \to Y$ between indecomposable modules X, Y. If $g : B \to Y$ is minimal right almost split, and $B = \bigoplus_{i=1}^{n} B_i$ with indecomposable modules B_i, then we can factor f as follows:

$$\begin{tikzcd} B_i \arrow[r, hook] & B \arrow[r, "g"'] & Y \\ & X \arrow[ul, dotted, "h_i"] \arrow[u, dotted, "h"'] \arrow[ur, "f"'] & \end{tikzcd} \qquad f = gh = \sum_{i=1}^{n} g|_{B_i} \circ h_i \quad \text{with irreducible maps } g|_{B_i}.$$

Moreover, if h_i is not an isomorphism, we can repeat the argument. But this procedure will stop eventually, because we know from the assumption that there is a bound on the length of nonzero compositions of non-isomorphisms between indecomposable modules (e. g. by the Lemma of Harada and Sai, see [16, VI.1.3]). So after a finite number of steps we see that f has the desired shape. $\square$

Remark 4.0.6. (1) In [7], Auslander also proved the converse of the first statement in Theorem 4.0.5. Combining this with a result of Zimmermann-Huisgen [35], we obtain that an Artin algebra is of finite representation type if and only if every left module is a direct sum of indecomposable left modules. The question whether the same holds true for any left artinian ring is known as the *Pure-Semisimple Conjecture*. We refer e. g. to [22] for a discussion of this problem.
(2) A proof for the converse of the second statement in Theorem 4.0.5 has been announced by Alvares-Coelho [1].

5 HEREDITARY ARTIN ALGEBRAS

As we have seen in the last section, the information on the indecomposable finite length modules is encoded in the AR-quiver. We now describe its shape in the hereditary case (thus in particular for path algebras).

5.1 Hereditary rings

We briefly recall the notion of a hereditary ring. For a more detailed treatment, we refer to [30, p. 120], or [19, 3.7].

Definition. A ring R is *left hereditary* if it satisfies one of the following equivalent conditions:

(a) Every left ideal of R is projective.

(b) Every submodule of a projective left R-module is projective.

(c) Every factor module of an injective left R-module is injective.

A ring which is left and right hereditary is called *hereditary*.

Example: Let Q be a finite quiver without oriented cycles, and let $\Lambda = kQ$ be the path algebra of Q over a field k. Let Q_0 be the set of vertices of Q.
(1) The Jacobson radical $J = J(\Lambda)$ is the ideal of Λ generated by all arrows.
(2) The empty paths $e_i = (i\|i)$, $i \in Q_0$, are orthogonal idempotents as in 3.1(5).
(3) For each vertex $i \in Q_0$, the paths starting in i form a k-basis of Λe_i.
(4) Let $i \in Q_0$ be a vertex, and denote by $\alpha_1, \ldots, \alpha_t$ the arrows $i\bullet \xrightarrow{\alpha_k} \bullet\, j_k$ of Q which start in i. Then $Je_i = \bigoplus_{k=1}^{t} \Lambda e_{j_k}\alpha_k$. Hence $Je_i \cong \bigoplus_{k=1}^{t} \Lambda e_{j_k}$ is projective for each $i \in Q_0$. In particular, Λ is hereditary.

Proposition 5.1.1. Let Λ be a hereditary Artin algebra.

(1) Let ${}_\Lambda M$ be an indecomposable Λ-module. Then every non-zero map $f \in \mathrm{Hom}_\Lambda(M, P)$ with ${}_\Lambda P$ projective is a monomorphism.

(2) If P is an indecomposable projective Λ-module, then $\mathrm{End}_\Lambda P$ is a skew field.

(3) If $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, then $\mathrm{Hom}_\Lambda(M, P) = 0$ for all projective modules ${}_\Lambda P$.

