ON SOME PRECOVERS AND PREENVELOPES

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Habilitationsschrift
zur Feststellung der Lehrbefähigung
für das Fach Mathematik
in der Fakultät für Mathematik und Informatik
der Ludwig-Maximilians-Universität München

Mai 2000

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Introduction

Precovers and preenvelopes were introduced in the early eighties by Enochs [48] and, independently, by Auslander and Smalø [16]. Enochs gave a general definition in terms of commutative diagrams for modules over arbitrary rings, whereas Auslander and Smalø, mainly concerned with the case of finitely generated modules over finite-dimensional algebras, stressed the functorial viewpoint and coined the terminology of contravariantly and covariantly finiteness. Both approaches turned out to be extremely fruitful for general module theory as well as for representation theory.

The idea behind these concepts is to exploit interesting features of a special class of modules $\mathcal{X}$ for the study of the whole module category. The link between $\mathcal{X}$ and the other modules is given by some universal homomorphisms, namely the precovers and the preenvelopes.

The work of Enochs and other authors provided a common frame for a number of classical notions, such as injective envelopes or projective covers, and on the other hand, it also raised many challenging problems that are still object of current research. For instance, the well-known Flat Cover Conjecture, asserting that each module admits a flat cover, remained open for almost twenty years and was settled only very recently [22], [47].

Similarly, contravariantly and covariantly finite subcategories had a significant impact on the representation theory of finite-dimensional algebras. Auslander and Reiten pointed out in [15] that these concepts are intimately related to the notion of a tilting module introduced by Brenner and Butler [25] and Happel and Ringel [54]. This connection provided a better understanding of some problems in tilting theory, and further, it led to notable progress on quasi-hereditary algebras [80] and on the homological conjectures.

Let us focus for a moment on two applications of the theory of contravariantly and covariantly finite subcategories. The first concerns the homological conjectures. For a finite-dimensional algebra Λ the little finitistic dimension $\mathrm{findim}\Lambda$ is defined as the supremum of the projective dimensions attained on the category $\mathcal{P}^{<\infty}$ of all finitely generated Λ-modules of finite projective dimension, while the big finitistic dimension $\mathrm{Findim}\Lambda$ is defined correspondingly on the category of all Λ-modules of finite projective dimension. The Finitistic Dimension Conjectures ask when these dimensions coincide (this is known to fail in general), and moreover, whether the little finitistic dimension is always finite. Now, it was shown by Auslander and Reiten [15] that $\mathrm{findim}\Lambda$ is finite provided the category $\mathcal{P}^{<\infty}$ is contravariantly finite. Indeed, it is enough to consider the projective dimensions of the $\mathcal{P}^{<\infty}$-covers of the (finitely many) simple Λ-modules; the projective dimension of any other finitely generated Λ-module will then be either infinite or bounded by the maximum of these numbers. Huisgen-Zimmermann and Smalø [60] later gave a direct limit argument which allowed to extend this result to arbitrary Λ-modules, and so they could prove that contravariantly finiteness of $\mathcal{P}^{<\infty}$ also yields $\mathrm{Findim}\Lambda = \mathrm{findim}\Lambda$.

The second result is related to the concept of a partial tilting module. A module M of finite projective dimension with $\mathrm{Ext}^k_\Lambda(M, M) = 0$ for all $k > 0$ is called a partial tilting module. The question whether M admits a complement, that is, a module N turning $M \oplus N$ into a tilting module, is of interest for tilting theory and also because of its relationship with the homological conjectures, see [55]. While such complements always exist in case the projective dimensions involved are at most one [24], the same does not hold in general for higher projective dimensions. However, the situation changes once we allow also infinite-dimensional tilting modules as studied in [40] and [6]. Applying a general result on the existence of preenvelopes due to Eklof and Trlifaj [46], we have recently proved in a joint work with Coelho [7] that every finitely generated partial tilting module M over a finite-dimensional algebra admits a (possibly infinite-dimensional) complement N. Here N is obtained as a preenvelope of Λ with respect to a class of modules related to M.

Both results show how precovers and preenvelopes can be employed. Moreover, they illustrate the interaction between finite-dimensional and infinite-dimensional modules, between the representation-theoretical approach and general module theory.

This interaction is also one of the main aspects of the present work. We consider an arbitrary ring R and discuss the existence of precovers and preenvelopes with respect to certain classes $\mathcal{X}$ of (right) R-modules. Actually, the only constrain on our classes $\mathcal{X}$ concerns their size. More precisely, we will assume that there is a set of modules $\{M_i \mid i \in I\}$ such that every module in $\mathcal{X}$ is isomorphic to a direct summand of a coproduct of copies of the module $M = \coprod_{i \in I} M_i$. Such classes are denoted by $\operatorname{Add} M$. For example, the class of all projective modules, but also the class of all pure-projective modules, can be written in this way.

Our starting point is an easy observation. Namely, a finitely generated module A_R has an $\operatorname{Add} M$-preenvelope $A \to X$ if and only if the left $\operatorname{End}_R M$-module $\operatorname{Hom}_R(A, M)$ is finitely generated. This allows us to translate the categorical notion of a preenvelope into an endoproperty of M, that is, into a module-theoretical property of M over its endomorphism ring $S = \operatorname{End}_R M$. We will thus handle infinite categories $\mathcal{X}$ in terms of finiteness conditions on a single module ${}_SM$. On the other hand, we will investigate non-finitely generated modules M_R and even big products of copies of M_R by studying the category $\mathcal{X}$ given by the (perhaps finitely generated or indecomposable) direct summands of M_R.

One of the main endoproperties we will consider is coherence. We show for example that ${}_SM$ being coherent entails the existence of $\operatorname{Add} M$-preenvelopes for all finitely presented right R-modules. We also investigate some stronger coherence properties related to S-submodules of M of some special kind, namely the matrix subgroups studied by Zimmermann [98]. By specializing to the case $M = R$, we then rediscover some known features of coherent rings and of the strongly coherent rings introduced by Zimmermann-Huisgen in [92]. With similar techniques, we further relate the existence of left almost split morphisms in our category to noetherianness of ${}_SM$.

Furthermore, we will deal with $\operatorname{Add} M$-precovers. Every module C_R has an $\operatorname{Add} M$-precover $g : B \to C$. But is there even an $\operatorname{Add} M$-cover? That is, can we choose g in such a way that it also satisfies a minimality condition ensuring that B is uniquely determined by C up to isomorphism? For the case $M = R$, the answer is given by Bass's work on perfect rings. We will see that Bass's results extend to the general situation. In fact, we prove that $\operatorname{Add} M$-covers exist for instance when $\operatorname{Add} M$ consists of modules with semiregular endomorphism ring, or when the endomorphism ring of each M_i is local and ${}_SM$ satisfies the descending chain condition on cyclic S-submodules. By using some results on perfect functor categories due to Harada and Simson [56],

[83], we then obtain that these perfectness conditions even characterize the existence of $\operatorname{Add} M$-covers in case that all M_i are finitely presented.

We then turn to $\operatorname{Add} M$-envelopes, that is, $\operatorname{Add} M$-preenvelopes satisfying a dual minimality condition. It was shown by Krause and Saorín that $\operatorname{Add} M$-envelopes exist if and only if all products of copies of M have well-behaved direct sum decompositions. We refine this relationship between preenvelopes and direct summands of products by describing when an indecomposable direct summand of a product of modules $\prod_{k \in K} X_k$ is isomorphic to a direct summand of one of the factors X_k, and we answer a question posed by Auslander in [13]. Furthermore, we see that closure of $\operatorname{Add} M$ under direct products can be characterized by combining the coherence and perfectness properties mentioned above. This extends a well-known theorem of Chase asserting that all products of projective modules are projective if and only if R is left coherent and right perfect. As a consequence, we will also obtain some new characterizations of endofinite modules.

So, the theory of precovers and preenvelopes allows to extend some classical results on rings to arbitrary modules, and, at the same time, to put these old results in a new context by stressing their representation-theoretic meaning.

Let us now describe the contents of the paper in more detail.

We begin in Chapter 1 by reviewing the notions of a (pre)cover and a (pre)envelope together with some known results on their existence.

Chapter 2 is devoted to the existence of $\operatorname{Add} M$-covers. We use Harada's concept of T-nilpotency to investigate some endoproperties of M which in the case $M = R$ correspond to perfectness. Further, we recall Simson's characterization of perfect functor categories and employ some results of Lenzing to translate it into properties of $\operatorname{Add} M$. An important role in this context is played by Azumaya's notion of a locally split epimorphism. In case that all M_i are finitely presented, we then characterize the existence of $\operatorname{Add} M$-covers by a number of properties which can all be viewed as generalizations of Σ-pure-injectivity.

In chapter 3, we deal with the existence of $\operatorname{Add} M$-preenvelopes. A result of Zimmermann [101] on matrix subgroups together with some properties of the functor $\operatorname{Hom}_R(\ ,M) : \operatorname{Mod} R \to S \operatorname{Mod}$ will enable us to relate this problem to some coherence properties of ${}_S M$. We describe when all matrix subgroups, respectively all finite matrix subgroups, of M are finitely generated over S. Further, we discuss the case that ${}_S M$ is coherent or that even the finitely generated S-submodules of arbitrary products of copies of ${}_SM$ are finitely presented (Theorem 3.17). In the case $M = R$, we obtain known results on coherent rings due to several authors [48], [50], [92], [69], [27].

In chapter 4, we apply our previous results to the study of direct products of modules. We describe the case that M is product-rigid, that is, that every direct summand A of a product of copies of M with $\operatorname{End}_R A$ local is isomorphic to some indecomposable direct summand of M itself. We see that every endonoetherian module is product-rigid, and every endocoherent module has the corresponding property restricted to finitely presented A. We then discuss the notion of a product-complete module introduced by Krause and Saorín [66] and show that M is product-complete if and only if it has the perfectness and coherence properties of chapters 2 and 3. As an application, we characterize the pure-projective product-complete modules over an artin algebra.

In chapter 5, we study the existence of left and right almost split morphisms in our category by investigating finiteness conditions on the radicals ${}_S\,\mathrm{r}\,(A, M)$ and $\mathrm{r}\,(M, C)_S$. Moreover, we deal with the question of when the property product-rigid is inherited by direct summands. We show that this is equivalent to the existence of certain maps closely related to left almost split maps and answer a question posed by Auslander in [13]. We then combine our results to characterize endofinite modules in terms of product-rigidity as well as in terms of the existence of preenvelopes and left almost split maps.

We close the paper with chapter 6 devoted to applications and examples. We first give some applications of our results on endocoherence to tilting theory. Next, we address the question of the existence of $\operatorname{Prod} M$-precovers and answer it to the positive in case that M is Σ-pure-injective. Finally, we illustrate some of our results by considering the case that R is a finite-dimensional hereditary algebra or a pure-semisimple ring.

I would like to thank Wolfgang Zimmermann for his constant interest and his many valuable comments on my work. There are also several other persons I would like to thank for stimulating discussions on this or related topics, in particular Flávio Coelho, Riccardo Colpi, Henning Krause, Sverre Smalø, Alberto Tonolo, and Jan Trlifaj. Moreover, I acknowledge a research grant of the University of Munich inside the HSPIII-program. Finally, many many thanks to my family! Without their great patience and their constant support this work would not have been possible.

1 PRELIMINARIES

Let us start with some general notation which we will use throughout the paper. We denote by R an arbitrary ring with the Jacobson radical $\boldsymbol{J(R)}$ and write $\mathbf{Mod}\,\boldsymbol{R}$ and $\mathbf{mod}\,\boldsymbol{R}$ for the categories of all, respectively of the finitely presented, right R-modules. Moreover, we *fix for the whole paper* a skeletally small subcategory $\boldsymbol{\mathcal{M}}$ of $\mathrm{Mod}\,R$, and let $\{M_i \mid i \in I\}$ be a complete irredundant set of representatives of the isomorphism classes of $\mathcal{M}$. We put $\boldsymbol{M} = \coprod_{i\in I} M_i$ with $\boldsymbol{S} = \mathrm{End}_R\,M$, and $\boldsymbol{N} = \prod_{i\in I} M_i$ with $\boldsymbol{T} = \mathrm{End}_R\,N$.

We will say that M is **endonoetherian**, or **endofinite**, if ${}_SM$ is noetherian, or a module of finite length, respectively. Furthermore, we will consider a special class of S-submodules of M, the matrix subgroups [98]. Recall that, if Y_R is a module and U a subgroup of the abelian group Y, then U is said to be a **matrix subgroup** of Y if there is a module A_R and an element $x \in A$ such that U equals the set $\boldsymbol{H}_{\mathbf{A,x}}(\boldsymbol{Y}) = \{f(x) \mid f \in \mathrm{Hom}_R\,(A, Y)\}$. Of course, every matrix subgroup is a left submodule of Y over the endomorphism ring $\mathrm{End}_R\,Y$. Moreover, the functor $Y \mapsto \mathrm{H}_{A,x}(Y)$ commutes with products and coproducts. A matrix subgroup $\mathrm{H}_{A,x}(Y)$ is called **finite matrix subgroup** if A_R is finitely presented.

Finally, given a class of modules $\mathcal{X}$, we denote by $\mathbf{Add}\,\boldsymbol{\mathcal{X}}$ (respectively, $\mathbf{add}\,\boldsymbol{\mathcal{X}}$) the class consisting of all modules isomorphic to direct summands of (finite) direct sums of modules of $\mathcal{X}$. The class consisting of all modules isomorphic to direct summands of products of modules of $\mathcal{X}$ is denoted by $\mathbf{Prod}\,\boldsymbol{\mathcal{X}}$. If $\mathcal{X}$ consists just of one module X, then we write $\mathrm{Add}\,X$, respectively $\mathrm{add}\,X$, or $\mathrm{Prod}\,X$.

By a subcategory we always mean a full subcategory.

1.1 Precovers and preenvelopes

Let $\mathcal{X} \subset \operatorname{Mod} R$ and A, C be right R-modules. Following Enochs [48], we say that a homomorphism $a : A \to X$ is an **$\mathcal{X}$-preenvelope** if $X \in \mathcal{X}$ and the abelian group homomorphism $\operatorname{Hom}_R(a, X') : \operatorname{Hom}_R(X, X') \to \operatorname{Hom}_R(A, X')$ is surjective for each $X' \in \mathcal{X}$. Dually, a homomorphism $b : Y \to C$ is called an **$\mathcal{X}$-precover** if $Y \in \mathcal{X}$ and the abelian group homomorphism $\operatorname{Hom}_R(Y', b) : \operatorname{Hom}_R(Y', Y) \to \operatorname{Hom}_R(Y', C)$ is surjective for each $Y' \in \mathcal{X}$. In the representation theory of artin algebras, the usual terminology is left, respectively right, $\mathcal{X}$-approximation.
We will say that $\mathcal{X}$ is a **preenvelope class**, respectively a **precover class**, if every right R-module has an $\mathcal{X}$-preenvelope, respectively an $\mathcal{X}$-precover.

Rada and Saorín have characterized precover and preenvelope classes in [77]. According to [77], we will say that $\mathcal{X}$ is **locally initially small** if for every module A_R there is a set $\mathcal{X}_A \subset \mathcal{X}$ such that every map $A \to X$ where $X \in \mathcal{X}$ factors through $\operatorname{Prod} \mathcal{X}_A$. Dually, $\mathcal{X}$ is said to be **locally finally small** if for every module C_R there is a set $\mathcal{X}_C \subset \mathcal{X}$ such that every map $Y \to C$ where $Y \in \mathcal{X}$ factors through $\operatorname{Add} \mathcal{X}_C$. We have the following characterization of preenvelope, respectively precover classes.

Theorem 1.1 (Rada-Saorín [77, 3.3 and 3.4]) Let $\mathcal{X}$ be a class in $\operatorname{Mod} R$.
(I) $\mathcal{X}$ is a preenvelope class if and only if it is locally initially small and every product of modules in $\mathcal{X}$ is a direct summand of a module in $\mathcal{X}$.
(II) $\mathcal{X}$ is a precover class if and only if it is locally finally small and every coproduct of modules in $\mathcal{X}$ is a direct summand of a module in $\mathcal{X}$.

With the above notations, we have the following consequence.

Corollary 1.2 (Rada-Saorín [77, 3.5, 2.9 and 3.6]) (1) $\operatorname{Prod} \mathcal{M}$ is a preenvelope class, $\operatorname{Add} \mathcal{M}$ is a precover class.
(2) $\operatorname{Add} \mathcal{M}$ is locally initially small. More precisely, every map $A \to X$ where $X \in \operatorname{Add} \mathcal{M}$ factors through $\coprod_{i \in I} M_i^{(H_i)}$ where $H_i = \operatorname{Hom}(A, M_i)$. Thus $\operatorname{Add} \mathcal{M}$ is a preenvelope class if and only if it is closed under products.

Following Krause and Saorín [66], we will call M **product-complete** if $\operatorname{Add} \mathcal{M}$ is closed under products.

We turn to a further application of Theorem 1.1. Using a result of Kiełpiński on purity, Rada and Saorín have shown that every class $\mathcal{X}$ in $\operatorname{Mod} R$ which is closed under pure submodules is locally initially small [77, 2.8]. So, they have obtained the following useful result.

Proposition 1.3 (Rada-Saorín [77, 3.5]) A class $\mathcal{X}$ in $\mathrm{Mod}\, R$ which is closed under pure submodules is a preenvelope class if and only if it is closed under products.

For our skeletally small subcategory $\mathcal{M}$, the existence of $\mathrm{add}\,\mathcal{M}$-preenvelopes and $\mathrm{add}\,\mathcal{M}$-precovers can be interpreted in terms of the functor category as observed by Auslander and Smalø [16]. More precisely, a module A has an $\mathrm{add}\,\mathcal{M}$-preenvelope if and only if the covariant functor $\mathrm{Hom}_R(A,\ )\mid_{\mathcal{M}}$ from $\mathcal{M}$ to the category Ab of all abelian groups is finitely generated. Similarly, C has an $\mathrm{add}\,\mathcal{M}$-precover if and only if the contravariant functor $\mathrm{Hom}_R(\ ,C)\mid_{\mathcal{M}}$ is finitely generated. So, if $\mathcal{Y} \subset \mathrm{Mod}\, R$ is a subcategory containing $\mathcal{M}$, then $\mathcal{M}$ is said to be **covariantly finite** in $\mathcal{Y}$ if every $A \in \mathcal{Y}$ has an $\mathrm{add}\,\mathcal{M}$-preenvelope, and $\mathcal{M}$ is **contravariantly finite** in $\mathcal{Y}$ if every $C \in \mathcal{Y}$ has an $\mathrm{add}\,\mathcal{M}$-precover.

We now give a more elementary interpretation.

Proposition 1.4 (1) A module A_R has an $\mathrm{add}\,\mathcal{M}$-preenvelope if and only if the left T-module ${}_T\mathrm{Hom}_R\,(A,N)$ is generated by finitely many maps whose images are contained in a finite subproduct $\prod_{i\in I_0} M_i$ of N.
(2) A module C_R has an $\mathrm{add}\,\mathcal{M}$-precover if and only if the right S-module $\mathrm{Hom}_R\,(M,C)_S$ is generated by finitely many maps whose kernels contain a cofinite subcoproduct $\coprod_{i\in I\setminus I_0} M_i$ of M.