(4) Tr induces a duality $\Lambda\,\mathrm{mod}_{\mathcal{P}} \to \mathrm{mod}\,\Lambda_{\mathcal{P}}$ which is isomorphic to the functor $\mathrm{Ext}^1_\Lambda(-, \Lambda)$, and there is an equivalence $\tau : \Lambda\,\mathrm{mod}_{\mathcal{P}} \longrightarrow \Lambda\,\mathrm{mod}_{\mathcal{I}}$ with inverse τ^-.

Proof: We sketch the argument for (4). By (3) we have $P(M, N) = 0$ for all $M, N \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, and similarly, $I(M, N) = 0$ for all $M, N \in \Lambda\,\mathrm{mod}_{\mathcal{I}}$. Moreover, if $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, then a minimal projective presentation $0 \to P_1 \to P_0 \to M \to 0$ yields a long exact sequence $0 \to M^* \to P_0^* \to P_1^* \to \mathrm{Ext}^1_\Lambda(M, \Lambda) \to 0$ where $M^* = 0$, so $\mathrm{Ext}^1_\Lambda(M, \Lambda) \cong \mathrm{Tr}\, M$. $\square$

The Auslander-Reiten formulae now read as follows.

Corollary 5.1.2. *Let Λ be a hereditary Artin algebra. Let $A \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$ and $C \in \Lambda\mathrm{Mod}$.*

$$\text{(I)}\quad \mathrm{Hom}_\Lambda\,(C, \tau\, A) \cong D\,\mathrm{Ext}^1_\Lambda\,(A, C)$$
$$\text{(II)}\quad D\,\mathrm{Hom}_\Lambda\,(A, C) \cong \mathrm{Ext}^1_\Lambda\,(C, \tau\, A)$$

5.2 Preprojective, preinjective and regular components

Throughout this section, Λ denotes a (basic) hereditary finite dimensional algebra over an algebraically closed field k with AR-translation $\tau = D\,\mathrm{Tr}$, and AR-quiver Γ. Then Λ is isomorphic to the path algebra of a quiver Q without oriented cycles. We will assume that Q is a connected quiver, that is, Λ is indecomposable (this is a harmless assumption, cf.[16, II.5]). Then all indecomposable projective modules lie in the same component of Γ, and dually, the same holds true for the indecomposable injectives.

Definition. The component $\mathbf{p}$ of Γ containing all projective indecomposable modules is called the *preprojective component*. The component $\mathbf{q}$ of Γ containing all injective indecomposable modules is called the *preinjective component*. The remaining components of Γ are called *regular*.

Recall that the preprojective and preinjective components are obtained from Q by applying the Knitting Procedure discussed in [17]. This can be formalized by employing the following notion.

Definition. Let Q be a locally finite quiver without loops. For two vertices $i, j \in Q_0$, let d_{ji} be the number of arrows $i \to j$.

(1) We construct a quiver $\mathbb{Z}Q$ as follows:

The set of vertices is $\mathbb{Z} \times Q_0$.

The set arrows is given by the following rule: if $n \in \mathbb{Z}$ and $i, j \in Q_0$ with $d_{ji} \neq 0$, then we put d_{ji} arrows

$$\begin{tikzcd}
\cdots & (n-1,j) \arrow[dr] & & (n,j) \arrow[dr] & & (n+1,j) & \cdots \\
& & (n,i) \arrow[ur] & & (n+1,i) \arrow[ur] & &
\end{tikzcd}$$

(2) We denote by $\mathbb{N}Q$, respectively $-\mathbb{N}Q$, the full subquivers of $\mathbb{Z}Q$ with vertices $\{0, 1, 2, \dots\} \times Q_0$, respectively $\{0, -1, -2, \dots\} \times Q_0$.

(3) Finally, we denote by Q^{op} the quiver obtained from Q by reverting the arrows.

Such infinite quivers are called *translation quivers*. For a more precise treatment, we refer to [16, VII.4].

Example: Consider the quiver $\mathbb{A}_3\ \underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$ and compute $\mathbb{Z}\,\mathbb{A}_3^{op}$.

$$\begin{tikzcd}
& & (-1,1) \arrow[dr] & & (0,1) \arrow[dr] & & (1,1) \arrow[dr] & & (2,1) \ldots \\
& (-1,2) \arrow[ur] \arrow[dr] & & (0,2) \arrow[ur] \arrow[dr] & & (1,2) \arrow[ur] \arrow[dr] & & (2,2) \arrow[ur] & \\
\ldots (-1,3) \arrow[ur] & & (0,3) \arrow[ur] & & (1,3) \arrow[ur] & & (2,3) \arrow[ur] & &
\end{tikzcd}$$

We see that $\mathbb{Z}\,\mathbb{A}_3^{op}$ contains the AR-quiver $\Gamma(k\,\mathbb{A}_3)$ of the path algebra of $\mathbb{A}_3$ as a subquiver, cf. the Example in 3.5.

The following result was proved in [21].

Theorem 5.2.1 (Gabriel-Riedtmann 1979). *Let Λ and Q be as above.*

(1) If Q is a Dynkin quiver, then $\Gamma = \mathbf{p} = \mathbf{q}$ is a full finite subquiver of $\mathbb{N}Q^{\mathrm{op}}$.

(2) If Q is not a Dynkin quiver, then $\mathbf{p} = \mathbb{N}Q^{\mathrm{op}}$, and $\mathbf{q} = -\mathbb{N}Q^{\mathrm{op}}$, and the modules in $\mathbf{p}$ and $\mathbf{q}$ are uniquely determined by their dimension vectors. Moreover $\mathbf{p}\cap\mathbf{q} = \emptyset$, and $\mathbf{p}\cup\mathbf{q} \subsetneqq \Gamma$.

So, regular components only occur when Λ is of infinite representation type. They have a rather simple shape, as we are going to see next.

Lemma 5.2.2. *Let $\mathcal{C}$ be a regular component of Γ. Let further M, N be modules in $\mathcal{C}$, and let $f : \tau N \to M$ and $g : M \to N$ be irreducible maps. Then g is a monomorphism (respectively, an epimorphism) if and only if f is an epimorphism (respectively, a monomorphism).*

Proof: If f and g were both monomorphisms, then by 5.1.1(4) also $\tau^n f$ and $\tau^n g$ would be proper monomorphisms for each $n \in \mathbb{N}$. But then we would obtain an infinite descending chain of proper monomorphisms

$$\cdots\ \tau^2 N \overset{\tau f}{\hookrightarrow} \tau M \overset{\tau g}{\hookrightarrow} \tau N \overset{f}{\hookrightarrow} M \overset{g}{\hookrightarrow} N$$

contradicting the fact that N has finite length. □

The following was shown independently in [12] and [27].

Theorem 5.2.3 (Auslander-Bautista-Platzeck-Reiten-Smalø; Ringel 1979). *Let Λ be of infinite representation type. Let $\mathcal{C}$ be a regular component of Γ. For each $[M]$ in $\mathcal{C}$ there are at most two arrows ending in $[M]$.*

More precisely, if $g : B \longrightarrow M$ is a minimal right almost split map with $M \in \mathcal{C}$, and if we denote by $\alpha(M)$ the number of summands in an indecomposable decomposition $B = B_1 \oplus \ldots \oplus B_{\alpha(M)}$, then it was proven that $\alpha(M) \leq 2$.