Proof: (1) Let $a : A \to X$ be an $\mathrm{add}\,\mathcal{M}$-preenvelope. By the universal property of products, every map $f : A \to N$ factors through a. Moreover, there is a split monomorphism $\iota : X \to \bigoplus_{k=1}^n X_k$ for some $X_1,\ldots,X_n \in \mathcal{M}$, and for each $1 \le k \le n$ there is $i_k \in I$ such that $X_k \cong M_{i_k}$, giving rise to a split monomorphism $\alpha_k : X_k \to N$. Then the maps $a_k : A \xrightarrow{a} X \xrightarrow{\iota} \bigoplus_{k=1}^n X_k \xrightarrow{pr_k} X_k \xrightarrow{\alpha_k} N$, $1 \le k \le n$, form a generating set of $\mathrm{Hom}_R\,(A,N)$ over T with the required property. Conversely, assume that $c_k : A \to N$, $1 \le k \le n$, is a generating set of $\mathrm{Hom}_R\,(A,N)$ over T and that there is a finite subset $I_0 \subset I$ such that $\mathrm{Im}\, c_k \subset X = \prod_{i\in I_0} M_i$ for all k. Then one can easily check that the map $c : A \to X^n$ induced by the c_k is an $\mathrm{add}\,\mathcal{M}$-preenvelope of A.
(2) is proven dually. □

In case we are considering the single module M_R, we immediately obtain the following useful criterion.

Corollary 1.5 (1) A_R has an $\operatorname{add} M$-preenvelope if and only if the left S-module ${}_S\mathrm{Hom}_R\,(A, M)$ is finitely generated.
(2) C_R has an $\operatorname{add} M$-precover if and only if the right S-module $\mathrm{Hom}_R\,(M, C)_S$ is finitely generated. $\square$

Throughout the paper, we will freely use the fact that for a finitely generated module A_R, every homomorphism $A \to X \in \operatorname{Add} \mathcal{M}$ factors through $\operatorname{add} \mathcal{M}$. In particular, since we always have $\operatorname{add} \mathcal{M} \subset \operatorname{add} M \subset \operatorname{Add} \mathcal{M}$, we obtain that for a finitely generated module the existence of an $\operatorname{add} M$-preenvelope is equivalent to the existence of an $\operatorname{add} \mathcal{M}$-preenvelope, and also to the existence of an $\operatorname{Add} \mathcal{M}$-preenvelope.

1.2 Covers and envelopes

So far, we have only reviewed the notions of a precover and a preenvelope. Let us now add the following minimality conditions. A homomorphism $a : A \to X$ is said to be **left minimal** if every endomorphism $h : X \to X$ such that $h\,a = a$ is an isomorphism. Dually, a homomorphism $b : Y \to C$ is **right minimal** if every endomorphism $h : Y \to Y$ such that $b\,h = b$ is an isomorphism. Left minimal preenvelopes are called **envelopes**, and right minimal precovers are called **covers**. Envelopes and covers are uniquely determined up to isomorphism [91, 1.2.1 and 1.2.6]. We will say that $\mathcal{X} \subset \operatorname{Mod} R$ is an **envelope class**, respectively a **cover class**, if every right R-module has an $\mathcal{X}$-envelope, respectively an $\mathcal{X}$-cover.

We say that a homomorphism $b : Y \to C$ has a **right minimal version** if there is a decomposition $Y = Y' \oplus K$ such that $b(K) = 0$ and the restriction $b' = b|_{Y'} : Y' \to C$ of b on Y' is right minimal. Of course, if b is an $\mathcal{X}$-precover and $Y' \in \mathcal{X}$, then b' is an $\mathcal{X}$-cover. Moreover, if b is an $\mathcal{X}$-precover and C admits an $\mathcal{X}$-cover, then there exists an $\mathcal{X}$-cover which is a right minimal version of b, see [91, 1.2.7]. **Left minimal versions** are defined dually and have the dual properties [91, 1.2.2].

When we are dealing with finite length modules, we can always find right or left minimal versions by choosing suitable modules of minimal length [16, 1.2 and 1.4]. This is the situation usually considered in the representation theory of artin algebras. In general, however, a module may have an $\mathcal{X}$-precover (or an $\mathcal{X}$-preenvelope) without having an $\mathcal{X}$-cover (respectively, an $\mathcal{X}$-envelope), as in the case $\mathcal{X} = \operatorname{Add} R$ with R non-perfect. The following result is a useful criterion for the existence of left, respectively right, minimal versions. It contains the finite length case and also a result of Krause and Saorín [66, 1.3], see [101]. We include a proof for the reader's convenience, since we are going to apply it in the sequel.

Theorem 1.6 (Zimmermann [101]) (I) Let $f : A \to X$ be a homomorphism, $E = \operatorname{End}_R X$ and $\ell_E(f) = \{h \in E \mid h\, f = 0\}$. Then f has a left minimal version provided that (i) there is a left ideal $\mathcal{I}$ in E such that $\ell_E(f) + \mathcal{I} = E$ and $\ell_E(f) \cap \mathcal{I} \subset J(E)$; and (ii) idempotents lift modulo $J(E)$.
(II) Let $g : Y \to C$ be a homomorphism, $E = \operatorname{End}_R Y$ and $r_E(g) = \{h \in E \mid g\, h = 0\}$. Then g has a right minimal version provided that (i) there is a right ideal $\mathcal{I}$ in E such that $r_E(g) + \mathcal{I} = E$ and $r_E(g) \cap \mathcal{I} \subset J(E)$; and (ii) idempotents lift modulo $J(E)$.

Proof: We show statement (II), the proof of (I) is dual. Let us denote by $\overline{x}$ the equivalence class in $E/J(E)$ of an element $x \in E$. By condition (i) there are $a \in r_E(g)$ and $b \in \mathcal{I}$ such that $\overline{a}, \overline{b}$ are orthogonal idempotents in $E/J(E)$ with $\overline{a} + \overline{b} = \overline{1_E}$.
Let us first prove that we can assume b idempotent in E. Indeed, by condition (ii), we can find an idempotent $b' \in E$ with $\overline{b'} = \overline{b}$. Then the element $u = 1_E - (b' - b)$ is invertible in E, and, using that $(u - b)\, b' = 0$, we obtain that the idempotent $b'' = u\, b'\, u^{-1} = b\, b'\, u^{-1}$ is an element of $\mathcal{I}$ satisfying $\overline{b''} = \overline{b''\, u} = \overline{u\, b'} = \overline{b'} = \overline{b}$.
Next, we deduce from $1_E - (a + b) \in J(E)$ that we can find an element $v \in E$ such that $(a + b)v = 1_E$, hence $av = 1_E - bv$. Then it is easy to check that $e = b + bv(1_E - b) \in \mathcal{I}$ is an idempotent with $1_E - e \in r_E(g)$ and $r_E(g) \cap e\, E \subset e\, J(E)$. This implies in particular that $r_{eEe}(g) \subset e\, J(E)\, e$.
Now it follows that every element $h \in eEe$ with $g\, h = g\, e$ satisfies $e - h \in e\, J(E)\, e$ and thus is invertible in eEe. But then the restriction $g' = g|_{Y'}$ of g on the direct summand $Y' = eY$ of Y is right minimal. Since $g((1_E - e)(Y)) = 0$, we conclude that g' is a right minimal version of g. $\square$

2 PERFECTNESS

As we have seen in Corollary 1.2, the category $\operatorname{Add}\mathcal{M}$ is always a precover class. When is it even a cover class? For the case $\mathcal{M} = \{R_R\}$, the answer is given by Bass's work on perfect rings. We are now going to discuss how Bass's results extend to the general case.

In section 2.1, we start by applying the criterion 1.6 for the existence of right minimal versions and prove that $\operatorname{Add}\mathcal{M}$ is a cover class if it consists of modules with semiregular endomorphism ring. Harada's notion of T-nilpotency will then enable us to relate this sufficient condition to the descending chain condition for finitely generated S-submodules of M and also to the case where the functor category $((\operatorname{add}\mathcal{M})^{\mathrm{op}}, Ab)$ is perfect (Proposition 2.7). We recall Simson's characterization of perfect functor categories in section 2.2 and translate it into properties of $\operatorname{Add}\mathcal{M}$ in case that $\mathcal{M}$ consists of finitely presented modules by applying some results of Lenzing. An important role in this context is played by Azumaya's notion of a locally split epimorphism, and more precisely, by the category $\mathcal{G}(M)$ of all locally split epimorphic images of modules in $\operatorname{Add}\mathcal{M}$, which we introduce in section 2.3. Our main result (Theorem 2.13) finally characterizes the subcategories $\mathcal{M} \subset \operatorname{mod} R$ providing for $\operatorname{Add}\mathcal{M}$-covers in terms of the two endoproperties mentioned above as well as in terms of closure properties under direct limits.

2.1 Semiregular endomorphism rings

A ring R is said to be **semiregular** if $R/J(R)$ is von Neumann regular and idempotents lift modulo $J(R)$. These rings were introduced by Oberst and Schneider [75] under the name F-semiperfect and were later studied by Nicholson [74] and other authors. We recall some equivalent conditions.

Theorem 2.1 (Oberst-Schneider [75, Satz 1.2]) The following statements are equivalent.
(1) R is semiregular.
(2) Every finitely presented right (or left) R-module has a projective cover.
(3) Every finitely generated right (or left) ideal $\mathcal{A}$ has an additive complement, i. e. a right (respectively, left) ideal $\mathcal{I}$ which satisfies $\mathcal{A}+\mathcal{I} = R$ and is minimal with respect to this property.

Applying Theorem 1.6, we obtain a criterion for the existence of $\operatorname{Add}\mathcal{M}$-covers.

Proposition 2.2 If $\operatorname{End}_R A$ is semiregular for all $A \in \operatorname{Add}\mathcal{M}$, then $\operatorname{Add}\mathcal{M}$ is a cover class.

Proof: Let C_R be a module and $g : M^{(H)} \to C$ an $\operatorname{Add}\mathcal{M}$-precover. Set $E = \operatorname{End}_R M^{(H)}$, and assume that $r_E(g) = \{h \in E \mid g\,h = 0\}$ is a finitely generated right ideal. Then it admits an additive complement $\mathcal{I}$ by the above Theorem 2.1. It is well known that $r_E(g) \cap \mathcal{I}$ is then superfluous in $\mathcal{I}$ and therefore also in E. Thus condition (i) in Theorem 1.6 is satisfied, and condition (ii) is satisfied by the definition of semiregular rings. Hence g has a right minimal version, and we are done.
So, our task is to show that we can assume $r_E(g)$ finitely generated without loss of generality. First of all, putting $K = \operatorname{Ker} g$, we deduce from the exact sequence $0 \to \operatorname{Hom}_R(M^{(H)}, K)_E \longrightarrow E_E \xrightarrow{\operatorname{Hom}_R(M^{(H)}, g)} \operatorname{Hom}_R(M^{(H)}, C)_E \to 0$ that $r_E(g) \simeq \operatorname{Hom}_R(M^{(H)}, K)$. Let us now consider an $\operatorname{Add}\mathcal{M}$-precover $f : M^{(J)} \to K$. If the cardinality of J is less or equal the cardinality of H, then the epimorphism $\operatorname{Hom}_R(M^{(H)}, f) : \operatorname{Hom}_R(M^{(H)}, M^{(J)})_E \to \operatorname{Hom}_R(M^{(H)}, K)_E$ shows that $r_E(g)$ is even a cyclic ideal. If the cardinality of J is greater than the cardinality of H, then J has a subset L such that $M^{(L)} \oplus M^{(H)} \cong M^{(J)}$, and we can replace g and f by the $\operatorname{Add}\mathcal{M}$-precovers $g' = (0, g) : M^{(J)} \cong M^{(L)} \oplus M^{(H)} \to C$ and $f' = \operatorname{pr} \oplus f : M^{(J)} \oplus M^{(J)} \to M^{(L)} \oplus K = \operatorname{Ker} g'$, where pr denotes the canonical projection. Then, putting $E' = \operatorname{End}_R M^{(J)}$, we infer from the epimorphism $\operatorname{Hom}_R(M^{(J)}, f') : E'^2{}_{E'} \to \operatorname{Hom}_R(M^{(J)}, \operatorname{Ker} g')_{E'}$ that $r_{E'}(g')$ is a finitely generated right ideal of E', and the proof is complete. □

Our aim is now to prove that the converse of Proposition 2.2 holds true when $\mathcal{M}$ consists of finitely presented modules. To this end, we first need a better understanding of modules with semiregular endomorphism ring. These modules have been characterized by Harada in terms of the notion of semi-T-nilpotency.

We say that the family $(M_i)_{i\in I}$ is **locally** (right) **semi-T-nilpotent** if for each sequence of non-isomorphisms $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} M_{i_3} \dots$ with pair-wise different indices $(i_n)_{n\in IN}$ from I, and each element $x \in M_{i_1}$, there exists $m = m_x \in IN$ such that $f_m f_{m-1} \dots f_1(x) = 0$. If the same condition is satisfied also when we allow repetitions in the sequence of indices $(i_n)_{n\in IN}$ involved, then the family $(M_i)_{i\in I}$ is called **locally** (right) **T-nilpotent**. If, furthermore, the index m does not depend on the element x, that is, if for each sequence of non-isomorphisms $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} M_{i_3} \dots$ there exists $m \in IN$ such that $f_m f_{m-1} \dots f_1 = 0$, then the family $(M_i)_{i\in I}$ is said to be (right) **T-nilpotent**. In Auslander's terminology [12], this means that all families of non-isomorphisms between modules from $\mathcal{M}$ are noetherian. Of course, if all M_i are finitely generated, then $(M_i)_{i\in I}$ is T-nilpotent whenever it is locally T-nilpotent.

Recall moreover that a submodule X of a module Y is called a **local direct summand** of Y if it has a decomposition $X = \coprod_{k\in K} X_k$ with the property that $\coprod_{k\in K_0} X_k$ is a direct summand of Y for every finite subset $K_0 \subset K$. Finally, we refer to [96] for a definition of the exchange property.

Theorem 2.3 (Harada, Zimmermann-Huisgen and Zimmermann [57, 7.3.15] [96, Corollary 6]) Assume that the endomorphism ring of each M_i is local. Then the following statements are equivalent.
(1) S is semiregular.
(2) The family $(M_i)_{i\in I}$ is locally semi-T-nilpotent.
(3) Every local direct summand of M is a direct summand.
(4) M has the (finite) exchange property.

2.2 Perfect functor categories

The concept of T-nilpotency can also be interpreted in terms of the functor category. Let $\mathcal{X}$ be a skeletally small subcategory of $\operatorname{Mod} R$, and let $\{X_\beta \mid \beta \in B\}$ be a complete irredundant set of representatives of the isomorphism classes of $\mathcal{X}$. We denote by $(\boldsymbol{\mathcal{X}^{\mathrm{op}}, Ab})$ the category of all additive contravariant functors $\mathcal{X} \to Ab$ from $\mathcal{X}$ to the category Ab of abelian groups. It is well known that $(\mathcal{X}^{\mathrm{op}}, Ab)$ is a locally finitely presented Grothendieck category and that the functors $H_\beta = \operatorname{Hom}_R(\ , X_\beta) \mid_{\mathcal{X}}$, $\beta \in B$, form a family of finitely presented projective generators.

The functors $F \in (\mathcal{X}^{\mathrm{op}}, Ab)$ which are direct limits of finitely generated free objects, that is, of finite coproducts of some H_β, are called **flat**. Furthermore, a direct system $(F_j, \varphi_{kj})_{k,j\in K}$ in $(\mathcal{X}^{\mathrm{op}}, Ab)$ is said to be **factorizable** if for every $j \in J$ there exists $l \geq j$ such that φ_{lj} factors through each φ_{kj} with $k \geq j$. Extending the concept of a Mittag-Leffler module due to Gruson and Raynaud [53], Simson has studied in [83] the functors $F \in (\mathcal{X}^{\mathrm{op}}, Ab)$ which are colimits of factorizable direct systems consisting of finitely presented objects. He has called them **Mittag-Leffler objects**. They coincide with the **Mittag-Leffler modules** of [53] when the category $\mathcal{X}$ just consists of the module R_R. For further details we refer to [83] and [72].

Flat objects and Mittag-Leffler objects play a prominent role in Simson's characterization of perfect functor categories. We say that $(\mathcal{X}^{\mathrm{op}}, Ab)$ is **perfect** if each of its objects admits a projective cover.

Theorem 2.4 (Simson [83, Theorem 5.4]) The following statements are equivalent for a skeletally small subcategory $\mathcal{X} \subset \mathrm{Mod}\, R$.
(1) The category $(\mathcal{X}^{\mathrm{op}}, Ab)$ is perfect.
(2) Every flat object in $(\mathcal{X}^{\mathrm{op}}, Ab)$ has a projective cover.
(3) Every flat object in $(\mathcal{X}^{\mathrm{op}}, Ab)$ is projective.
(4) Every flat object in $(\mathcal{X}^{\mathrm{op}}, Ab)$ is a Mittag-Leffler object.

Corollary 2.5 [83] Let $\mathcal{X} \subset \mathrm{Mod}\, R$ be a skeletally small subcategory. If $(\mathcal{X}^{\mathrm{op}}, Ab)$ is perfect, then each $X \in \mathcal{X}$ has a finite decomposition in modules with local endomorphism ring.

Proof: If all flat objects in $(\mathcal{X}^{\mathrm{op}}, Ab)$ are projective, then we know from [83, Corollary 5.2] that every finitely generated projective object in $(\mathcal{X}^{\mathrm{op}}, Ab)$ has a semiperfect endomorphism ring. So, we deduce from Yoneda's Lemma that each $X \in \mathcal{X}$ has a semiperfect endomorphism ring, which proves our claim by [49, 3.14]. □

We now turn to the connection with T-nilpotency.

Theorem 2.6 (Harada [56, Theorem 5]) Assume that the endomorphism ring of each M_i is local. Denote by $\mathcal{M}'$ the category of all finite coproducts of modules which are isomorphic to some M_i. Then the family $(M_i)_{i\in I}$ is T-nilpotent if and only if the category $(\mathcal{M}'^{\,\mathrm{op}}, Ab)$ is perfect.

Combining Theorem 2.3 with Theorem 2.6, we now establish a relationship between perfectness of $((\operatorname{add}\mathcal{M})^{\mathrm{op}}, Ab)$ and coperfectness of ${}_SM$. Recall that a left module ${}_BX$ over a ring B is said to be **coperfect** if it satisfies the descending chain condition for finitely generated (or equivalently, for cyclic) B-submodules [23].

Proposition 2.7 Assume that the endomorphism ring of each M_i is local. Then the following statements are equivalent.
(1) $\operatorname{End}_R A$ is semiregular for all $A \in \operatorname{Add}\mathcal{M}$.
(2) The family $(M_i)_{i\in I}$ is locally T-nilpotent.
(3) ${}_SM$ is coperfect.
Moreover, the conditions (1)-(3) are satisfied provided that $((\operatorname{add}\mathcal{M})^{\mathrm{op}}, Ab)$ is perfect, and the converse implication holds when all M_i are finitely generated.

Proof: (1)⇒(2): Let $M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} M_{i_3} \ldots$ be a sequence of non-isomorphisms with possible repetitions in the indices $(i_n)_{n\in I\!N}$ taken from I. Roughly speaking, we will now force the indices to become pairwise different by considering the (external) direct sum $A = \coprod_{n\in I\!N} M_{i_n}$. More precisely, we denote by $e_n : M_{i_n} \to A$ the canonical embeddings and set $A_n = e_n(M_{i_n})$. Since the endomorphism ring of $A \in \operatorname{Add}\mathcal{M}$ is semiregular, the family $(A_n)_{n\in I\!N}$ is locally semi-T-nilpotent by Theorem 2.3. Moreover, our sequence of non-isomorphisms corresponds to a sequence of non-isomorphisms between the A_n where none of the modules involved occurs more than once. Then for any element $x \in M_{i_1}$ there exists $n \in I\!N$ such that $f_n\, f_{n-1} \ldots f_1(x) = 0$.
(2)⇒(1): If a ring E is semiregular and $e \in E$ is idempotent, then eEe is semiregular as well [74, 2.3]. So, it is enough to consider modules A of the form $M^{(K)}$ for some set K. But $M^{(K)} = \coprod_{(i,k)\in I\times K} M_{(i,k)}$ where $M_{(i,k)} \simeq M_i$ for all $i \in I$ and $k \in K$, and the family $(M_{(i,k)})_{(i,k)\in I\times K}$ is locally semi-T-nilpotent by assumption. Thus the claim follows from Theorem 2.3.
(2)⇔(3) is proven in a recent paper by Huisgen-Zimmermann, Krause and Saorín [59, Proposition F].
For the last statement, we note that the category $\mathcal{M}'$ of all finite coproducts of modules which are isomorphic to some M_i coincides with $\operatorname{add}\mathcal{M}$. Indeed, every module $X \in \operatorname{add}\mathcal{M}$ is isomorphic to some direct summand U of a module in $\mathcal{M}'$. But then, by using for instance [2, 12.7], we deduce that also U has a finite decomposition in modules with local endomorphism ring which must be isomorphic to some M_i. So, U and X are in $\mathcal{M}'$ as well. Theorem 2.6 now completes the proof. □

2.3 Locally split homomorphisms

In order to discuss the existence of $\operatorname{Add}\mathcal{M}$-covers, we still need some preliminary results on two subcategories of $\operatorname{Mod}R$ related to $\mathcal{M}$.

Following Azumaya [17], we will say that a homomorphism $g : B \to A$ is a **locally split epimorphism** if for each $x \in A$ there is a map $\varphi = \varphi_x : A \to B$ such that $x = g\varphi(x)$. Dually, a homomorphism $f : A \to B$ is said to be a **locally split** (or strongly pure [78]) **monomorphism** if for each $x \in A$ there is a map $\psi = \psi_x : B \to A$ such that $x = \psi f(x)$. Of course, every split epimorphism (monomorphism) is locally split. Further, given a locally split epimorphism $g : B \to A$ (or a locally split monomorphism $f : A \to B$) and a finite number of elements $x_1, \ldots, x_n \in A$, we can use an argument of Villamayor (see [28, Proposition 2.2] or [19, Corollary 2]) to construct a map $\varphi : A \to B$ such that $x_i = g\varphi(x_i)$ for all $1 \le i \le n$ (or, respectively, a map $\psi : B \to A$ such that $x_i = \psi f(x_i)$ for all $1 \le i \le n$). This shows that every locally split epimorphism $g : B \to A$ (or every locally split monomorphism $f : A \to B$) is a pure epimorphism (respectively, a pure monomorphism), and moreover, that g splits whenever A is countably generated (respectively, f splits whenever A is finitely generated), see [67, Exercises 38 and 39 in Chapter II, §4].

A submodule X of a module Y is said to be a **locally split** (or strongly pure [78]) **submodule** if the embedding $X \subset Y$ is locally split. Examples for locally split submodules are the local direct summands considered in section 2.1.

Furthermore, a module X is called **locally pure-projective** if every pure epimorphism $Y \to X$ is locally split, and X is called **locally pure-injective** if every pure monomorphism $X \to Y$ is locally split.

Locally split submodules and locally pure-injective modules are studied in [78] and in a recent paper by Zimmermann [102]. As far as locally pure-projective modules are concerned, it was shown by Azumaya [19, Proposition 8] that they coincide with the strict Mittag-Leffler modules studied by Gruson and Raynaud in [53]. Indeed, they can be characterized as follows.

Proposition 2.8 (Azumaya [19, Proposition 4]) *A module A is locally pure-projective if and only if for each $x \in A$ there are a finitely presented module F and homomorphisms $f : A \to F$ and $\varphi : F \to A$ such that $\varphi f(x) = x$.*

Here we are mainly interested in modules A admitting a locally split epimorphism $g : M^{(K)} \to A$ for some set K. We will denote by $\boldsymbol{\mathcal{G}(M)}$ the category of all such modules.

In the case $M = R$, these modules were extensively studied by several authors, namely by Gruson and Raynaud [53] under the name 'flat strict Mittag-Leffler modules', by Ohm and Rush [76] under the name 'trace modules', by Garfinkel [50] under the name 'universally torsionless' and by Zimmermann-Huisgen [92] under the name **locally projective**, which is the terminology we will adopt here. Observe that a module A is locally projective if and only if every epimorphism $Y \to A$ is locally split, or equivalently, if A is flat and locally pure-projective.

In the general case, the modules in $\mathcal{G}(M)$ can be characterized as follows.

Lemma 2.9 The following statements are equivalent for a module A_R.
(1) $A \in \mathcal{G}(M)$.
(2) Every $\mathrm{Add}\,\mathcal{M}$-precover of A is a locally split epimorphism.
(3) For all X_R and $x \in X$ we have $\mathrm{H}_{X,x}(A) = \sum_{g\in \mathrm{Hom}_R(M,A)} g(\mathrm{H}_{X,x}(M))$.
(4) For each $x \in A$ there are $n \in I\!N$ and homomorphisms $f : A \to M^n$ and $\varphi : M^n \to A$ such that $\varphi\, f(x) = x$.

Proof: (1)⇒(2): This follows immediately from the fact that the locally split epimorphism $g : M^{(K)} \to A$ factors through any $\mathrm{Add}\,\mathcal{M}$-precover of A. Since every module has an $\mathrm{Add}\,\mathcal{M}$-precover, we then conclude (1)⇔(2).
(1)⇒(3): Let $h \in \mathrm{Hom}_R(X, A)$ and $g : M^{(K)} \to A$ a locally split epimorphism. Then there is a map $\varphi : A \to M^{(K)}$ such that $h(x) = g\varphi h(x) = \sum_{k\in K_0}(ge_k)\,(\mathrm{pr}_k\varphi h)(x)$ where K_0 is a finite subset of K and $e_k : M \to M^{(K)}$ and $\mathrm{pr}_k : M^{(K)} \to M$, $k \in K_0$, are the canonical inclusions and projections, respectively. Since $ge_k \in \mathrm{Hom}_R(M, A)$ and $(\mathrm{pr}_k\varphi h)(x) \in \mathrm{H}_{X,x}(M)$, we have shown the inclusion $\subset$. The other inclusion is always true.
(3)⇒(4): Every $x \in A$ can be written as $x = \mathrm{id}_A(x) \in \mathrm{H}_{A,x}(A)$ and therefore as $x = \sum_{i=1}^n g_i(y_i)$ for some $g_i \in \mathrm{Hom}_R(M, A)$ and $y_i \in \mathrm{H}_{A,x}(M)$, or, more precisely, as $x = \sum_{i=1}^n g_i\, f_i(x)$ where $f_i \in \mathrm{Hom}_R(A, M)$. But this means that $x = \varphi\, f(x)$ where $f : A \to M^n$ is the product map induced by the f_i and $\varphi : M^n \to A$ is the coproduct map induced by the g_i.
(4)⇒(1): For all $x \in A$ choose $n_x \in I\!N$, $f_x : A \to M^{n_x}$ and $\varphi_x : M^{n_x} \to A$ such that $\varphi_x\, f_x(x) = x$. Then it is easy to check that the coproduct map $g : \coprod_{x\in A} M^{n_x} \to A$ induced by the φ_x is a locally split epimorphism. □

Let us investigate the closure properties of $\mathcal{G}(M)$.

Lemma 2.10 (1) A module $A \in \mathcal{G}(M)$ belongs to $\operatorname{Add}\mathcal{M}$ provided it is countably generated or pure-projective or has a local endomorphism ring.
(2) $\mathcal{G}(M)$ is closed under coproducts, locally split submodules and locally split epimorphic images.

Proof: (1) The first two statements follow from the considerations at the beginning of this section. Assume that A has a local endomorphism ring and that $g : M^{(K)} \to A$ is a locally split epimorphism. Take an element $0 \neq x \in A$ and a homomorphism $\varphi : A \to M^{(K)}$ such that $x = g\varphi(x)$. The endomorphism $\mathrm{id}_A - g\,\varphi \in \operatorname{End}_R A$ then maps the nonzero element x to zero and therefore is a non-isomorphism. This implies that $g\,\varphi$ is an isomorphism and g is a split epimorphism.
(2) The closure under coproducts and locally split epimorphic images is straightforward. The closure under locally split submodules is easily verified using condition (4) in Lemma 2.9. $\square$

When $\mathcal{M}$ consists of finitely presented modules, $\mathcal{G}(M)$ has some further nice properties which we are going to discuss now. To this aim we first need some knowledge on the category $\vec{\mathcal{M}}$ of all modules which are the direct limit of some direct system of modules in $\operatorname{add}\mathcal{M}$.

Lemma 2.11 (Lenzing [70, 2.1]) Assume that $\mathcal{M}$ is a subcategory of $\operatorname{mod} R$. Then the following statements are equivalent for a module A_R.
(1) $A \in \vec{\mathcal{M}}$.
(2) There is a pure epimorphism $\coprod_{k\in K} X_k \to A$ for some modules X_k in $\mathcal{M}$.
(3) Every homomorphism $h : F \to A$ where F is finitely presented factors through a module in $\operatorname{add}\mathcal{M}$.

We then obtain the following result.

Proposition 2.12 Assume that $\mathcal{M}$ is a subcategory of $\operatorname{mod} R$. Then $\mathcal{G}(M)$ consists of all modules in $\vec{\mathcal{M}}$ which are locally pure-projective. In particular, $\mathcal{G}(M)$ is closed under pure submodules.

Proof: If A admits a locally split epimorphism $g : M^{(K)} \to A$ for some set K, then it satisfies condition (2) in the above Lemma and therefore lies in $\vec{\mathcal{M}}$. Further, since $M^{(K)}$ is pure-projective by assumption, the map g factors through any pure epimorphism $h : Y \to A$, which thus has to be locally split

as well. This shows that A is locally pure-projective. Conversely, it is straightforward that any locally pure-projective module satisfying condition (2) in the above Lemma belongs to $\mathcal{G}(M)$.
It remains to show that $\mathcal{G}(M)$ is closed under pure submodules. From condition (3) in Lemma 2.11 it follows that $\vec{\mathcal{M}}$ is closed under pure submodules, see [65, 3.10]. So, we have only to verify that every pure submodule $A' \subset A$ of a locally pure-projective module A is locally pure-projective. To this end, we apply Azumaya's characterization 2.8. Let $x \in A'$. Then there are a finitely presented module F and homomorphisms $f : A \to F$ and $\varphi : F \to A$ such that $\varphi\, f(x) = x$. Setting $y = f(x) \in F$, we see that x is contained in the finite matrix subgroup $\mathrm{H}_{F,y}(A)$. But $A' \subset A$ being pure implies that $\mathrm{H}_{F,y}(A) \cap A' = \mathrm{H}_{F,y}(A')$, see [98, p. 1088]. Thus we can find a map $\varphi' : F \to A'$ such that $x = \varphi'(y)$, and we conclude that the maps φ' and $f' = f|_{A'} : A' \to F$ satisfy $x = \varphi'\, f'(x)$, as required. $\square$

Further properties of $\mathcal{G}(M)$ will be discussed in section 3.1.

2.4 Add$\mathcal{M}$-covers

We are finally in a position to describe when a category $\mathcal{M}$ of finitely presented modules provides for $\operatorname{Add}\mathcal{M}$-covers.

Theorem 2.13 Assume that $\mathcal{M} \subset \operatorname{mod} R$. Then the following statements are equivalent.
(1) $\operatorname{Add}\mathcal{M}$ is a cover class.
(2) $((\operatorname{add}\mathcal{M})^{\mathrm{op}}, Ab)$ is perfect.
(3) $\operatorname{Add}\mathcal{M}$ is closed under direct limits.
(4) $\mathcal{G}(M)$ is closed under direct limits.
(5) All modules in $\vec{\mathcal{M}}$ are Mittag-Leffler modules.
(6) $\operatorname{End}_R A$ is semiregular for all $A \in \operatorname{Add}\mathcal{M}$.
(7) ${}_S M$ is coperfect, and M has a decomposition in modules with local endomorphism ring.
(8) Every pure submodule (or every locally split submodule, or every local direct summand) of a module $A \in \operatorname{Add}\mathcal{M}$ is a direct summand.
If $\mathcal{M}$ is a finite category, then (1)-(8) are further equivalent to
(9) S is right perfect.

Proof: We will employ the functor $H : \operatorname{Mod} R \longrightarrow ((\operatorname{add}\mathcal{M})^{\mathrm{op}}, Ab)$ which maps every module A_R to the functor $H_A = \operatorname{Hom}_R(\ ,A)\,|_{\operatorname{add}\mathcal{M}}$ and every

homomorphism $g \in \mathrm{Hom}_R(B, A)$ to the natural transformation $\mathrm{Hom}_R(\ , g) : H_B \to H_A$. Since $\mathcal{M} \subset \mathrm{mod}\, R$, we know that H commutes with coproducts and direct limits. As a consequence, H induces an equivalence between $\vec{\mathcal{M}}$ and the category of the flat objects in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$, which restricts to an equivalence between $\mathrm{Add}\,\mathcal{M}$ and the category of the projective objects in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$, see [70, 2.4]. This shows immediately that condition (3) is satisfied if and only if every flat object in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$ is projective, which is equivalent to condition (2) by Theorem 2.4.
Further, it is straightforward that a homomorphism $g : B \to A$ is an $\mathrm{Add}\,\mathcal{M}$-precover if and only if $\mathrm{Hom}_R(\ , g) : H_B \to H_A$ is an epimorphism with H_B being projective. Also, as an easy consequence of Yoneda's Lemma, we see that g is right minimal if and only if so is $\mathrm{Hom}_R(\ , g)$. But an epimorphism in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$ of the form $H_B \to F$ is right minimal if and only if it is superfluous, see [14, 2.1]. So, we conclude that under the action of H, the $\mathrm{Add}\,\mathcal{M}$-covers correspond to the projective covers. Thus we have (2)$\Rightarrow$(1). Conversely, condition (1) implies that all flat objects in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$ have a projective cover, which again by Theorem 2.4 is equivalent to (2). We remark that (3)$\Rightarrow$(1) can also be obtained as a special case of [91, Theorem 2.2.8].
Next, we recall that $\mathcal{G}(M)$ consists of the modules in $\vec{\mathcal{M}}$ which are locally pure-projective by Proposition 2.12. In particular, this implies that (3)$\Rightarrow$(4) is true. Moreover, if (4) is satisfied, then all modules in $\vec{\mathcal{M}}$ are locally pure-projective and therefore Mittag-Leffler modules by a result of Azumaya [19, Theorem 5]. This shows (4)$\Rightarrow$(5).
Assume now condition (5). Using again Yoneda's Lemma, we see that a direct system in $\mathrm{Mod}\, R$ is factorizable if and only if so is the direct system in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$ which corresponds to it under the action of H. Hence, as observed by Simson [83, p. 105], a module $A \in \vec{\mathcal{M}}$ is a Mittag-Leffler module if and only if H_A is a Mittag-Leffler object in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$. But then we deduce from (5) that all flat objects in $((\mathrm{add}\,\mathcal{M})^{\mathrm{op}}, Ab)$ are Mittag-Leffler objects, which is equivalent to (2) by Theorem 2.4.
So, we have proven the equivalence of the first five conditions. For the next equivalences, we first remind Corollary 2.5 which asserts that under condition (2), each M_i has a decomposition in modules N_{ij}, $j = 1 \dots n_i$, with local endomorphism ring. We are then in a position to apply Proposition 2.7 to the family (N_{ij}), and obtain that (6) and (7) are satisfied. Conversely, condition (7) implies (6), and (6) implies (1) by Proposition 2.2.

Let us now turn to condition (8). Observe that the cokernel C of a pure monomorphism $f : A' \to A$ with $A \in \operatorname{Add}\mathcal{M}$ belongs to $\vec{\mathcal{M}}$ by Lemma 2.11. So, if $\operatorname{Add}\mathcal{M}$ is closed under direct limits, the module C is pure-projective and f splits. This shows that (3) implies the strongest version of (8). Then also the special cases hold true, that is, the locally split submodules, and in particular, the local direct summands, of modules in $\operatorname{Add}\mathcal{M}$ split. Conversely, it is enough to require that all local direct summands of modules in $\operatorname{Add}\mathcal{M}$ are direct summands to obtain condition (6). In fact, it was recently proven by J. L. Gómez Pardo and P. A. Guil Asensio [51, 2.3] that under this assumption M has a decomposition in modules with local endomorphism ring. But then we can apply Theorem 2.3 to the modules of the form $A = M^{(K)}$ for some set K and conclude that A has a semiregular endomorphism ring. Since this implies by [74, 2.3] that also the direct summands of A have a semiregular endomorphism ring, we have just proven (8)⇒(6).
Finally, let us consider the case that $\mathcal{M}$ is a finite category. Then the well-known argument of Bass [2, pp. 316] shows that under condition (7) the Jacobson radical $J(S)$ of S is right T-nilpotent, cp. [56, Theorem 5] or [83, 5.4]. Moreover, S is then obviously semiperfect and therefore right perfect. Conversely, (9)⇒(8) is proven by a standard argument applying the functor $\operatorname{Hom}_R(M, \ ) : \operatorname{Mod} R \to \operatorname{Mod} S$ on pure-exact sequences $0 \longrightarrow A' \xrightarrow{f} A \longrightarrow C \longrightarrow 0$ in $\operatorname{Mod} R$ with $A \in \operatorname{Add}\mathcal{M}$. □

Of course, in the case $M = R$, Theorem 2.13 contains Theorem P of Bass [21] as well as some other characterizations of perfect rings due to Nicholson [74, 3.9], Huisgen-Zimmermann [94, p. 60] and to Azumaya and Facchini [20, Corollary 2], [18, Theorem 5].

In case that M is an arbitrary projective module, we rediscover and improve a characterization of perfect modules due to Stock. According to [71], a projective module P is called **perfect** if all modules generated by P have a projective cover.

Corollary 2.14 (cp. [85, 3.9]) Assume that M is projective. Then the following statements are equivalent.
(1) M is perfect.
(2) $\operatorname{End}_R M^{(K)}$ is semiregular for all sets K.
If M is finitely generated, (1) is further equivalent to
(3) S is right perfect.

Proof: (1)$\Rightarrow$(2) was already proven by Mares [71, 2.4 and 4.5]. Conversely, condition (2) implies by our Proposition 2.2 that every module A has a right minimal $\operatorname{Add} M$-precover $g : B \to A$. If A is M-generated, then g is an epimorphism and hence a projective cover, see [91, 1.2.12]. Thus (2)$\Rightarrow$(1). If, moreover, M is finitely generated, the equivalence of (6) and (9) in Theorem 2.13 yields (2)$\Leftrightarrow$(3). $\square$

Finally, in the case $\mathcal{M} = \operatorname{mod} R$, our Theorem 2.13 contains known characterizations of pure-semisimple rings which we will recall in section 6.4.

3 COHERENCE

We have seen in chapter 2 that the existence of Add $\mathcal{M}$-covers can be interpreted in terms of perfectness. Similarly, the existence of Add $\mathcal{M}$-preenvelopes is related to coherence.

Indeed, it was already shown by Enochs that the category of all finitely generated projective modules is covariantly finite in mod R if and only if R is left coherent, and more generally, it was proven by Crawley-Boevey that covariantly finiteness of $\mathcal{M}$ in mod R is characterized by closure under products of the category $\vec{\mathcal{M}}$.