For a proof, we refer to [16, VIII.4]. Here we only explain the

Construction of a regular component $\mathcal{C}$: Let us start with a module $C_0 \in \mathcal{C}$ of minimal length. Such a module is called *quasi-simple* (or *simple regular*).
Note that $\alpha(C_0) = 1$. Otherwise there is an almost split sequence of the form $0 \to \tau C_0 \overset{(f_1,f_2)^t}{\longrightarrow} X_1 \oplus X_2 \overset{(g_1,g_2)}{\longrightarrow} C_0 \to 0$ with non-zero modules X_1, X_2 of length $l(X_i) \geq l(C_0)$, and g_1, g_2 must be epimorphisms. On the other hand, if g_1 is an epimorphism, then

so is f_2, see [16, I, 5.7]. But then it follows from 5.2.2 that g_2 is a monomorphism, a contradiction.
Now $\alpha(C_0) = 1$ implies that in Γ there is a unique arrow $[X] \to [C_0]$ ending in C_0, and therefore by 3.5.4, also a unique arrow starting in $[C_0]$. So we have an almost split sequence $0 \longrightarrow C_0 \xrightarrow{f_0} C_1 \xrightarrow{g_0} \tau^- C_0 \longrightarrow 0$ with C_1 being indecomposable. Moreover, we have an almost split sequence $0 \longrightarrow \tau C_1 \longrightarrow C_0 \oplus Y \xrightarrow{(f_0,h)} C_1 \longrightarrow 0$ where $Y \neq 0$ because f_0 is an irreducible monomorphism. Hence $\alpha(C_1) = 2$ and Y is indecomposable. Furthermore, one checks that h must be an irreducible epimorphism.
Setting $C_2 = \tau^- Y$ and $g_1 = \tau^- h$, we obtain an almost split sequence $0 \longrightarrow C_1 \xrightarrow{(f_1,g_0)^t} C_2 \oplus \tau^- C_0 \xrightarrow{(g_1,\tau^- f_0)} \tau^- C_1 \longrightarrow 0$ where g_0, g_1 are irreducible epimorphisms and $f_1, \tau^- f_0$ are irreducible monomorphisms.
Proceeding in this manner, we obtain a chain of irreducible monomorphisms $C_0 \hookrightarrow C_1 \hookrightarrow C_2 \ldots$ with almost split sequences $0 \longrightarrow C_i \longrightarrow C_{i+1} \oplus \tau^- C_{i-1} \longrightarrow \tau^- C_i \longrightarrow 0$ for all i.
The component $\mathcal{C}$ thus has the shape

$$\begin{tikzcd}[column sep=small]
 & & & \vdots & & \vdots & & \vdots \\
 & & & Y = \tau C_2 \arrow[dr, "h"] & & C_2 \arrow[dr, "g_1"] & & \tau^- C_2 \\
 & \cdots & \tau C_1 \arrow[ur] \arrow[dr] & & C_1 \arrow[ur, "f_1"] \arrow[dr, "g_0"] & & \tau^- C_1 \arrow[ur] & \cdots \\
 \cdots & \tau C_0 \arrow[ur] & & C_0 \arrow[ur, "f_0"] & & \tau^- C_0 \arrow[ur, "\tau^- f_0"] & \cdots &
\end{tikzcd}$$

and every module in $\mathcal{C}$ has the form $\tau^r C_i$ for some i and some $r \in \mathbb{Z}$.
Observe that if $\tau^r C_i \cong C_i$ for some i and r, then $\tau^r C \cong C$ for all C in $\mathcal{C}$.

Corollary 5.2.4. *Let $\mathbb{A}_\infty$ be the infinite quiver $\bullet \to \bullet \to \bullet \to \bullet \cdots$
Then $\mathcal{C}$ has either the form $\mathbb{Z}A_\infty$ or it has the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ where $n = \min\{r \in \mathbb{N} \mid \tau^r C \cong C$ for some $C \in \mathcal{C}\}$.*

Definition. We call $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ a *(stable) tube*, and we call it *homogeneous* if $n = 1$.

Stable tubes do not occur in the wild case. In the tame case, the regular components form a family of tubes $\mathbf{t} = \bigcup \mathbf{t}_\lambda$ indexed over the projective line $\mathbb{P}_1 k$, and all but at most three $\mathbf{t}_\lambda$ are homogeneous, see [28, 3.6] and [27, 2.4].

Let us illustrate the considerations above by an example.

5.3 The Kronecker Algebra

Consider the quiver

$$Q = \widetilde{\mathbb{A}_1}: \qquad \overset{1}{\bullet} \rightrightarrows \overset{2}{\bullet}$$

A finite dimensional representation of Q is given as

$$V_1 \underset{f_\beta}{\overset{f_\alpha}{\rightrightarrows}} V_2$$

where V_1, V_2 are finite dimensional k-vectorspaces and $f_\alpha, f_\beta : V_1 \to V_2$ are k-linear.

In other words, every finite dimensional representation of Q corresponds to a pair of matrices (A, B) with $A, B \in k^{n\times m}$ and $n, m \in \mathbb{N}_0$. Moreover, it is easy to see that isomorphism of two representations, in terms of matrix pairs (A, B) and (A', B') corresponds to the existence of two invertible matrices $X \in GL_n(K)$ and $Y \in GL_m(K)$ such that $A' = XAY^{-1}$ and $B' = XBY^{-1}$. So, the classification of the finite dimensional representations of Q translates into the classification problem of matrix pencils considered by Kronoceker in [25]. For this reason $\Lambda = kQ$ is called the *Kronecker algebra*.

Let us now collect some information on Λ. For unexplained terminology, we refer to [17, 18].

The Cartan Matrix. We have two indecomposable projectives Λe_1 and Λe_2 with dimension vectors

$$\underline{p_1} = \underline{\dim}\Lambda e_1 = (1, 2)$$
$$\underline{p_2} = \underline{\dim}\Lambda e_2 = (0, 1)$$

Hence the Cartan matrix is

$$C = \begin{pmatrix} \underline{p_1} \\ \underline{p_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

with inverse

$$C^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

The Coxeter transformation. The map

$$c\colon \mathbb{Z}^n \to \mathbb{Z}^n, \qquad \underline{x} \mapsto -\underline{x}\, C^{-1}\, C^t$$

is called Coxeter transformation. We have

$$c(\underline{x}) = -\underline{x}\, C^{-1}\, C^t = \underline{x} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$$

Setting $\underline{v} = (1, 1)$, we can write

$$c(\underline{x}) = \underline{x} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right) = \underline{x} + 2(x_1 - x_2)\underline{v}$$

and since $c(\underline{v}) = \underline{v}$, we have

$$c^m \underline{x} = \underline{x} + 2m(x_1 - x_2)\underline{v} \quad \text{for each } m.$$

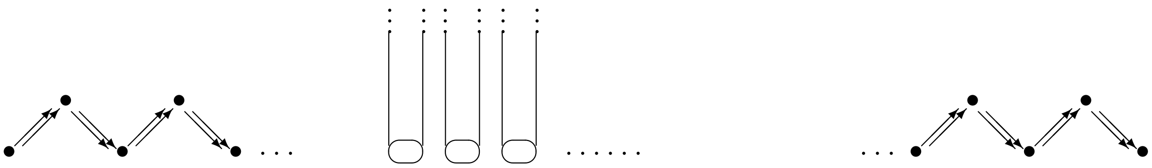
Note that c can be used to compute τ.

Proposition 5.3.1. [16, VIII,2.2]

(1) If $A \in \Lambda\,mod$ is indecomposable non-projective, then $c(\underline{\dim}A) = \underline{\dim}\,\tau A$.

(2) An indecomposable module $A \in \Lambda\,mod$ is projective if and only if $c(\underline{\dim}A)$ is negative.

The AR-quiver. We are now ready to compute Γ:



p **t** **q**

For the shape of **p** and **q** we refer to Theorem 5.2.1. Let us compute the dimension vectors. For example, from the first two arrows on the left we deduce that there is an almost split sequence $0 \longrightarrow P_2 \longrightarrow P_1 \oplus P_1 \longrightarrow C \longrightarrow 0$ and $\underline{\dim}C = (1,2) + (1,2) - (0,1) = (2,3)$. In this way we observe the following.