We now interpret these results in terms of endoproperties of M. More precisely, we use the functor $\mathrm{Hom}_R(\ ,M) : \mathrm{Mod}\,R \to S\,\mathrm{Mod}$ to investigate the connection between the existence of Add $\mathcal{M}$-preenvelopes and coherence properties of ${}_SM$ (Theorem 3.17). Further, we employ a result of Zimmermann to show that a subcategory $\mathcal{M}$ of mod R is covariantly finite in mod R if and only if all finite matrix subgroups of M_R are finitely generated over S (Corollary 3.3). We then discuss when *all* matrix subgroups of M_R are finitely generated over S. This turns out to be equivalent to closure under products of $\mathcal{G}(M)$, in analogy to Crawley-Boevey's result mentioned above.

Our results contain known characterizations of two special classes of coherent rings, the strongly coherent and the π-coherent rings (Corollaries 3.12 and 3.18).

In the last section, we compare the different coherence properties of ${}_SM$ and relate them to coherence properties of S.

3.1 When matrix subgroups are finitely generated

We begin by establishing a connection between Add $\mathcal{M}$-preenvelopes and matrix subgroups of M_R. In fact, as we will now see, the existence of an Add $\mathcal{M}$-

preenvelope for a module A implies that all matrix subgroups of M induced by A are finitely generated modules over the endomorphism ring of M.

Lemma 3.1 Let A_R be a module and assume there is a map $a : A \to X$ such that $X \in \operatorname{Add} \mathcal{M}$ and all maps $h \in \operatorname{Hom}_R(A, Y)$ where $Y \in \operatorname{add} \mathcal{M}$ factor through a. Then there is a family $(g_k)_{k\in K}$ in $\operatorname{Hom}_R(A, M)$, and for every $x \in A$ there is a finite subset $K_x \subset K$ such that the matrix subgroup $\mathrm{H}_{A,x}(M)$ is generated by $(g_k(x))_{k\in K_x}$ as a left module over $S = \operatorname{End}_R M$.

Proof: Since $X \in \operatorname{Add} \mathcal{M}$, there is a family $(X_k)_{k\in K}$ in $\mathcal{M}$ and a split monomorphism $e : X \to \coprod_{k\in K} X_k$. Moreover, for any $k \in K$ there is a split monomorphism $e_k : X_k \to M$. Taking $g_k : A \xrightarrow{a} X \xrightarrow{e} \coprod_{k\in K} X_k \xrightarrow{pr_k} X_k \xrightarrow{e_k} M$ where pr_k are the canonical projections, we obtain a family in $\operatorname{Hom}_R(A, M)$ with the stated property. Indeed, for every $x \in A$ there is a finite subset $K_x \subset K$ such that $g_k(x) = pr_k\, ea(x) = 0$ for all $k \in K \setminus K_x$, and so $\mathrm{H}_{A,x}(M) = \sum_{k\in K_x} S \cdot g_k(x)$. $\square$

The following result of Zimmermann will enable us to find a sort of global converse for finitely generated modules. We include the proof for the reader's convenience.

Proposition 3.2 (Zimmermann [101]) Let A_R be a module, and let $\mathcal{A}$ denote the category of all modules X which are isomorphic to A/A' for some finitely generated submodule $A' \subset A$. Assume that all matrix subgroups of the form $\mathrm{H}_{X,x}(M)$ with $X \in \mathcal{A}$ and $x \in X$ are finitely generated over S. Then for all $n \in I\!N$, all $X \in \mathcal{A}$ and $\underline{x} = (x_1, \ldots, x_n) \in X^n$ we have that the image $\mathrm{H}_{X,\underline{x}}(M)$ of the S-homomorphism $\varepsilon_{\underline{x}} : \operatorname{Hom}_R(X, M) \longrightarrow M^n$, $f \mapsto (f(x_1), \ldots, f(x_n))$ is finitely generated. In particular, if A_R is finitely generated, then the left S-module $\operatorname{Hom}_R(A, M)$ is finitely generated.

Proof: If $X \in \mathcal{A}$, then the module $Y = X/x_1 R$ is in $\mathcal{A}$ as well, and the exact sequence $0 \longrightarrow x_1 R \longrightarrow X \xrightarrow{\nu} Y \longrightarrow 0$ together with the element $\underline{y} = (\nu(x_2), \ldots, \nu(x_n)) \in Y^{n-1}$ give rise to the following commutative diagram with S-linear maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(Y,M) & \xrightarrow{\mathrm{Hom}_R(\nu,M)} & \mathrm{Hom}_R(X,M) & \xrightarrow{\varepsilon_{x_1}} & \mathrm{H}_{X,x_1}(M) & \longrightarrow & 0 \\
 & & \downarrow\varepsilon_{\underline{y}} & & \varepsilon_{\underline{x}}\downarrow & & \| & & \\
0 & \longrightarrow & {}_S\mathrm{H}_{Y,\underline{y}}(M) & \longrightarrow & {}_S\mathrm{H}_{X,\underline{x}}(M) & \longrightarrow & {}_S\mathrm{H}_{X,x_1}(M) & \longrightarrow & 0 \\
 & & \cap & & \cap & & \cap & & \\
0 & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \xrightarrow{\mathrm{pr}_1} & M & \longrightarrow & 0
\end{array}$$

where pr_1 denotes the canonical projection on the first component. So, the claim follows by induction on n. Finally, if A_R is generated by the elements $a_1, \ldots, a_n$ and $\underline{a} = (a_1, \ldots, a_n) \in A^n$, then the S-homomorphism $\varepsilon_{\underline{a}}$ is injective, and we obtain that ${}_S\mathrm{Hom}_R(A, M)$ is finitely generated. □

We now obtain a useful criterion for the existence of $\mathrm{Add}\,\mathcal{M}$-preenvelopes in terms of finiteness conditions on S-submodules. In particular, we see that a subcategory $\mathcal{M} \subset \mathrm{mod}\,R$ is covariantly finite in $\mathrm{mod}\,R$ if and only if all finite matrix subgroups of M are finitely generated over S.

Corollary 3.3 Every finitely generated (finitely presented) module has an $\mathrm{Add}\,\mathcal{M}$-preenvelope if and only if all matrix subgroups of the form $\mathrm{H}_{A,x}(M)$ with A_R finitely generated (finitely presented) and $x \in A$ are finitely generated over S.

Proof: Combine the above results with Corollary 1.5, keeping in mind that the category $\mathcal{A}$ in Proposition 3.2 consists of finitely generated (finitely presented) modules if A_R is finitely generated (finitely presented). □

Note that covariantly finiteness can also be characterized in terms of the category $\vec{\mathcal{M}}$ by employing Lemma 2.11 and Proposition 1.3, as observed by Crawley-Boevey and Krause.

Theorem 3.4 (Crawley-Boevey, Krause [42, 4.2] [65, 3.11]) The following statements are equivalent for a subcategory $\mathcal{M}$ of $\mathrm{mod}\,R$.
(1) $\mathcal{M}$ is covariantly finite in $\mathrm{mod}\,R$.
(2) $\vec{\mathcal{M}}$ is closed under products.
(3) $\vec{\mathcal{M}}$ is a preenvelope class.

Our aim in this section will be to give an analogous characterization of the case that *all* matrix subgroups of M are finitely generated over S. We will see that the role of $\vec{\mathcal{M}}$ will now be played by $\mathcal{G}(M)$. But let us first point out that

Theorem 3.4 contains two known characterizations of coherence. Recall that a ring R is said to be **left coherent** if all finitely generated left ideals of R are finitely presented, or equivalently, if the category of all flat right R-modules is closed under products. But the latter is nothing else than condition (2) in the above theorem for the case that $\mathcal{M}$ just consists of the regular module R, and so we rediscover results of Enochs and Zimmermann.

Corollary 3.5 (ENOCHS, ZIMMERMANN [48, 5.1] [98, 1.3]) The following statements are equivalent for a ring R.
(1) R is left coherent.
(2) Every right R-module has a flat preenvelope.
(3) All finite matrix subgroups of R_R are finitely generated left ideals.

In order to discuss when all matrix subgroups of M are finitely generated over S, we start out by showing that the matrix subgroups of M induced by modules A in $\mathcal{G}(M)$ are always finitely generated over S.

Lemma 3.6 Let $A \in \mathcal{G}(M)$, $x \in A$ and $Y \in \operatorname{Mod} R$. Then there is an $n \in I\!N$ and an element $m \in M^n$ such that $\mathrm{H}_{A,x}(Y) = \mathrm{H}_{M^n,m}(Y)$. In particular, if $Y = M$, then there are $m_1, \ldots, m_n \in M$ which generate $\mathrm{H}_{A,x}(M)$ over S.

Proof: We know from Lemma 2.9 that there are $n \in I\!N$ and homomorphisms $f : A \to M^n$ and $\varphi : M^n \to A$ such that $\varphi f(x) = x$. Then, putting $m = f(x)$, we have for all $h \in \operatorname{Hom}_R(A, Y)$ that $h(x) = h'(m)$ with $h' = h\varphi \in \operatorname{Hom}_R(M^n, Y)$. This shows $\mathrm{H}_{A,x}(Y) \subset \mathrm{H}_{M^n,m}(Y)$. The other implication follows from the fact that $h(m) = hf(x)$ for all $h \in \operatorname{Hom}_R(M^n, Y)$. Assume now $Y = M$ and write $m = (m_1, \ldots, m_n) \in M^n$. Then for every element $h(x) \in \mathrm{H}_{A,x}(M)$ there is $h' \in \operatorname{Hom}_R(M^n, M)$ with components $s_1, \ldots, s_n \in S$ such that $h(x) = h'(m) = \sum_{j=1}^n s_j(m_j)$. $\square$

We will now need a category which can be viewed as the dual counterpart of $\mathcal{G}(M)$, namely the category $\boldsymbol{\mathcal{C}(M)}$ of all modules which are isomorphic to some locally split submodule of a product of copies M. Dually to Lemma 2.9, we can characterize $\mathcal{C}(M)$ as follows.

Lemma 3.7 The following statements are equivalent for a module C_R.
(1) $C \in \mathcal{C}(M)$.
(2) Every $\operatorname{Prod} M$-preenvelope of C is a locally split monomorphism.
(3) For all X_R and $x \in X$ we have $\mathrm{H}_{X,x}(C) = \bigcap_{f \in \operatorname{Hom}_R(C,M)} f^{-1}(\mathrm{H}_{X,x}(M))$.

Proof: (1)⇔(2) is proven with arguments dual to those in Lemma 2.9.
(1)⇔(3): If $C \in \mathcal{C}(M)$, then there is a locally split monomorphism $\tau : C \to M^J$ for some set J. Thus every element $c \in \bigcap_{f\in \mathrm{Hom}_R(C,M)} f^{-1}(\mathrm{H}_{X,x}(M))$ satisfies $\tau(c) \in \prod_{j\in J} \mathrm{H}_{X,x}(M) = \mathrm{H}_{X,x}(M^J)$ and admits a map $\psi : M^J \to C$ such that $c = \psi\tau(c)$, hence belongs to $\mathrm{H}_{X,x}(C)$. Since the inclusion $\subset$ is always true, we have thus proven (1)⇒(3). Conversely, if C has property (3), we consider the product map $\tau : C \to M^J$ induced by all homomorphisms in $J = \mathrm{Hom}_R(C, M)$. Then, if we choose $X = M^J$ in (3), it is easy to check that every $c \in C$ belongs to $\mathrm{H}_{M^J,\tau(c)}(C)$ and therefore has the form $c = \psi\tau(c)$ for some $\psi : M^J \to C$. Thus τ is a locally split monomorphism, and $C \in \mathcal{C}(M)$. □

Let us collect further properties of $\mathcal{C}(M)$.

Proposition 3.8 (1) A module $C \in \mathcal{C}(M)$ belongs to $\mathrm{Prod}\, M$ provided it is finitely generated or pure-injective or has a local endomorphism ring.
(2) $\mathcal{C}(M)$ is closed under locally split submodules and locally split epimorphic images, and moreover, under products and coproducts. In particular, $\mathcal{G}(M) \subset \mathcal{C}(M)$.
(3) If M is pure-injective, then $\mathcal{C}(M)$ consists of all modules which are isomorphic to some pure submodule of a product of copies M and are locally pure-injective.

Proof: (1) is proven with arguments dual to those in Lemma 2.10 (1).
(2) The closure under products and locally split submodules is straightforward. Since $\coprod_{k\in K} X_k$ is a locally split submodule of $\prod_{k\in K} X_k$, we then also have that $\mathcal{C}(M)$ is closed under coproducts. Let us now show that $C' \in \mathcal{C}(M)$ whenever there is a locally split epimorphism $g : C \to C'$ with $C \in \mathcal{C}(M)$. Observe first that for every $c' \in C'$ there is an element $c \in C$ such that $g(c) = c'$ and $\mathrm{H}_{C',c'}(M) = \mathrm{H}_{C,c}(M)$. We assume that $c' \in \bigcap_{f'\in \mathrm{Hom}_R(C',M)} {f'}^{-1}(\mathrm{H}_{X,x}(M))$ for some X_R and $x \in X$ and choose an element $c \in C$ as above. Then for every $f \in \mathrm{Hom}_R(C, M)$ we have that $f(c)$ belongs to $\mathrm{H}_{C',c'}(M)$ and therefore has the form $f(c) = f'(c')$ for some $f' \in \mathrm{Hom}_R(C', M)$. This shows that c lies in $\bigcap_{f\in \mathrm{Hom}_R(C,M)} f^{-1}(\mathrm{H}_{X,x}(M))$, which coincides with $\mathrm{H}_{X,x}(C)$ by assumption. Since $c' = g(c)$, it then follows that c' lies in $\mathrm{H}_{X,x}(C')$. So, since the inclusion $\subset$ in condition (3) of Lemma 3.7 is always true, we have proven that $C' \in \mathcal{C}(M)$. Observe that the inclusion $\mathcal{G}(M) \subset \mathcal{C}(M)$ can also be verified by using condition (4) in 2.9 to construct a locally split monomorphism $A \to \prod_{x\in A} M^{n_x}$.
(3) is proven with arguments dual to those in Proposition 2.12. □

The following examples show that $\mathcal{C}(M)$ in general is neither closed under pure submodules nor under pure epimorphic images.

Example 3.9 (1) Let A_R be a finitely generated module which is not pure-injective. Such modules exist for example over von Neumann regular, non-semisimple rings. Let further M be a pure-injective envelope of A. Then A is a pure submodule of M. But A is not contained in $\mathcal{C}(M)$, because otherwise it would even belong to $\operatorname{Prod} M$ by Proposition 3.8 and would therefore be pure-injective.
(2) Let R be a left noetherian ring which is not right perfect. By Corollary 3.5 the class of all flat R-modules is a preenvelope class. Moreover, we will see in Theorem 3.11 that $\mathcal{C}(R)$ coincides with the class $\mathcal{G}(R)$ of all locally projective modules and is a preenvelope class as well. However, $\mathcal{G}(R)$ is not closed under direct limits by Theorem 2.13. □

The next result is inspired by an argument due to Ringel [81, Proposition 2].

Lemma 3.10 Let A_R be a module, $x \in A$, and assume that there are a family $(X_k)_{k\in K}$ in $\mathcal{M}$ and homomorphisms $\mu : A \to \prod_{k\in K} X_k$, and $\varepsilon : \prod_{k\in K} X_k \to A$ with $\varepsilon\mu(x) = x$. Further, let λ be a cardinal such that the matrix subgroup of M of the form $\mathrm{H}_{A,x}(M)$ is a λ-generated left S-module. Then there is a family $(Y_l)_{l\in L}$ in $\mathcal{M}$, where the index set L is finite if λ is finite or else has cardinality λ, and there are homomorphisms $f : A \to \prod_{l\in L} Y_l$ and $\varphi : \prod_{l\in L} Y_l \to A$ such that $\varphi f(x) = x$.

Proof: Let $(y_j)_{j\in J}$ be a generating set of $\mathrm{H}_{A,x}(M)$ over S where J is a set of cardinality λ. Since $\mathrm{H}_{A,x}(M) \cong \coprod_{i\in I} \mathrm{H}_{A,x}(M_i)$, for each $j \in J$ we have a finite subset $I_j \subset I$ and maps $f_{jl} \in \operatorname{Hom}_R(A, M_l)$ such that $y_j = \sum_{l\in I_j} \iota_l\, f_{jl}(x)$ where the $\iota_l : M_l \to M$ are the canonical embeddings. Then our homomorphism f will be the product map $f : A \to M' = \prod_{j\in J} \prod_{l\in I_j} M_l$ given by all f_{jl}.
Further, for any $k \in K$ there are maps $e_k : X_k \to M$ and $p_k : M \to X_k$ with $p_k\, e_k = \mathrm{id}_{X_k}$. We set $p = \prod_{k\in K} p_k : M^K \to \prod_{k\in K} X_k$ and denote by $pr_k : \prod_{k\in K} X_k \to X_k$ the canonical projections.
Our aim is to construct a map $\psi : M' \to M^K$ such that for the composition $\varphi : M' \xrightarrow{\psi} M^K \xrightarrow{p} \prod_{k\in K} X_k \xrightarrow{\varepsilon} A$ we have $\varphi f(x) = x$. To this end, we consider $\mu(x) = (x_k)_{k\in K}$ and write each $e_k(x_k) = e_k pr_k \mu(x) \in \mathrm{H}_{A,x}(M)$ as $e_k(x_k) = \sum_{j\in J_k} s_{kj} y_j$ for some finite subset $J_k \subset J$ and some $s_{kj} \in S$. Then

we have $e_k(x_k) = \sum_{j\in J_k}\sum_{l\in I_j} s_{kj}\iota_l f_{jl}(x)$. We now define maps $\psi_k : M' \to M$, $z = (z_{jl})_{j\in J, l\in I_j} \mapsto \sum_{j\in J_k}\sum_{l\in I_j} s_{kj}\iota_l(z_{jl})$, for each k and take the product map $\psi : M' \to M^K$, $z \mapsto (\psi_k(z))_{k\in K}$. Then it is easy to check that $\varphi f(x) = x$, and the proof is complete. □

We are now ready to prove the main result of this section.

Theorem 3.11 The following statements are equivalent.
(1) All matrix subgroups of M are finitely generated over S.
(2) $\mathcal{G}(M) = \mathcal{C}(M)$.
(3) $\mathcal{G}(M)$ is closed under products.
(4) $M^M \in \mathcal{G}(M)$.
If $\mathcal{M} \subset \operatorname{mod} R$, then the following statement is further equivalent.
(5) $\mathcal{G}(M)$ is a preenvelope class.

Proof: (1)⇒(2): By Proposition 3.8, we have only to prove that $\mathcal{C}(M) \subset \mathcal{G}(M)$. Let $A \in \mathcal{C}(M)$ and $\mu : A \to M^K$ a locally split monomorphism. For every $x \in A$ there is a map $\varepsilon : M^K \to A$ such that $x = \varepsilon\,\mu(x)$. But then condition (1) and Lemma 3.10 tell us that we can choose homomorphisms $f : A \to M^n$ and $\varphi : M^n \to A$ with $n \in I\!N$ such that $x = \varphi\, f(x)$. In other words, A satifies condition (4) in Lemma 2.9 and is therefore contained in $\mathcal{G}(M)$.
(2)⇒(3) follows immediately from Proposition 3.8; (3)⇒(4) is clear.
(4)⇒(1): It is well known that every matrix subgroup $\mathrm{H}_{A,x}(M)$ of M can be written in the form $\mathrm{H}_{M^M,y}(M)$ by taking the element $y = (y_m)_{m\in M} \in M^M$ defined by $y_m = m$ if $m \in \mathrm{H}_{A,x}(M)$ and $y_m = 0$ otherwise, see for instance [92, p. 241]. But now we can apply Lemma 3.6 to conclude that $\mathrm{H}_{A,x}(M)$ is finitely generated over S.
Finally, if $\mathcal{M} \subset \operatorname{mod} R$, then we know from Proposition 2.12 that $\mathcal{G}(M)$ is closed under pure submodules. Hence Proposition 1.3 tells us that $\mathcal{G}(M)$ is a preenvelope class if and only if it is closed under products. This completes the proof. □

If we restrict to the case $M = R$, then condition (3) in the above theorem asserts that products of (locally) projective right R-modules are locally projective. Rings with this property have been called **left strongly coherent** by Zimmermann-Huisgen [92]. The above theorem was actually inspired by her characterization of these rings, which was also independently obtained by

Garfinkel [50] and which we are now going to recall. The last condition in our Corollary, however, seems to be new.

Corollary 3.12 (Garfinkel, Zimmermann-Huisgen [50, 5.4] [92, 4.2]) The following statements are equivalent for a ring R.
(1) R is left strongly coherent.
(2) All matrix subgroups of R_R are finitely generated left ideals.
(3) All pure (or locally split) submodules of direct products of copies of R are locally projective.
(4) The module R^R is locally projective.
(5) Every right R-module has a locally projective preenvelope.

As pointed out in [92, p. 242], the commutative ring $R = K^{I\!N}$ over a field K is an example of a strongly coherent ring which is not noetherian. For the relationship between coherent and strongly coherent rings, we refer to section 3.3. Let us now close by exhibiting a rather natural example of modules with all matrix subgroups being finitely generated.

Example 3.13 Let M_R be a locally projective module over a left strongly coherent ring R. Then all matrix subgroups of M_R are finitely generated over S if and only if ${}_SM$ is finitely generated. Indeed, the only-if part is always true, for instance by applying Proposition 3.2 to $A = R$. For the if-part, we need the fact that the matrix subgroups of a locally projective module have the form $\mathrm{H}_{A,x}(M) = M \cdot \mathrm{H}_{A,x}(R_R)$, see [92, Theorem 2.1]. Then, if ${}_SM$ and ${}_R\mathrm{H}_{A,x}(R)$ are finitely generated, it follows immediately that ${}_S\mathrm{H}_{A,x}(M)$ is finitely generated as well. $\Box$

3.2 Endocoherence

This section is devoted to investigating coherence properties of a module over its endomorphism ring. Some of the results will be applied to tilting theory in section 6.1.

Let ${}_BQ_A$ be a bimodule. Recall that a module X_A is said to be Q_A-**reflexive** if the evaluation morphism $\boldsymbol{\delta}_X\colon X \to \mathrm{Hom}_B\left(\mathrm{Hom}_A\left(X, Q_A\right), {}_BQ\right)$ given by $\delta_X(x) : \alpha \mapsto \alpha(x)$ is an isomorphism. Of course, since $\mathrm{Ker}\,\delta_X$ coincides with the **reject** $\mathrm{Rej}_Q(X)$ of Q in X, all reflexive modules are in the category $\mathbf{Cogen}\,\boldsymbol{Q}$ of Q-cogenerated modules. We denote further by $\mathbf{cogen}\,\boldsymbol{Q}$

the category of all finitely Q-cogenerated modules, by **copres $\boldsymbol{Q}$** (respectively, by **sfcopres $\boldsymbol{Q}$**) the category of all **(semi-)finitely $\boldsymbol{Q}$-copresented modules**, that is, of all modules X admitting an exact sequence $0 \longrightarrow X \longrightarrow Q^n \longrightarrow L \longrightarrow 0$ where $n \in I\!N$ and L is finitely Q-cogenerated (respectively, Q-cogenerated). Dually, we write **gen $\boldsymbol{Q}$** for the category of all finitely Q-generated modules, and **pres $\boldsymbol{Q}$** for the category of all **finitely $\boldsymbol{Q}$-presented modules**, that is, of all modules X admitting an exact sequence $0 \longrightarrow K \longrightarrow Q^n \longrightarrow X \longrightarrow 0$ where $n \in I\!N$ and K is finitely Q- generated. Finally, we denote by $\boldsymbol{\mathcal{K}}(\boldsymbol{Q}_A)$ the subcategory of $\mathrm{Mod}\,A$ consisting of all modules K_A which admit an exact sequence $0 \longrightarrow K \longrightarrow A^n \longrightarrow Y_A \longrightarrow 0$ where $n \in I\!N$ and Y_A is Q_A-cogenerated, and by $\boldsymbol{\mathcal{K}}({}_B\boldsymbol{Q})$ the corresponding subcategory of $B\,\mathrm{Mod}$. We will be interested in the case where Q is our bimodule ${}_SM_R$. Then S is obviously ${}_SM$-reflexive, and we have the following result.

Lemma 3.14 (1) ${}_S\mathrm{Hom}_R\,(A, M) \in \mathrm{sfcopres}\,{}_SM$ for all finitely generated modules A_R, and ${}_S\mathrm{Hom}_R\,(A, M) \in \mathrm{copres}\,{}_SM$ for all finitely presented modules A_R.
(2) The functor $\mathrm{Hom}_R\,(\ , M) : \mathrm{Mod}\,R \longrightarrow S\,\mathrm{Mod}$ induces dense functors $\mathrm{gen}\,M_R \longrightarrow \mathcal{K}({}_SM)$ and $\mathrm{pres}\,M_R \longrightarrow \mathrm{copres}\,{}_SS$.

Proof: (1) Let A_R be finitely generated with an exact sequence $0 \longrightarrow K \xrightarrow{f} R^n \longrightarrow A \longrightarrow 0$. We then have an exact sequence $0 \to {}_S\mathrm{Hom}_R\,(A, M) \to {}_S\mathrm{Hom}_R\,(R^n, M) \xrightarrow{\mathrm{Hom}_R\,(f,M)} {}_S\mathrm{Hom}_R\,(K, M)$ where ${}_S\mathrm{Hom}_R\,(K, M)$ is a submodule of ${}_S\mathrm{Hom}_R\,(R^{(J)}, M) \simeq {}_SM^J$ for some set J. Further, if A_R is finitely presented, then K is finitely generated, and ${}_S\mathrm{Hom}_R\,(K, M)$ is a submodule of ${}_S\mathrm{Hom}_R\,(R^n, M) \simeq {}_SM^n$ for some $n \in I\!N$.
(2) As in (1), we show that $A \in \mathrm{gen}\,M_R$ gives rise to an exact sequence $0 \to {}_S\mathrm{Hom}_R\,(A, M) \to {}_S\mathrm{Hom}_R\,(M^n, M) \to_S \mathrm{Hom}_R\,(K, M)$ where ${}_S\mathrm{Hom}_R\,(K, M)$ is ${}_SM$-cogenerated, and moreover, that we can assume ${}_S\mathrm{Hom}_R\,(K, M)$ finitely cogenerated by S provided that $A \in \mathrm{pres}\,M_R$. So, it remains to prove that the functors are dense. Any exact sequence $0 \longrightarrow K \longrightarrow S^n \longrightarrow {}_SY \longrightarrow 0$ with $Y \in \mathrm{Cogen}\,{}_SM$ yields an exact sequence $0 \to \mathrm{Hom}_S\,(Y, M) \to \mathrm{Hom}_S\,(S^n, M)$ $\xrightarrow{g} \mathrm{Hom}_S\,(K, M)$ where $L_R = \mathrm{Im}\,g$ is an epimorphic image of M^n. We obtain the commutative diagram

$$\begin{array}{ccccccc}
0 \to & K & \longrightarrow & S^n & \longrightarrow & Y & \to 0 \\
 & \downarrow\alpha & & \delta_{S^n}\downarrow & & \downarrow\delta_Y & \\
0 \to & \mathrm{Hom}_R\,(L, M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n, M), M)) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(Y, M), M) &
\end{array}$$

where α and δ_Y are monomorphisms and δ_{S^n} is an isomorphism. Then by the snake lemma α is an isomorphism, hence ${}_SK \cong \mathrm{Hom}_R(L, M)$ with $L \in \mathrm{gen}\, M_R$.

Assume further that there is a monomorphism $i : Y \to S^m$ for some $m \in I\!N$. Then we also have a map $f = \mathrm{Hom}_S(i, M) : \mathrm{Hom}_S(S^m, M) \to \mathrm{Hom}_S(Y, M)$ with $A_R = \mathrm{Im}\, f \in \mathrm{gen}\, M_R$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \mathrm{Hom}_S(S^n, M) & \longrightarrow & L' & \longrightarrow & 0 \\ & & \bigcap e & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(Y, M) & \longrightarrow & \mathrm{Hom}_S(S^n, M) & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

where $L' \in \mathrm{pres}\, M_R$. Since δ is a natural transformation, $\mathrm{Hom}_R(f, M)\,\delta_Y = \delta_{S^m}\, i$ is a monomorphism, and therefore $\mathrm{Hom}_R(e, M)\,\delta_Y$ is a monomorphism as well. So, we conclude as above from the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & K & \longrightarrow & S^n & \longrightarrow & Y & \to & 0 \\ & \downarrow \alpha & & \delta_{S^n} \downarrow & & \downarrow \delta_Y & & \\ 0 \to & \mathrm{Hom}_R(L, M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n, M), M)) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(Y, M), M) & & \\ & \downarrow \beta & & \| & & \downarrow \mathrm{Hom}_R(e,M) & & \\ 0 \to & \mathrm{Hom}_R(L', M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S^n, M), M)) & \longrightarrow & \mathrm{Hom}_R(A, M) & & \end{array}$$

that $\beta\alpha$ is an isomorphism, hence ${}_SK \cong \mathrm{Hom}_R(L', M)$ with $L' \in \mathrm{pres}\, M_R$. $\Box$

Let us remark that if ${}_SM_R$ is faithfully balanced, then by similar arguments, the functor $\mathrm{Hom}_R(\ \ , M) : \mathrm{Mod}\, R \longrightarrow S\,\mathrm{Mod}$ induces dense functors $\mathrm{gen}\, R \longrightarrow \mathrm{sfcopres}\, {}_SM$ and $\mathrm{mod}\, R \longrightarrow \mathrm{copres}\, {}_SM$.

We now obtain a characterization of left coherent endomorphism rings.

Proposition 3.15 *S is left coherent if and only if every $A \in \mathrm{pres}\, M$ has an $\mathrm{add}\, M$-preenvelope.*

Proof: Of course, S is left coherent if and only if every module in $\mathrm{copres}\, {}_SS$ is finitely generated over S. By Lemma 3.14 the latter means that ${}_S\mathrm{Hom}_R(A, M)$ is finitely generated for all modules $A_R \in \mathrm{pres}\, M$. Combining this with Corollary 1.5, we obtain the claim. $\Box$

Note that Lenzing has described left coherence in terms of annihilators of matrix rings [68, §4, Korollar 1]. More precisely, denoting by $S^{n\times n}$ the $n \times n$ matrix ring over S, he has proven that S is left coherent if and only if for every $n \in I\!N$ and every $f \in S^{n\times n}$ the left annihilator $\ell_{S^{n\times n}}(f)$ of f in $S^{n\times n}$ is a finitely generated left ideal. We are now going to investigate the annihilators of subsets of M^n in $S^{n\times n} \cong \mathrm{End}_R M^n$, and we are going to see that they are related to a stronger coherence property.

Proposition 3.16 The following statements are equivalent.
(1) Every finitely generated left S-module which is cogenerated by ${}_SM$ is finitely presented.
(2) Every $A \in \operatorname{gen} M$ has an $\operatorname{add} M$-preenvelope.
(3) For every $n \in I\!N$ and every $X \subset M^n$ the annihilator $\mathrm{ann}_{S^{n\times n}}(X)$ of X in $S^{n\times n}$ is a finitely generated left ideal.

Proof: (1) $\Leftrightarrow$ (2) is shown combining Lemma 3.14 with Corollary 1.5 as in the proof of Proposition 3.15.
(2)$\Rightarrow$(3): Let $X \subset M^n$, put $K = X \cdot R$ and $A_R = M^n/K$, and denote by $\nu : M^n \to A$ the canonical surjection. By assumption, $A_R \in \operatorname{gen} M$ has an $\operatorname{add} M$-preenvelope $a : A \to M^m$, and we can consider the maps $f_i : M^n \xrightarrow{\nu} A \xrightarrow{a} M^m \xrightarrow{\mathrm{pr}_i} M \xrightarrow{\iota} M^n$, $1 \le i \le m$, where pr_i and ι denote the canonical projections and a canonical injection, respectively. Obviously, $f_1, \ldots, f_m$ are contained in $\mathrm{ann}_{S^{n\times n}}(X)$, and since every other map $h \in \mathrm{ann}_{S^{n\times n}}(X)$ factors through ν and hence through $a\nu$, they are generators of $\mathrm{ann}_{S^{n\times n}}(X)$ over $S^{n\times n}$.
(3)$\Rightarrow$(2): Consider an exact sequence $0 \longrightarrow K \longrightarrow M^n \xrightarrow{g} A \longrightarrow 0$ and a generating set $f_1, \ldots, f_m$ of $\mathrm{ann}_{S^{n\times n}}(K)$ over $S^{n\times n}$. Then K is contained in the kernel of the product map $f : M^n \to M^{nm}$ induced by the f_i, and so there is a map $a : A \to M^{nm}$ such that $f = a\,g$. Let us verify that a is an $\operatorname{add} M$-preenvelope. In fact, if we denote again by $M \xrightarrow{\iota} M^n$ a canonical injection, then for every homomorphism $h : A \to M$ the composition $\iota\, h\, g$ lies in $\mathrm{ann}_{S^{n\times n}}(K)$ and therefore has the form $\sum_{i=1}^m t_i\, f_i$ for some $t_1, \ldots, t_m \in S^{n\times n}$. This shows that $h\, g$ factors through $a\, g$, and hence h factors through a. $\square$

Recall that a left module ${}_BX$ over a ring B is **coherent** if it is finitely presented and every finitely generated submodule of ${}_BX$ is finitely presented. Inspired by Lenzing's and Camillo's work on a special class of coherent rings [68], [27], we will further say that ${}_BX$ is $\boldsymbol{\pi}$**-coherent** if it is finitely presented and every finitely generated left B-module which is cogenerated by ${}_BX$ is finitely presented. Then the ring B is left π-coherent in the sense of [27] (or 'stark links-kohärent' in the sense of [68]) if and only if the regular left module ${}_BB$ is π-coherent.

Theorem 3.17 (1) If ${}_SM$ is π-coherent, then every finitely generated module has an $\operatorname{add} M$-preenvelope. The converse holds if M_R is finitely generated.

(2) If ${}_SM$ is coherent, then every finitely presented module has an $\operatorname{add} M$-preenvelope. The converse holds if M_R is finitely presented.

Proof: (1) If A_R is finitely generated, then by Lemma 3.14 there is an exact sequence $0 \longrightarrow {}_S\mathrm{Hom}_R(A, M) \longrightarrow {}_SM^n \longrightarrow L \longrightarrow 0$ where $n \in I\!N$ and $L \in \operatorname{Cogen} {}_SM$. By assumption L is then finitely generated and even finitely presented, so ${}_S\mathrm{Hom}_R(A, M)$ is finitely generated, and A has an $\operatorname{add} M$-preenvelope by Corollary 1.5. Conversely, if M_R is finitely generated and every finitely generated module has an $\operatorname{add} M$-preenvelope, then we deduce that R and every $A \in \operatorname{gen} M$ have an $\operatorname{add} M$-preenvelope. But this implies by 1.5 that ${}_SM \cong_S \mathrm{Hom}_R(R, M)$ is finitely generated, and moreover, that ${}_SM$ satisfies condition (1) in Proposition 3.16. So, we conclude that ${}_SM$ is π-coherent.
(2) We show as in (1) that Lemma 3.14 and Corollary 1.5 yield the existence of an $\operatorname{add} M$-preenvelope for every finitely presented module A_R. Conversely, if M_R is finitely presented and every finitely presented module has an $\operatorname{add} M$-preenvelope, then we deduce that R and every $A \in \operatorname{pres} M$ have an $\operatorname{add} M$-preenvelope. In particular, S is then left coherent by Proposition 3.15. Moreover, if $a : R \to M^n$ is an $\operatorname{add} M$-preenvelope with cokernel L, then also L_R is finitely presented, and therefore ${}_S\mathrm{Hom}_R(L, M)$ is finitely generated by 1.5. So, we infer from the exact sequence $0 \longrightarrow {}_S\mathrm{Hom}_R(L, M) \longrightarrow {}_S\mathrm{Hom}_R(M^n, M) \longrightarrow {}_S\mathrm{Hom}_R(R, M) \longrightarrow 0$ that ${}_SM$ is finitely presented and hence coherent. □

The above results also include known characterizations of π-coherent rings.

Corollary 3.18 ([68, Satz 4], [94, p. 36], [27], [77, 5.3]) The following statements are equivalent.
(1) R is left π-coherent.
(2) For every $n \in I\!N$ and every $X \subset R^n$ the annihilator $\mathrm{ann}_{R^{n\times n}}(X)$ of X in $R^{n\times n}$ is a finitely generated left ideal.
(3) All matrix subgroups of the form $\mathrm{H}_{A,x}(R_R)$ for some finitely generated module A_R are finitely generated left ideals.
(4) ${}_R\mathrm{Hom}_R(A, R)$ is finitely generated for every finitely generated module A_R.
(5) Every finitely generated module A_R has a projective preenvelope. □

If R is semiregular, then it was shown by Asensio Mayor-Martinez Hernandez [10, Corollary 3] and by Rada-Saorin [77, Corollary 5.4] that R being left (π-)coherent even implies the existence of projective envelopes for the finitely presented (respectively, finitely generated) modules. We can now use the criterion for the existence of minimal versions in Theorem 1.6 to generalize those results.

Corollary 3.19 Let S be semiregular.
(1) If ${}_SM$ is π-coherent, then every finitely generated module has an $\mathrm{Add}\,\mathcal{M}$-envelope.
(2) If ${}_SM$ is coherent and M_R is finitely presented, then every finitely presented module has an $\mathrm{Add}\,\mathcal{M}$-envelope.

Proof: By Theorem 3.17 we have only to show the existence of a left minimal version for an $\mathrm{Add}\,\mathcal{M}$-preenvelope $f : A \to M^n$ with A finitely generated or finitely presented, respectively. Note that in both cases the cokernel $L = \mathrm{Coker}\, f$ has an $\mathrm{add}\, M$-preenvelope $g : L \to M^m$, too. Indeed, in case (1) this follows from Proposition 3.16 and the fact that $L \in \mathrm{gen}\, M$, and in case (2) we have only to remind that M_R, and therefore also L_R, are finitely presented. Observe further that $E = \mathrm{End}_R\, M^n$ is semiregular by [74, 2.7]. From the exact sequence ${}_E\mathrm{Hom}_R\,(M^m, M^n) \longrightarrow {}_EE \xrightarrow{\mathrm{Hom}_R\,(f,M^n)} {}_E\mathrm{Hom}_R\,(A, M^n) \longrightarrow 0$ we deduce that the left annihilator $\ell_E(f)$ is finitely generated. Then, as in the proof of Proposition 2.2, we conclude that the criterion in Theorem 1.6 is satisfied and f has a left minimal version. □

We have just seen that the existence of $\mathrm{add}\, M$-preenvelopes is related to coherence properties of ${}_SM$. Let us now investigate the existence of $\mathrm{add}\, M$-precovers in terms of coherence properties of the dual module $\boldsymbol{M}^*{}_S = \mathrm{Hom}_R\,(M, W)\,{}_S$, where $\boldsymbol{W}_R$ denotes a minimal injective cogenerator of $\mathrm{Mod}\, R$. Here we have to consider the covariant functor $\mathbf{F} = \mathrm{Hom}_R\,(M,\ ) : \mathrm{Mod}\, R \to \mathrm{Mod}\, S$ together with its left adjoint $\mathbf{G} = \ \otimes_S M : \mathrm{Mod}\, S \to \mathrm{Mod}\, R$. Again, we have a natural homomorphism $\boldsymbol{\sigma}_Y \colon Y_S \to FG(Y_S)$ given by $\sigma_Y(y) : m \mapsto y \otimes m$, which is a monomorphism if and only if $Y_S \in \mathrm{Cogen}\, M^*_S$ (see for instance [33, 3.2]), and of course, σ_S is an isomorphism. The following results are then obtained by arguments dual to those used above, and we leave the details to the reader.