Dimension vectors.
p consists of the modules X with $\underline{\dim}X = (m, m+1)$.
q consists of the modules X with $\underline{\dim}X = (m+1, m)$.
The modules in **t** are precisely the modules X with $\underline{\dim}X = (m, m)$.

Let us check the last statement. Let $X \in \mathbf{t}$ and $\underline{\dim}X = (l, m)$. If $l < m$, then

$$c^m(\underline{\dim}X) = (l, m) + 2m(l - m, l - m)$$

is negative. By 5.3.1 we have $c^m(\underline{\dim}X) = c(\underline{\dim}\tau^{m-1}X)$, thus $\tau^{m-1}X$ is projective, and $X \in \mathbf{p}$. Dually, $l > m$ implies $X \in \mathbf{q}$. Hence we conclude $l = m$.

We now explain the shape of **t**.

The regular components. First of all, the quasi-simple modules, that is, the indecomposable regular modules of minimal length, are precisely the modules X with $\underline{\dim}X = \underline{v} = (1,1)$. A complete irredundant set of quasi-simples is then given by

$$V_\lambda : K \underset{\lambda}{\overset{1}{\rightrightarrows}} K,\ \lambda \in K, \quad \text{and} \quad V_\infty : K \underset{1}{\overset{0}{\rightrightarrows}} K$$

Note that each V_λ is sincere with composition factors S_1, S_2.
Furthermore, applying $\mathrm{Hom}(-, V_\mu)$ on the projective resolution $0 \to \Lambda e_2 \to \Lambda e_1 \to V_\lambda \to 0$ we see that V_λ, V_ν are "perpendicular":

$$\dim_k \mathrm{Hom}_\Lambda(V_\lambda, V_\mu) = \dim_k \mathrm{Ext}^1_\Lambda(V_\lambda, V_\mu) = \begin{cases} 1 & \mu = \lambda \\ 0 & \text{else} \end{cases}$$

Next, we check that each V_λ defines a homogeneous tube $\mathbf{t}_\lambda$.
In fact, $\tau V_\lambda \cong V_\lambda$ for all $\lambda \in K \cup \{\infty\}$:

$$\begin{aligned} &\underline{\dim}\tau V_\lambda = c(\underline{\dim}V_\lambda) = (1,1)\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = (1,1)\,, \\ &\text{hence} \quad \tau V_\lambda \cong V_\mu \text{ with } \mathrm{Ext}^1(V_\lambda, V_\mu) \neq 0\,, \text{thus } \mu = \lambda\,. \end{aligned}$$

So, for each $\lambda \in K \cup \{\infty\}$ there is a chain of irreducible monomorphisms

$$V_\lambda = V_{\lambda,1} \hookrightarrow V_{\lambda,2} \hookrightarrow \dots$$

that gives rise to a homogeneous tube $\mathbf{t}_\lambda \cong \mathbb{Z}A_\infty \setminus \langle \tau \rangle$ consisting of modules $V_{\lambda,j}$ with $\tau V_{\lambda,j} \cong V_{\lambda,j}$, $\underline{\dim}V_{\lambda,j} = (j,j)$, and $V_{\lambda,j+1}/V_{\lambda,j} \cong V_\lambda$.
Moreover, there are neither nonzero maps nor extensions between different tubes $\mathbf{t}_\lambda$.
Finally, let us indicate how to show that every indecomposable regular module X is contained in some tube $\mathbf{t}_\lambda$. We already know that X has the form $X : K^m \underset{\beta}{\overset{\alpha}{\rightrightarrows}} K^m$.
Now, suppose that α is a isomorphism. Then, since k is algebraically closed, $\alpha^{-1}\beta$ has an eigenvalue λ, and, as explained in [16, VIII.7.3], it is possible to embed $V_\lambda \subset X$. This proves that X belongs to $\mathbf{t}_\lambda$. Similarly, if $\mathrm{Ker}\,\alpha \neq 0$, it is possible to embed $V_\infty \subset X$, which proves that X belongs to t_∞.