Lemma 3.20 (1) $\mathrm{Hom}_R\,(M, C)_S \in \mathrm{sfcopres}\, M^*_S$ for all finitely W-cogenerated modules C_R, and $\mathrm{Hom}_R\,(M, C)_S \in \mathrm{copres}\, M^*_S$ for all finitely W-copresented modules C_R.

(2) The functor $F = \mathrm{Hom}_R(M, \;) : \mathrm{Mod}\, R \to \mathrm{Mod}\, S$ induces dense functors $\mathrm{cogen}\, M_R \longrightarrow \mathcal{K}(M_S^*)$ and $\mathrm{copres}\, M_R \longrightarrow \mathrm{copres}\, S_S$. □

Again, if we assume that the natural homomorphism $\rho_W : GF(W) = \mathrm{Hom}_R(M, W) \otimes_S M \to W$ given by $\alpha \otimes m \mapsto \alpha(m)$ is an isomorphism, we also obtain that F induces dense functors $\mathrm{copres}\, W \to \mathrm{copres}\, M_S^*$ and $\mathrm{cogen}\, W \to \mathrm{sfcopres}\, M_S^*$.

Proposition 3.21 S is right coherent if and only if every $C \in \mathrm{copres}\, M$ has an $\mathrm{add}\, M$-precover. □

Proposition 3.22 The following statements are equivalent.
(1) Every finitely generated right S-module which is cogenerated by M_S^* is finitely presented.
(2) Every $C \in \mathrm{cogen}\, M$ has an $\mathrm{add}\, M$-precover. □

Theorem 3.23 (1) If M_S^* is π-coherent, then every finitely W-cogenerated module has an $\mathrm{add}\, M$-precover. The converse holds if M_R is finitely W-cogenerated.
(2) If M_S^* is coherent, then every finitely W-copresented module has an $\mathrm{add}\, M$-precover. The converse holds if M_R is finitely W-copresented. □

If R is **a right Morita ring**, i. e. a right artinian ring such that the minimal injective cogenerator W_R of $\mathrm{Mod}\, R$ is finitely generated, then we obtain a characterization of contravariantly finiteness. This and other consequences are collected in the following corollary. Note that the last statement generalizes a result proven by Auslander for finitely generated projective modules [11, 6.6].

Corollary 3.24 (1) Assume that M is a finitely generated module over a right Morita ring R. Then M_S^* is (π-)coherent if and only if $\mathcal{M}$ is contravariantly finite in $\mathrm{mod}\, R$.
(2) Assume that M_R is a finitely generated module over a right noetherian ring R. If $\mathcal{M}$ is contravariantly finite in $\mathrm{mod}\, R$, then every finitely generated right S-module which is cogenerated by M_S^* is finitely presented. In particular, S is then a right π-coherent ring.
(3) Assume that M_R is a coherent module. If all finitely generated modules have an $\mathrm{add}\, M$-precover, then S is a right coherent ring.

Proof: (1) By assumption every finitely generated module is finitely W-copresented and therefore has an add $\mathcal{M}$-precover provided that M_S^* is coherent. Conversely, assume that $\mathcal{M}$ is contravariantly finite in mod R. Then every finitely W-cogenerated module, being finitely presented by assumption, has an add M-precover. Moreover, the finitely generated module M_R is finitely W-cogenerated, and we conclude from Theorem 3.23 that M_S^* is π-coherent.
(2) Under the given assumptions, all modules in cogen M are finitely presented and therefore have an add M-precover whenever $\mathcal{M}$ is contravariantly finite in mod R. The claim then follows from Proposition 3.22. That S is right π-coherent follows from the fact that S_S is M_S^*-cogenerated.
(3) Under the given assumption, all modules in copres M are finitely generated and therefore have an add M-precover. The claim then follows from Proposition 3.21. □

3.3 Comparing coherence

We now collect a couple of results comparing the different notions of coherence occurring in the previous sections. First of all, if M_R is a finitely generated module with all matrix subgroups being finitely generated over the endomorphism ring S, then it follows immediately from Theorem 3.17 and Corollary 3.3 that ${}_SM$ is π-coherent and in particular coherent. Examples for the failure of the converse implications even in the case $M = R$ are given in [94, Example 29], [50, Example 5.2] and [27]. In particular, every commutative von Neumann regular ring which is not self-injective is coherent but not π-coherent, and the ring $R = K[X_1, X_2, \ldots]$ over a field K is π-coherent but not strongly coherent.

Next, we recall that the finitely generated S-submodules of a module M_R are matrix subgroups, and observe further that the class of all matrix subgroups is closed under arbitrary intersections [98, p. 1088]. So, if all matrix subgroups are finitely generated over S, we obtain that

(F) all finite matrix subgroups and all intersections of finitely generated S-submodules of M are finitely generated over S.

For $M = R$, condition (F) is equivalent to a sharpening of left coherence considered in [50, pp. 136] and [92, pp. 241]. It was pointed out in [92, Example 4.4] that a ring satisfying (F) need not be left strongly coherent. On the other hand, of course, all matrix subgroups of a module M_R are finitely generated over S whenever M satisfies (F) and

(I) every matrix subgroup is an intersection of finite matrix subgroups.

Note that the latter property is shared by all locally pure-injective modules, see [102, Theorem 2.1]. Moreover, if M is pure-projective, then the finitely generated S-submodules of M are even finite matrix subgroups [97, p. 706]. So, we conclude that for pure-projective or locally pure-injective modules M, all matrix subgroups are finitely generated over S if and only if (F) and (I) hold true. In particular, this yields a characterization of left strongly coherent rings which answers a question raised by Garfinkel [50, Question (2) on p. 137].

Since the locally projective modules coincide with the flat locally pure-projectives, a ring is left strongly coherent if and only if it is left coherent and all products of projective modules are locally pure-projective. The general result for modules reads as follows.

Proposition 3.25 If all finite matrix subgroups of M_R are finitely generated over S and all products of copies of M_R are locally pure-projective, then all matrix subgroups are finitely generated over S. The converse holds provided that $\mathcal{M} \subset \operatorname{mod} R$.

Proof: As in Theorem 3.11, it is actually enough to consider M^M. Indeed, by Azumaya's characterization of locally pure-projective modules 2.8, we then have for each $x \in M^M$ a finitely presented module F and homomorphisms $f : M^M \to F$ and $\varphi : F \to M^M$ such that $x = \varphi\, f(x)$. Moreover, Corollary 3.3 yields the existence of an $\operatorname{Add}\mathcal{M}$-preenvelope $a : F \to M^n$ with $n \in I\!N$. We deduce that there is a homomorphism $\varphi' : M^n \to M^M$ such that $\varphi = \varphi'\, a$. Thus the maps $f' = a\, f : M^M \to M^n$ and φ' satisfy $\varphi'\, f'(x) = x$. This verifies condition (4) in 2.9 and shows $M^M \in \mathcal{G}(M)$. By Theorem 3.11 we conclude that all matrix subgroups are finitely generated over S. Conversely, if $\mathcal{M} \subset \operatorname{mod} R$, then we know from Proposition 2.12 that all modules in $\mathcal{G}(M)$ are locally pure-projective, and so the claim follows immediately from Theorem 3.11. $\square$

We now turn to a similar characterization of π-coherent rings, where the role of the locally pure-projective modules is played by the R-Mittag-Leffler (or finitely pure-projective) modules studied in [52], [29], [63] and [17]. Recall that a Mittag-Leffler module X_R is characterized by the property that the canonical map $X \otimes_R (\prod_{j \in J} Y_j) \to \prod_{j \in J}(X \otimes_R Y_j)$ is a monomorphism for every family of left R-modules $(Y_j)_{j\in J}$. Similarly, a module X_R is said to be an ***R*-Mittag-Leffler module** if the canonical map $X \otimes_R R^J \to X^J$ is a monomorphism for every set J, or equivalently, if for every finitely generated submodule A_R the embedding $A \subset X$ factors through a finitely presented module. The relationship with the locally pure-projective modules is explained in [17, Proposition 7]. Jones showed in [63, p. 104] that a ring is left π-coherent if and only if it is left coherent and all products of copies of R (on either side) are R-Mittag-Leffler modules. Note that since the class of R-Mittag-Leffler modules is closed under pure submodules [17, Proposition 9], the latter property amounts to saying that all products of projective modules are R-Mittag-Leffler modules. We now prove the general statement for modules.

Proposition 3.26 The following statements are equivalent.
(1) ${}_SM$ is π-coherent.
(2) S is left (π-)coherent, ${}_SM$ is finitely presented, and all products of copies of ${}_SM$ are S-Mittag-Leffler modules.
If M_R is finitely presented, the following statement is further equivalent.
(3) S is left (π-)coherent, ${}_SM$ is finitely presented, and all products of copies of M_R are R-Mittag-Leffler modules.

Proof: (1)⇒(2): Any epimorphism $R^{(K)} \to M$ gives rise to a monomorphism ${}_SS \simeq \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(R^{(K)}, M) \simeq {}_SM^K$, showing that S is left (π-)coherent. Moreover, all finitely generated submodules of products of copies of ${}_SM$ are finitely presented by definition, and so the claim is proven.
(2)⇒(1): Let ${}_SA$ be a finitely generated submodule of a product of copies of ${}_SM$. By assumption, ${}_SA$ is contained in a finitely presented module ${}_SY$, which is coherent since so is the ring S. Hence ${}_SA$ is finitely presented, and we have verified that ${}_SM$ is π-coherent.
(1)⇒(3): Let A_R be a finitely generated submodule of M^J for some set J. By Theorem 3.17, the embedding $A \subset M^J$ factors through an $\mathrm{add}\,M$-preenvelope $A \to M^n$, and M^n is finitely presented if so is M_R.
(3)⇒(1): We claim that every finitely generated module has an $\mathrm{add}\,M$-preenvelope. The claim then follows from Theorem 3.17 whenever M_R is finitely generated. So, let A_R be finitely generated. By possibly considering $A/\mathrm{Rej}_M(A)$, we can assume without loss of generality that A is M-cogenerated. Then the product map $f : A \to M^J$ induced by all maps in $J = \mathrm{Hom}_R(A, M)$ is a monomorphic $\mathrm{Prod}\,M$-preenvelope and therefore factors through a homomorphism $f' : A \to F$ where F is finitely presented. But since ${}_SM$ is

coherent by assumption, we obtain from Theorem 3.17 the existence of an $\operatorname{add} M$-preenvelope $a : F \to M^n$. Now it is easy to check that the composition $a\, f' : A \to M^n$ is an $\operatorname{add} M$-preenvelope as well. $\square$

In the above proof, we have already used the well-known fact that ${}_SM$ is coherent provided that it is finitely presented and S is left coherent. When M_R is finitely generated, then ${}_SS$ is finitely cogenerated by ${}_SM$, and therefore we also have the converse implication. A similar relationship holds for strong coherence.

Proposition 3.27 Assume that M_R is finitely generated. If all matrix subgroups of M_R are finitely generated over S, then ${}_SM$ is finitely presented and S is left strongly coherent. The converse implication holds if M_R is finitely presented.

Proof: Assume first that M_R is finitely presented. Then it is well-known (see for instance [90, 51.10]) that the functors F and G defined on page 35 induce mutually inverse equivalences between $\vec{\mathcal{M}}$ and the category of all flat right S-modules. We claim that these equivalences can be restricted to equivalences between $\mathcal{G}(M)$ and the category of all locally projective right S-modules.
To this end, we start by showing that a locally split epimorphism $g : M^{(K)} \to A$ gives rise to a locally split epimorphism $S^{(K)} \to F(A)$. Indeed, F commutes with coproducts since M_R is finitely generated. Moreover, taking a generating set $m_1, \ldots, m_n$ of M_R, we know that for any element α in $F(A)$ there is a homomorphism $\varphi : A \to M^{(K)}$ such that $g\,\varphi(\alpha(m_j)) = \alpha(m_j)$ for all $1 \leq j \leq n$. But this means that $\alpha = g\,\varphi\,\alpha = F(g)\,F(\varphi)(\alpha)$, and so we are done. Thus $F(A)$ is locally projective for all $A_R \in \mathcal{G}(M)$. Conversely, if Y_S is a locally projective module with a locally split epimorphism $\gamma : S^{(K)} \to Y$, then it is easy to check that $G(\gamma) = \gamma \otimes \mathrm{id}_M$ is a locally split epimorphism, and so the flat module Y_S is isomorphic to $F(G(Y))$ where $G(Y) \in \mathcal{G}(M)$. This proves our claim.
Of course, the functor F commutes with products. If we assume that also G commutes with products, it is now straightforward that $\mathcal{G}(M)$ is closed under products if and only if so is the category of all locally projective right S-modules. In other words, using Theorem 3.11, we have that all matrix subgroups of M_R are finitely generated over S if and only if S is left strongly coherent and ${}_SM$ is finitely presented.
One implication, however, can also be proven when M_R is just finitely gen-

erated. In fact, we can employ a result of Zimmermann [101] asserting that every matrix subgroup $\mathrm{H}_{Y,y}(S_S)$ of S_S is isomorphic as a left S-module to the S-submodule $\mathrm{H}_{Y\otimes_S M,\underline{y}}(M_R)$ of M^n defined in Proposition 3.2 where $\underline{y} = (y \otimes m_1, \ldots, y \otimes m_n)$ and $m_1, \ldots, m_n$ is a generating set of M_R. Recall that these submodules are finitely generated over S if so are all matrix subgroups of M_R, and thus we can then conclude from Corollary 3.12 that S is left strongly coherent. $\square$

4 DIRECT SUMMANDS OF PRODUCTS

In this chapter, we employ our previous results to investigate direct products of modules. One of the main problems in this context is to describe the indecomposable direct summands of a product $\prod_{k \in K} X_k$. It is well known that they need not be isomorphic to any of the direct summands of the X_k. Over artin algebras, for instance, this phenomenon has important representation-theoretical consequences, as shown by the work about generic modules of Crawley-Boevey and Krause [41], [64].

Section 4.1 is devoted to modules M with the property that every (finitely presented) direct summand of a product of copies of M having a local endomorphism ring is isomorphic to some indecomposable direct summand of M itself. We call such modules (finitely) product-rigid. They can be characterized in terms of the existence of preenvelopes and in terms of finiteness conditions on matrix subgroups. In particular, we see that every endonoetherian module is product-rigid, and every endocoherent module is finitely product-rigid.

The product-rigid modules which are Σ-pure-injective coincide with the product-complete modules considered by Krause and Saorín in [66]. We discuss this notion in section 4.2. We show (in Proposition 4.10) that a module is product-complete if and only if it satisfies the perfectness conditions of Theorem 2.13 together with the coherence properties over the endomorphism ring considered in section 3.1. This extends a well-known theorem of Chase. As an application, we characterize the pure-projective product-complete modules over an artin algebra (Theorem 4.12).

4.1 When summands of products are summands of a factor

When is an indecomposable direct summand of $\prod_{i \in I} M_i$ isomophic to an indecomposable direct summand of some M_i? The results of section 3.1 provide an answer to this question.

Theorem 4.1 Let A be a module in $\operatorname{Prod}\mathcal{M}$ with local endomorphism ring. Then the following statements are equivalent.
(1) A is isomorphic to a direct summand of M_i for some i.
(2) A has an $\operatorname{Add}\mathcal{M}$-preenvelope.
(2') A has an $\operatorname{add}\mathcal{M}$-preenvelope.
(3) Every matrix subgroup of M of the form $\mathrm{H}_{A,x}(M)$ with $x \in A$ is a finitely generated left S-module.
(3') There is a matrix subgroup of M of the form $\mathrm{H}_{A,x}(M)$ with $0 \neq x \in A$ which is a finitely generated left S-module.

Proof: The implications (1)⇒(2), (1)⇒(2'), (3)⇒(3') are clear. (2)⇒(3) and (2')⇒(3) follow from Lemma 3.1.
(3')⇒(1): From Lemma 3.10 we know that there are $Y_1, \dots, Y_n \in \mathcal{M}$ and maps $f : A \to \prod_{k=1}^n Y_k$ and $\varphi : \prod_{k=1}^n Y_k \to A$ such that $\gamma = \varphi f \in \operatorname{End}_R A$ satisfies $\gamma(x) = x$. Hence $\mathrm{id}_A - \gamma \in \operatorname{End}_R A$ maps the nonzero element $x \in A$ to zero and is therefore a nonisomorphism. For the canonical embeddings and projections $i_k : Y_k \to \prod_{k=1}^n Y_k$ and $pr_k : \prod_{k=1}^n Y_k \to Y_k$ we have $\sum_{k=1}^n \varphi i_k pr_k f + \mathrm{id}_A - \gamma = \mathrm{id}_A$, and using that $\operatorname{End}_R A$ is local, we conclude that some $pr_k f : A \to Y_k \in \mathcal{M}$ must be a split monomorphism. This completes the proof. □

Observe that if $\operatorname{End}_R A$ is not local, we still have the equivalences $A \in \operatorname{Add}\mathcal{M} \Leftrightarrow$ (2) and $A \in \operatorname{add}\mathcal{M} \Leftrightarrow$ (2'). In fact, we can just use the fact that there are a family $(X_k)_{k\in K}$ in $\mathcal{M}$ and a split monomorphism $\mu : A \to \prod_{k\in K} X_k$ which will factor through the preenvelope.

Let us say that $\mathcal{M}$ is **(finitely) product-rigid** if the equivalent conditions of Theorem 4.1 are satisfied for every (finitely presented) $A \in \operatorname{Prod}\mathcal{M}$ with local endomorphism ring. The module M_R will be called (finitely) product-rigid if so is the subcategory $\{M\}$. Also, a ring R is said to be **right (finitely) product-rigid** if the right module R_R is (finitely) product-rigid.

Since $\mathcal{M}$ is (finitely) product-rigid if so is M (cp. Lemma 4.7), we now restrict our attention to product-rigid modules.

Corollary 4.2 If every (finite) matrix subgroup of M is finitely generated over S, then M is (finitely) product-rigid.

In particular, as already observed by Huisgen-Zimmermann [58], every endonoetherian module is product-rigid. Moreover, by Theorem 3.17, every module which is coherent over its endomorphism ring is finitely product-rigid.

As an immediate consequence of Lemma 2.10 or 3.6, we obtain a characterization of product-rigid modules which is analogous to Theorem 3.11.

Proposition 4.3 A module M is (finitely) product-rigid if and only if every (finitely presented) $A \in \operatorname{Prod} M$ with local endomorphism ring belongs to $\mathcal{G}(M)$. $\square$

We now show that the converse of Corollary 4.2 is not true.