The tubular family t is separating, that is:

(a) $\mathrm{Hom}(\mathbf{q}, \mathbf{p}) = \mathrm{Hom}(\mathbf{q}, \mathbf{t}) = \mathrm{Hom}(\mathbf{t}, \mathbf{p}) = 0$

(b) Any map from a module in $\mathbf{p}$ to a module in $\mathbf{q}$ factors through any $\mathbf{t}_\lambda$.

To verify (b), let us consider a homomorphism $f\colon P \to Q$ with $P \in \mathbf{p}$, and $Q \in \mathbf{q}$. The argument is taken from [28, p.126].
Let $\lambda \in K \cup \{\infty\}$ be arbitrary, and let $\underline{\dim}P = (l, l+1)$ and $\underline{\dim}Q = (m+1, m)$. Choose an integer $j \geq l + m + 1$. We are going to show that f factors through $V_{\lambda,j}$.

Note that $\mathrm{Ext}^1_\Lambda(P, V_{\lambda,j}) = 0$ by 5.1.2. We employ the bilinear form

$$B\colon \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q},\ (\underline{x}, \underline{y}) \mapsto \underline{x}\, C^{-1} \underline{y}^t$$

in order to compute $\dim_k \mathrm{Hom}_\Lambda(P, V_{\lambda,j}) = \dim_k \mathrm{Hom}_\Lambda(P, V_{\lambda,j}) - \dim_k \mathrm{Ext}^1_\Lambda(P, V_{\lambda,j}) = B(\underline{\dim} P, \underline{\dim} V_{\lambda,j}) = (l, l+1) \left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right) \binom{j}{j} = j$.
So, the k-spaces $\mathrm{Hom}_\Lambda(P, V_{\lambda,j})$, $j \geq 0$, form a strictly increasing chain. Hence there exists a map $g\colon P \to V_{\lambda,j}$ such that $\mathrm{Im}\, g \not\subset V_{\lambda,j-1}$, and by length arguments we infer that $\mathrm{Im}\, g$ is a proper submodule of $V_{\lambda,j}$. Thus $\mathrm{Im}\, g$ is not regular. Then it must contain a preprojective summand P', and we deduce that g is a monomorphism. Consider the exact sequence

$$0 \longrightarrow P \xrightarrow{g} V_{\lambda,j} \longrightarrow Q' \longrightarrow 0$$

The module Q' cannot have regular summands, so it is a direct sum of preinjective modules. We claim that even $Q' \in q$. To see this, we use the $\mathbb{Q}$-linear map

$$\delta\colon \mathbb{Q}^2 \to \mathbb{Q},\ \underline{x} \mapsto B(\underline{v}, \underline{x}) = x_1 - x_2$$

called the *defect*. Note that $\mathbf{p}$ consists of the indecomposable modules X with $\delta(\underline{\dim} X) = -1$, while $\mathbf{q}$ consists of the indecomposable modules X with $\delta(\underline{\dim} X) = 1$, and $\mathbf{t}$ consists of the indecomposable modules X with $\delta(\underline{\dim} X) = 0$. So, by computing

$$\delta(\underline{\dim} Q') = \delta(\underline{\dim} V_{\lambda,j}) - \delta(\underline{\dim} P) = 1$$

we conclude that Q' is indecomposable.
Furthermore, $\underline{\dim} Q' = (s+1, s)$ with $s = j - (l+1) \geq m$, which proves $\mathrm{Ext}^1_\Lambda(Q', Q) = 0$. Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[rr, "g"] \arrow[dr, "f"'] & & V_{\lambda,j} \arrow[r] \arrow[dl, dotted] & Q' \arrow[r] & 0 \\
& & Q & & &
\end{tikzcd}$$

proving the statement.

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