Example 4.4 Let R be a local left perfect ring which is not right artinian, and let J be the Jacobson radical of R. Then the injective envelope M_R of R/J_R is product-rigid, but not all matrix subgroups of M are finitely generated over $S = \operatorname{End}_R M$. In fact, if A is a module in $\operatorname{Prod} M$ with local endomorphism ring, then A is indecomposable injective, and since R is left perfect, it has a non-zero socle, hence it contains a module isomorphic to R/J, which implies $A \cong M$. Thus M is product-rigid. Assume now that all matrix subgroups of M are finitely generated over S. From 3.3 we know that in this case every finitely generated module X_R has an $\operatorname{add} M$-preenvelope $a : X \to M^n$, which must be a monomorphism since M is a cogenerator. The injective envelope E of X is then isomorphic to a direct summand of the injective module M^n. Using again that R is left perfect and therefore every right module has an essential socle, we infer that $E \cong M^m$ for some m. From [2, 18.18] we then conclude that every finitely generated module X_R is finitely cogenerated. But this means that R is right artinian, a contradiction. $\square$

Observe that by a result of Auslander [13], every module over an artin algebra Λ is finitely product-rigid. However, as shown by Krause in [64, 5.1 and 8.7], for any indecomposable Σ-pure-injective Λ-module M, we can always find an endofinite module in $\operatorname{Prod} M$, and therefore M is product-rigid if and only if it is endofinite. In particular, Crawley-Boevey's and Krause's results about generic modules [41] [64] yield examples of indecomposable Σ-pure-injective Λ-modules which are not product-rigid. We remark the following.

Remark 4.5 Assume that R is a finite-dimensional algebra over an algebraically closed field. Then the following statements are equivalent.
(1) R is of finite representation type.
(2) All R-modules are product-rigid.

(3) The direct sum of a complete irredundant set of representatives of the isomorphism classes of $\operatorname{mod} R$ is product-rigid.

Proof: (1)⇒(2): Of course, if R is of finite representation type, then every module is endofinite [97, Theorem 6] and hence product-rigid.
(3)⇒(1): Set $\mathcal{M} = \operatorname{mod} R$, and assume that M is product-rigid and R is representation-infinite. Then we know from [41, 7.3] that there is a generic, i. e. an indecomposable endofinite infinite-dimensional R-module A. Further, it is well-known that A is a pure submodule of a module of the form $\prod_{k\in K} D(X_k)$, where $D : R\text{mod} \to \operatorname{mod} R$ is the ordinary duality and the X_k are some modules in Rmod, see for instance [102, A.3]. But since A is Σ-pure-injective, this implies that $A \in \operatorname{Prod} M$, and by assumption A then belongs to $\operatorname{mod} R$, a contradiction. □

4.2 Product-complete modules

In the previous section, we have discussed when an indecomposable direct summand of a product $\prod_{i\in I} M_i$ of arbitrary modules M_i is isomophic to an indecomposable direct summand of some M_i. Krause and Saorín [66] studied this problem in the case that all M_i coincide and are Σ-pure-injective.

Theorem 4.6 (Krause-Saorín [66, Theorem 3.1]) The following statements are equivalent.
(1) Every product of copies of M is a coproduct of direct summands of M (with local endomorphism ring).
(2) $\operatorname{Prod} M \subset \operatorname{Add} M$.
(3) M is product-complete.
(4) $\operatorname{Add} \mathcal{M}$ is an envelope class.

Let us first compare the notion of product-rigid with product-completeness.

Lemma 4.7 (1) If M is (finitely) product-rigid, so is $\mathcal{M}$. The converse holds if $M \in \operatorname{Prod} N$, in particular, if M is pure-injective.
(2) If $\mathcal{M}$ is (finitely) product-rigid, so is N. The converse holds if $N \in \operatorname{Add} M$.

Proof: We have $\operatorname{add} \mathcal{M} \subset \operatorname{add} N \subset \operatorname{Prod} N = \operatorname{Prod} \mathcal{M} \subset \operatorname{Prod} M$, with $\operatorname{Prod} \mathcal{M} = \operatorname{Prod} M$ whenever $M \in \operatorname{Prod} N$. The latter assumption is satisfied provided M is pure-injective since M always is a pure submodule of

N. Further, we have $\operatorname{add}\mathcal{M} \subset \operatorname{add} M \subset \operatorname{Add} M = \operatorname{Add}\mathcal{M} \subset \operatorname{Add} N$, and $\operatorname{Add}\mathcal{M} = \operatorname{Add} N$ whenever $N \in \operatorname{Add} M$. The two claims then follow easily from Theorem 4.1. □

Proposition 4.8 The following statements are equivalent.
(1) M is product-complete.
(2) $\operatorname{Prod}\mathcal{M} \subset \operatorname{Add}\mathcal{M}$.
(3) N lies in $\operatorname{Add} M$ and is product-complete.
(4) $\mathcal{M}$ is product-rigid and M is Σ-pure-injective.
(5) M is product-rigid and Σ-pure-injective.

Proof: (1)⇔(5): Recall that M is Σ-pure-injective if and only if every product of copies of M is a coproduct of modules with local endomorphism ring [93]. Combining this with Theorem 4.6, we immediately obtain our claim.
(1)⇒(2) : Since $\operatorname{Add} M$ is closed under products, we have $\operatorname{Prod}\mathcal{M} \subset \operatorname{Add} M = \operatorname{Add}\mathcal{M}$.
(2)⇒(3) : $\operatorname{Prod} N = \operatorname{Prod}\mathcal{M} \subset \operatorname{Add}\mathcal{M} \subset \operatorname{Add} N$, hence N lies in $\operatorname{Add} M$ and is product-complete by Theorem 4.6.
(3)⇒(4) : From (1)⇔(5) we have that N is Σ-pure-injective and product-rigid. But then the pure submodule M of N is also Σ-pure-injective. Moreover, N being product-rigid and lying in $\operatorname{Add} M$ implies by Lemma 4.7 that $\mathcal{M}$ is product-rigid.
(4)⇒(5) : follows immediately from Lemma 4.7. □

Chase proved in [28] that R_R is product-complete if and only if R is right perfect and left coherent. Thus, every left noetherian ring which is not left artinian is product-rigid by Corollary 4.2, but not product-complete. However, the notions product-complete and product-rigid coincide for **π-projective** modules, that is, modules M with all products of copies of M being projective.

Proposition 4.9 Assume that M is π-projective. Then the following statements are equivalent.
(1) M is product-complete.
(2) M is (finitely) product-rigid.
(3) M is finitely generated over S.

Proof: (1)⇒(3) follows from [66, 4.2].
(3)⇒(2) : By assumption, every module $A \in \operatorname{Prod} M$ is projective, so if A is

indecomposable, we have $A \simeq e\,R$ for some local idempotent $e \in R$. But then $\mathrm{Hom}_R(A, M)$ is a direct summand of $\mathrm{Hom}_R(R, M) \simeq {}_SM$ and is therefore finitely generated over S. We deduce that every matrix subgroup of M of the form $\mathrm{H}_{A,x}(M)$ with $x \in A$ is a finitely generated left S-module. Hence M is product-rigid. Moreover, we have just seen that all indecomposable modules in $\mathrm{Prod}\, M$ are finitely presented, hence the notions of product-rigid and finitely product-rigid coincide for M.
(2)⇒(1) follows from Proposition 4.8 and the fact that every π-projective module is Σ-pure-injective by [99]. □

Chase's result asserting that perfectness on one side together with coherence on the other side mean product-completeness can be extended to modules by combining the results of Chapters 2 and 3. From Theorems 2.13 and 3.11 we then obtain a whole bunch of conditions characterizing product-complete modules. We will only mention the most salient ones.

Proposition 4.10 The following statements are equivalent.
(1) M is product-complete.
(2) $\mathrm{Add}\,\mathcal{M} = \mathcal{G}(M)$, and all matrix subgroups of M are finitely generated over S.
(3) ${}_SM$ is coperfect, and all finite matrix subgroups of M are finitely generated over S.
If $\mathcal{M} \subset \mathrm{mod}\, R$, then (1) is further equivalent to
(4) $\mathrm{Add}\,\mathcal{M}$ is a cover class, and $\mathcal{M}$ is covariantly finite in $\mathrm{mod}\, R$.

Proof: When M is Σ-pure-injective, every pure epimorphism of the form $M^{(K)} \to A$ splits, and hence $\mathcal{G}(M) = \mathrm{Add}\,\mathcal{M}$. Then (1)⇔(2) follows immediately from Theorem 3.11. Moreover, since every Σ-pure-injective module M satisfies the descending chain condition for matrix subgroups, and in particular for finitely generated S-submodules, we obtain (1)⇒(3). Conversely, if M satisfies the descending chain condition for finitely generated S-submodules and all finite matrix subgroups are finitely generated S-submodules, then M obviousy satisfies the descending chain condition for finite matrix subgroups and is therefore Σ-pure-injective. But matrix subgroups of Σ-pure-injective modules are always finite matrix subgroups [97, p. 704], and so we can even conclude that (3)⇒(2). Finally, (3)⇔(4) is just Theorem 2.13 together with Corollary 3.3. □

In particular, the above Proposition contains a module-theoretical proof for a result which Krause and Saorín have proven with functorial methods [66, 4.2]. We also obtain a new proof for their characterization of finitely generated product-complete modules.

Corollary 4.11 [66, 3.9] Assume that M_R is finitely generated. Then M_R is product-complete if and only if S is left coherent and right perfect, and ${}_SM$ is finitely presented.

Proof: The only-if part follows from Theorem 3.17 and from the fact that finitely generated Σ-pure-injective modules have a semiprimary endomorphism ring [69]. For the if-part, we observe that under the given assumptions, M has a finite decomposition in modules $N_1, \ldots, N_r$ with local endomorphism ring [49, 3.14] and $J(S)$ is right T-nilpotent. This implies that the family $(N_1, \ldots, N_r)$ is T-nilpotent, and thus ${}_SM$ is coperfect by Corollary 2.7. Since ${}_SM$ is coherent, we then conclude from Theorem 3.17 and Corollary 3.3 that condition (3) in Theorem 4.10 is satisfied, and the proof is complete. $\square$

We now describe the pure-projective product-complete modules over artin algebras.

Theorem 4.12 Assume that R is an artin algebra with ordinary duality $D : \operatorname{mod} R \to R\operatorname{mod}$ and that $\mathcal{M} \subset \operatorname{mod} R$. Let $\mathcal{M}^*$ be the subcategory of Rmod consisting of all duals $D(X)$ of modules $X \in \mathcal{M}$, and set $M^* = \coprod_{i \in I} D(M_i)$. Then the following statements are equivalent.
(1) M is product-complete.
(2) $\mathcal{G}(M)$ and $\mathcal{G}(M^*)$ are closed under products.
(3) $\mathcal{G}(M)$ is closed under products, and $\mathcal{M}$ is contravariantly finite in $\operatorname{mod} R$.
(4) M^* is product-complete.

Proof: (1)$\Rightarrow$(2): $\mathcal{G}(M)$ is closed under products by Proposition 4.10. As far as $\mathcal{G}(M^*)$ is concerned, we argue as in [97, Corollary 12] and use the fact that M is Σ-pure-injective to deduce from [97, Proposition 3] that M^* is endonoetherian. Of course, all matrix subgroups of M^* are then finitely generated over the endomorphism ring, and the claim follows from Theorem 3.11.
(2)$\Rightarrow$(3): When $\mathcal{G}(M^*)$ is closed under products, we know from Theorem 3.11 and Corollary 3.3 that $\mathcal{M}^*$ is covariantly finite in Rmod. Hence, applying D, we obtain that $\mathcal{M}$ is contravariantly finite in $\operatorname{mod} R$.

(3)⇒(1) : We claim that $\mathcal{G}(M)$ is closed under direct limits. This will imply that $\operatorname{Add}\mathcal{M}$ is a cover class by Theorem 2.13, and that $\vec{\mathcal{M}} = \mathcal{G}(M)$ is closed under products, hence $\mathcal{M}$ is covariantly finite in $\operatorname{mod} R$ by Theorem 3.4. So, M will be product-complete by Proposition 4.10.
To prove the claim, observe first that $\mathcal{M}^*$ is covariantly finite in $R\text{mod}$, hence $\vec{\mathcal{M}}^*$ is closed under products by Theorem 3.4. Let now A_R be a module in $\vec{\mathcal{M}}$. By Lemma 2.11 there is a pure-exact sequence $0 \longrightarrow K \longrightarrow \coprod_{k\in K} X_k \longrightarrow A \longrightarrow 0$ with X_k in $\mathcal{M}$. Applying D, we obtain a pure-exact sequence $0 \longrightarrow D(A) \longrightarrow \prod_{k\in K} D(X_k) \longrightarrow D(K) \longrightarrow 0$ with $D(X_k) \in \mathcal{M}^*$, and since $D(A)$ is pure-injective, we infer that $D(A)$ lies in $\operatorname{Prod}\mathcal{M}^*$, hence in $\vec{\mathcal{M}}^*$. Again by Lemma 2.11, we deduce that there is a pure-exact sequence $0 \longrightarrow K' \longrightarrow \coprod_{j\in J} Y_j \longrightarrow D(A) \longrightarrow 0$ with $Y_j \in \mathcal{M}^*$, which in turn implies that $D^2(A) \in \operatorname{Prod}\mathcal{M}$. Since $\mathcal{G}(M)$ is closed under products, $D^2(A)$ is then contained in $\mathcal{G}(M)$. But A is a pure submodule of $D^2(A)$, and $\mathcal{G}(M)$ is closed under pure submodules by Proposition 2.12. So, we conclude that A belongs to $\mathcal{G}(M)$, which proves our claim.
(2)⇔(4) holds by symmetry. □

However, we will see in 6.15 a situation as in Theorem 4.12 where M is product-rigid while M^* is not.

5 ENDOFINITENESS

Left and right almost split morphisms are usually studied for categories of finitely generated modules over artin algebras. In section 5.1, we consider left and right almost split morphisms for $\mathcal{M}$ and interpret their existence in terms of finiteness conditions for the modules ${}_SM$ and ${}_TN$. An important role in this context is played by the radicals $\mathrm{r}\,(M,C)_S$ and ${}_T\,\mathrm{r}\,(A,N)$ where A and C are R-modules. In case that $\mathcal{M}$ is a finite subcategory consisting of modules with local endomorphism ring, we can restrict ourselves to the Jacobson radical $J(S)$ of S, and we obtain for instance that $\mathcal{M}$ has left (respectively, right) almost split morphisms if and only if $J(S)$ is a finitely generated left (respectively, right) S-module (Corollary 5.9).

In Section 5.2, we then investigate when the property product-rigid is inherited by direct summands. We show that this is equivalent to the existence of certain maps closely related to left almost split maps. As a consequence, we obtain a description of the right artinian rings R which satisfy the condition

> If A and $\{B_k\}_{k\in K}$ are indecomposables in $\operatorname{mod} R$ such that A is a direct summand of $\prod_{k\in K} B_k$, then $A \cong B_k$ for some $k \in K$,

answering a question posed by Auslander in [13].

In Section 5.3, we then combine our results to give a characterization of endofinite modules in terms of product-rigidity, as well as in terms of the existence of preenvelopes and left almost split maps (Theorem 5.12).

5.1 Left and right almost split morphisms

The results in this section will appear in [5].

Recall that for two modules X_R, Y_R the radical $\mathbf{r}\,(\boldsymbol{X},\boldsymbol{Y})$ denotes the collection of all homomorphisms $f : X \to Y$ such that there is no isomorphism of the form $Z \to X \xrightarrow{f} Y \to Z$ where Z is a module with local endomorphism ring. Then $\mathrm{r}\,(X,Y)$ is an $\mathrm{End}_R\,Y - \mathrm{End}_R\,X$ - subbimodule of $\mathrm{Hom}_R\,(X,Y)$. Let us collect some properties of this bimodule.

Lemma 5.1 Let Y_R be a module with endomorphism ring $E = \mathrm{End}_R\,Y$, and let X_R be an indecomposable module.
(1) $J(E) \subset \mathrm{r}\,(Y,Y)$, with equality if $Y = \coprod_{i=1}^n Y_i$ and all Y_i have local endomorphism ring.
(2) (cp. [14, 2.4]) $\mathrm{Hom}_R\,(X,Y)/\mathrm{r}\,(X,Y)$ is either zero or a simple left E-module.
(3) $\mathrm{r}\,(X,Y)$ is a noetherian left E-module if and only if $\mathrm{Hom}_R\,(X,Y)$ is a noetherian left E-module.

Proof: The first assertion in (1) is well-known. For the second assertion, assume that Y has a decomposition as stated. This means that E is semiperfect, that is, $\mathrm{id}_Y = \sum_{i=1}^n e_i$ for local idempotents $e_1,\ldots,e_n \in E$. Then every $f \in \mathrm{r}\,(Y,Y)$ has the form $f = \sum_{i=1}^n f\,e_i$ where $Ef\,e_i$ is properly contained in $E\,e_i$ and therefore lies in $J(E)\,e_i$, which shows $f \in J(E)$.
(2) Assume that there is a nonzero element $\overline{f} \in \mathrm{Hom}_R\,(X,Y)/\mathrm{r}\,(X,Y)$. Since X is indecomposable, f is a split monomorphism and therefore generates the left E-module $\mathrm{Hom}_R\,(X,Y)$. This yields the claim.
(3) Apply statement (2) on the exact sequence of E-modules $0 \longrightarrow {}_E\mathrm{r}\,(X,Y) \longrightarrow_E \mathrm{Hom}_R\,(X,Y) \longrightarrow_E \mathrm{Hom}_R\,(X,Y)/\mathrm{r}\,(X,Y) \longrightarrow 0$. $\square$

Following Auslander [14, §1], we say that a family of homomorphisms $(a_k : A \to X_k)_{k\in K}$ is **finitely cogenerated** if there is a finite subset $K_0 \subset K$ such that the product map $a : A \to \coprod_{k\in K_0} X_k$ induced by the a_k with $k \in K_0$ has the property that all a_k, $k \in K$, factor through a. We start out with two propositions describing when families of homomorphisms in $\bigcup_{i\in I} \mathrm{r}\,(A, M_i)$ are finitely cogenerated. The first one is proven as Proposition 1.4, cp. also [14, 1.10].

Proposition 5.2 The following statements are equivalent for a module A.
(1) The family of all homomorphisms in $\bigcup_{i\in I} \mathrm{r}\,(A, M_i)$ is finitely cogenerated.
(2) The left T-module $\mathrm{r}\,(A,N)$ is generated by finitely many maps whose images are contained in a finite subproduct $\prod_{i\in I_0} M_i$ of N.
(3) There is a map $a \in \mathrm{r}\,(A,X)$ such that $X \in \mathrm{add}\,\mathcal{M}$ and all maps $h \in \mathrm{r}\,(A,Y)$ where $Y \in \mathrm{add}\,\mathcal{M}$ factor through a.

If A is finitely generated, the following statement is further equivalent.
(4) $\mathrm{r}(A, M)$ is a finitely generated left S-module. $\square$

Proposition 5.3 The following statements are equivalent for a module A.
(1) Every family of homomorphisms in $\bigcup_{i\in I} \mathrm{r}(A, M_i)$ is finitely cogenerated.
(2) $\mathrm{r}(A, N)$ is a noetherian left T-module.
If A is finitely generated, then (1) and (2) are further equivalent to
(3) $\mathrm{r}(A, M)$ is a noetherian left S-module.

Proof: Denote by $p_i : N \to M_i$ and $e_i : M_i \to N$, $i \in I$, the canonical projections and injections, respectively.
(1)⇒(2): For a submodule ${}_T U \subset \mathrm{r}(A, N)$ we consider the family of homomorphisms $\{p_i f \mid f \in U, i \in I\}$ in $\bigcup_{i\in I} \mathrm{r}(A, M_i)$. By assumption there are indices $i_1, \ldots, i_n \in I$ and maps $f_k \in U$, $1 \le k \le n$, such that the product map $a : A \to \coprod_{k=1}^n M_{i_k}$ induced by the $p_{i_k} f_k$ has the property that all other maps of the form $p_i f$ with $i \in I$ and $f \in U$ factor through a. Then also all $f \in U$ factor through a, and so the f_k, $1 \le k \le n$, form a generating set of ${}_T U$.
(2)⇒(1): Let now $(a_k : A \to M_{i_k})_{k\in K}$ be a family in $\bigcup_{i\in I} \mathrm{r}(A, M_i)$, and consider the T-submodule $U = \sum_{k\in K} T f_k$ of $\mathrm{r}(A, N)$ given by the maps $f_k : A \xrightarrow{a_k} M_{i_k} \xrightarrow{e_{i_k}} N$ in $\mathrm{r}(A, N)$. By assumption ${}_T U = \sum_{k\in K_0} T f_k$ for some finite subset $K_0 \subset K$. This implies that all f_k, and therefore also all a_k, factor through the product map $a : A \to \coprod_{k\in K_0} M_{i_k}$ induced by the a_k with $k \in K_0$.
If A is finitely generated, the equivalence of (1) and (3) is proven with similar arguments, taking into account the fact that each map in $\mathrm{r}(A, M)$ has its image in a finite subcoproduct of M. $\square$

As observed by Huisgen-Zimmermann [58], the module N is endonoetherian if and only if for all finitely generated modules A_R every family of homomorphisms in $\bigcup_{i\in I} \mathrm{Hom}_R(A, M_i)$ is finitely cogenerated. Over a semilocal ring, we can now restrict ourselves to families of homomorphisms in $\bigcup_{i\in I} \mathrm{r}(A, M_i)$ with A indecomposable.

Proposition 5.4 Assume that R is semilocal. Then the following statements are equivalent.
(1) N is endonoetherian.
(2) M is endonoetherian.

(3) For all finitely generated (or equivalently, for all finitely presented) indecomposable modules A_R every family of homomorphisms in $\bigcup_{i \in I} \mathrm{r}\,(A, M_i)$ is finitely cogenerated.

Proof: (2)⇔(3): M is endonoetherian if and only if $\mathrm{Hom}_R\,(A, M)$ is a noetherian left S-module for every finitely generated (or equivalently, for every finitely presented) module A_R. Since R is semilocal, every finitely generated module has a finite decomposition in indecomposables [49, 1.14]. So, it suffices to consider finitely generated (or finitely presented) indecomposable modules A, and the statement follows from 5.3 and 5.1. The equivalence (1)⇔(3) is proven by the same arguments. □

We now consider the dual situation. A family of homomorphisms $(b_k : X_k \to C)_{k \in K}$ is said to be **finitely generated** [14, §1] if there is a finite subset $K_0 \subset K$ such that the coproduct map $b : \coprod_{k \in K_0} X_k \to C$ induced by the b_k with $k \in K_0$ has the property that all b_k, $k \in K$, factor through b. Let us describe when families of homomorphisms in $\bigcup_{i \in I} \mathrm{r}\,(M_i, C)$ are finitely generated. The arguments are dual to those employed above, and we will therefore omit the proofs.

Proposition 5.5 The following statements are equivalent for a module C.
(1) The family of all homomorphisms in $\bigcup_{i \in I} \mathrm{r}\,(M_i, C)$ is finitely generated.
(2) The right S-module $\mathrm{r}\,(M, C)$ is generated by finitely many maps whose kernels contain a cofinite subcoproduct $\coprod_{i \in I \setminus I_0} M_i$ of M.
(3) There is a map $b \in \mathrm{r}\,(X, C)$ such that $X \in \mathrm{add}\,\mathcal{M}$ and all maps $h \in \mathrm{r}\,(Y, C)$ where $Y \in \mathrm{add}\,\mathcal{M}$ factor through b. □

Proposition 5.6 The following statements are equivalent for a module C.
(1) Every family of homomorphisms in $\bigcup_{i \in I} \mathrm{r}\,(M_i, C)$ is finitely generated.
(2) $\mathrm{r}\,(M, C)$ is a noetherian right S-module. □

We now give a dual version of Proposition 5.4. In section 6.4, we will then apply both results to pure-semisimple rings.

Proposition 5.7 Assume that R is semilocal. Further, let W_R be a minimal injective cogenerator of $\mathrm{Mod}\,R$, and $M^*_S = \mathrm{Hom}_R\,(M, W)_S$. The following statements are equivalent.
(1) M^*_S is noetherian.

(2) For all finitely cogenerated indecomposable modules C_R every family of homomorphisms in $\bigcup_{i\in I} \mathrm{r}\,(M_i, C)$ is finitely generated.
If R is a right Morita ring, then following statement is further equivalent.
(3) For all finitely generated indecomposable modules C_R every family of homomorphisms in $\bigcup_{i\in I} \mathrm{r}\,(M_i, C)$ is finitely generated.

Proof: Since R is semilocal, there are only finitely many simple right R-modules $S_1, \cdots, S_n$ up to isomorphism. So, $W \cong \coprod_{i=1}^{n} C_i$ where $C_i = E(S_i)$ is an injective envelope of S_i.
(1)⇒(2): If C is finitely cogenerated, then $\mathrm{Hom}_R\,(M, C)_S$ is finitely M^*-cogenerated, and the claim follows from Proposition 5.6.
(2)⇒(1): By Proposition 5.6 we know that $\mathrm{r}\,(M, C_i)_S$ is noetherian for all $1 \leq i \leq n$, hence by the dual version of statement (3) in Lemma 5.1 also $\mathrm{Hom}_R\,(M, C_i)_S$ is noetherian for all $1 \leq i \leq n$, and M_S^* is noetherian.
Assume now that R is a right Morita ring. Since R is right artinian, all finitely generated modules are finitely cogenerated, which yields (2)⇒(3). Moreover, since all C_i are finitely generated, we deduce (3)⇒(1) as above from 5.6 and the dual version of statement (3) in Lemma 5.1. □

Assume now that $\mathcal{M}$ consists of modules with local endomorphism ring and let C be a module in $\mathcal{M}$. Recall that a homomorphism $b : X \to C$ with $X \in \mathrm{add}\,\mathcal{M}$ is said to be **right almost split in add $\mathcal{M}$** if b is not a split epimorphism and any homomorphism $h : Y \to C$ where $Y \in \mathrm{add}\,\mathcal{M}$ and h is not a split epimorphism factors through b. We have seen in Proposition 5.5 that C has a right almost split morphism in $\mathrm{add}\,\mathcal{M}$ if and only if the family of all homomorphisms in $\bigcup_{i\in I} \mathrm{r}\,(M_i, C)$ is finitely generated, see also [14, 1.9]. Inspired by Brune's work [26], we will further say that C has **generalized right almost split morphisms in add $\mathcal{M}$** if every family of homomorphisms in $\bigcup_{i\in I} \mathrm{r}\,(M_i, C)$ is finitely generated. **Left almost split morphisms** and **generalized left almost split morphisms in add $\mathcal{M}$** are defined dually. Finally, we say that $\mathcal{M}$ **has (generalized) right, respectively left, almost split morphisms** if (generalized) right, respectively left, almost split morphisms in $\mathrm{add}\,\mathcal{M}$ exist for every module belonging to $\mathcal{M}$.

It is well known that these concepts can be interpreted in terms of the functor category $(\mathcal{M}^{\mathrm{op}}, Ab)$. More precisely, a module $C \in \mathcal{M}$ admits a right almost split morphism in $\mathrm{add}\,\mathcal{M}$ if and only if the radical of the functor $H_C = \mathrm{Hom}_R\,(\ , C) \mid_{\mathcal{M}} : \mathcal{M} \to Ab$ is finitely generated, see [14, §2]. Moreover, by using an argument due to Brune [26, p. 444], we see that C has generalized right almost split morphisms in add $\mathcal{M}$ if and only if the functor H_C is noetherian, i. e. every subfunctor of H_C is finitely generated. So, $\mathcal{M}$ has generalized right (respectively, left) almost split morphisms if and only if it is a right (left) noetherian category in the sense of [72, p. 19].

On the other hand, the above results enable us to interpret these notions in terms of endoproperties. In particular, we have the following two Corollaries. Observe that statement (3) in Corollary 5.8 has independently been obtained by Dung [44, 2.3], [45, 3.11].

Corollary 5.8 Assume that the endomorphism ring of each M_i is local.
(1) $\mathcal{M}$ has generalized right almost split morphisms if and only if $\mathrm{r}(M, M_i)$ is a noetherian right S-module for all $i \in I$.
(2) $\mathcal{M}$ has generalized left almost split morphisms if and only if $\mathrm{r}(M_i, N)$ is a noetherian left T-module for all $i \in I$.
(3) If $\mathcal{M}$ consists of finitely generated modules, then $\mathcal{M}$ has (generalized) left almost split morphisms if and only if $\mathrm{r}(M_i, M)$ is a finitely generated (noetherian) left S-module for all $i \in I$. $\square$

Corollary 5.9 Assume that $\mathcal{M}$ is a finite subcategory consisting of modules with local endomorphism ring.
(1) $\mathcal{M}$ has left (respectively, right) almost split morphisms if and only if $J(S)$ is a finitely generated left (respectively, right) S-module.
(2) $\mathcal{M}$ has generalized left (respectively, right) almost split morphisms if and only if $J(S)$ is a noetherian left (respectively, right) S-module.
(3) Assume that all M_i have finite length. Then $\mathcal{M}$ has left (respectively, right) almost split morphisms if and only if S is left (respectively, right) artinian if and only if $\mathcal{M}$ has generalized left (respectively, right) almost split morphisms.

Proof: Note first that we have $M = N$ and $S = T$. Then (1) and (2) follow from 5.2 and 5.3, or 5.5 and 5.6, respectively, and the fact observed in Lemma 5.1 that $J(S) = \mathrm{r}(M, M) \simeq \bigoplus_{i=1}^{n} \mathrm{r}(M_i, M) \simeq \bigoplus_{i=1}^{n} \mathrm{r}(M, M_i)$.
(3) If all M_i have finite length, then so does M, and S is therefore semiprimary. Hence S is left artinian if and only if ${}_SJ(S)$ is finitely generated if and only if ${}_SJ(S)$ is noetherian, and the symmetric statement holds on the right side. This proves the claim. $\square$

We will see in 6.14 and 6.15 that there are categories with left and right almost split morphisms where neither ${}_SS$ nor ${}_SM$ are noetherian. Moreover, we will see in section 6.4 that pure-semisimple rings can be interpreted in terms of the existence of generalized left or right almost split morphisms.

5.2 Product-rigid subcategories

The fact that modules over artin algebras are finitely product-rigid was proven by Auslander [13] using the existence of left almost split maps. A closer look at his argument now yields a result describing when the property product-rigid is inherited by subcategories. Note that $\mathcal{M} = \{\mathbb{Z}_{p^\infty}, \mathbb{Q}\} \subset \mathrm{Mod}\,\mathbb{Z}$ is an example of a product-rigid category with a subcategory $\{\mathbb{Z}_{p^\infty}\}$ which is no longer product-rigid [66, 3.4], and another example for this phenomenon is given in 6.16.

Theorem 5.10 The following statements are equivalent for an indecomposable module A.
(1) If $\mathcal{M}'$ is subcategory of $\mathcal{M}$ and $A \in \mathrm{Prod}\,\mathcal{M}'$, then A is isomorphic to a direct summand of a module in $\mathcal{M}'$.
(2) There is a homomorphism $f : A \to Y$ which is not a split monomorphism and has the property that a homomorphism $h : A \to X$ with $X \in \mathcal{M}$ factors through f whenever A is not isomorphic to a direct summand of X.

Proof: We use Auslander's arguments from [13, 2.3 and 2.4].
(2)⇒(1): Consider a subcategory $\mathcal{M}'$ of $\mathcal{M}$ and take a split monomorphism $h : A \to \prod_{k\in K} X_k$ for some $X_k \in \mathcal{M}'$. Then there must be some $k \in K$ such that A is isomorphic to a direct summand of X_k, because otherwise h factors through f, contradicting the fact that f is not a split monomorphism.
(1)⇒(2): We put $H_j = \mathrm{Hom}_R(A, M_j)$ where j runs through the subset $J \subset I$ given by those M_i with no direct summand isomorphic to A. Then the product map $f : A \to \prod_{j\in J} {M_j}^{H_j} = Y$ defined by all homomorphisms in $H_j, j \in J$, has the required factorization property, and applying (1) to the subcategory $\mathcal{M}' = \{M_j \mid j \in J\}$ of $\mathcal{M}$ we see that f is not a split monomorphism. □

So, all subcategories of $\mathcal{M}$ are (finitely) product-rigid if and only if any (finitely presented) module A with local endomorphism ring admits a map $f : A \to Y$ as in condition (2) of the above Theorem 5.10. As a consequence, if R is right artinian, then the property that all categories of finitely generated indecomposables are finitely product-rigid is equivalent to the existence of maps located between left almost split maps in $\mathrm{Mod}\,R$ and left almost split maps in $\mathrm{mod}\,R$. This answers question (1) of [13].

Corollary 5.11 The following statements are equivalent for a right artinian ring R.
(1) If A and $\{B_k\}_{k\in K}$ are indecomposables in $\operatorname{mod} R$ such that A is a direct summand of $\prod_{k\in K} B_k$, then $A \cong B_k$ for some $k \in K$.
(2) Any indecomposable module A in $\operatorname{mod} R$ admits a map $f : A \to Y$ with $Y \in \operatorname{Mod} R$ such that the map $\operatorname{Hom}_R(f, X) : \operatorname{Hom}_R(Y, X) \to \operatorname{Hom}_R(A, X)$ satisfies $\operatorname{Im}\operatorname{Hom}_R(f, X) = \mathrm{r}\,(A, X)$ for each $X \in \operatorname{mod} R$.

Proof: Let $\mathcal{M}$ be the category of all indecomposables in $\operatorname{mod} R$ and apply Theorem 5.10. Then, if (1) is satisfied, any indecomposable module A in $\operatorname{mod} R$ admits a map $f \in \mathrm{r}\,(A, Y)$ with the property that a homomorphism $h : A \to X$ where $X \in \operatorname{mod} R$ is indecomposable, factors through f whenever X is not isomorphic to A. Suppose now that $h \in \mathrm{r}\,(A, X)$ and $X \cong A$. Then by length arguments we infer that $\operatorname{Im} h = \bigoplus_{j=1}^{n} X_j$ where $X_1, \ldots, X_n$ are indecomposable and not isomorphic to A, and so we conclude that h factors through f also in this case. This shows (1)⇒(2). The implication (2)⇒(1) follows immediately from the fact that $\operatorname{Hom}_R(A, X) = \mathrm{r}\,(A, X)$ whenever A and X are nonisomorphic indecomposables in $\operatorname{mod} R$. □

Abrams has given in [1] an example of a one-sided artinian ring where condition (1) in the above Corollary 5.11 fails. We will discuss his results in Corollary 5.13.

Let us also point out that the relationship between the existence of left almost split maps and the behaviour of products was already investigated by Krause in [65, 3.3]. His point of view, however, is different. He fixes a module A, which he assumes to be pure-injective, and shows that A satisfies condition (1) of Theorem 5.10 for any skeletally small subcategory $\mathcal{M}$ of $\operatorname{Mod} R$ consisting of indecomposable (pure-injective) modules if and only if there is a left almost split map $A \to Y$ in $\operatorname{Mod} R$.

5.3 Endofinite modules

Krause and Saorín have shown in [66, 4.1] that M is endofinite if and only if every direct summand of M is product-complete. So, our results can also be employed to characterize endofinite modules. We will focus on some equivalent conditions which relate endofiniteness to the question of when the property product-rigid is inherited by subcategories as well as to the existence of preenvelopes and left almost split morphisms.

Theorem 5.12 Assume that the endomorphism ring of each M_i is local. Then the following statements are equivalent.
(1) M is endofinite.
(2) All subcategories of $\mathcal{M}$ are product-rigid and M is Σ-pure-injective.
(3) ${}_SM$ is coperfect, and for all subcategories $\mathcal{M}' \subset \mathcal{M}$ and all finitely generated (finitely presented) modules A_R, there exists an $\operatorname{Add}\mathcal{M}'$-preenvelope.
If all M_i are finitely generated, then (1) is further equivalent to
(4) ${}_SM$ is coperfect, and all direct summands of M are (π-)coherent over their endomorphism ring.
(5) $\mathcal{M}$ has left almost split maps, ${}_SM$ is finitely generated, and S is a semiprimary ring.
On the other hand, if R is semilocal, then (1) is further equivalent to
(6) ${}_SM$ is coperfect, and for all subcategories $\mathcal{M}' \subset \mathcal{M}$ and all finitely generated (finitely presented) indecomposable modules A_R, there exists a map $a \in \mathrm{r}\,(A, X)$ such that $X \in \operatorname{add}\mathcal{M}'$ and all maps $h \in \mathrm{r}\,(A, Y)$ where $Y \in \operatorname{add}\mathcal{M}'$ factor through a.

Proof: (1)$\Leftrightarrow$(2): As mentioned above, M is endofinite if and only if every direct summand of M is product-complete. Observe that the direct summands of the Σ-pure-injective module M have all the form $M' \cong \coprod_{i\in I'} M_i$ for some subset $I' \subset I$. Hence condition (1) means that $\coprod_{i\in I'} M_i$ is product-complete for every subset $I' \subset I$, which is equivalent to (2) by Proposition 4.8.
(1)$\Leftrightarrow$(3): Assume that M is endofinite. Then M is Σ-pure-injective, and we deduce as in 4.10 that ${}_SM$ is coperfect. Moreover, M is endonoetherian, and it is well known that its direct summands are then endonoetherian as well. But by Corollary 3.3, the second statement in condition (3) just means that for every subset $I' \subset I$ all matrix subgroups of $M' = \coprod_{i\in I'} M_i$ of the form $\mathrm{H}_{A,x}(M')$ with A_R finitely generated (respectively, finitely presented) are finitely generated over $\operatorname{End}_R M'$. Thus (1) implies (3). For the converse implication, we note that coperfectness over the endomorphism ring is inherited to the modules M' by Proposition 2.7. So, from condition (3), we obtain for each M' that ${}_{\operatorname{End}_R M'}M'$ is coperfect and all finite matrix subgroups of M' are finitely generated over $\operatorname{End}_R M'$, which means that M' is product-complete by Proposition 4.10. In particular, we obtain that M is Σ-pure-injective, and we complete the proof as in (1)$\Leftrightarrow$(2).
(1)$\Rightarrow$(5): Since the index set I has to be finite by [41, 4.5], the module M is finitely generated and Σ-pure-injective. From [69] we then know that S is

semiprimary. Moreover, ${}_SM$ is noetherian, and we deduce that ${}_S\mathrm{Hom}_R\,(A, M)$ and its submodule ${}_S\mathrm{r}\,(A, M)$ are finitely generated for all finitely generated modules A. Since $\mathcal{M}$ consists of finitely generated modules, we thus obtain from Proposition 5.2 the existence of left almost split maps.
(5)⇒(4) : Since S is semiprimary, hence semiperfect, the index set I has to be finite and ${}_SJ(S)$ is therefore finitely generated by Corollary 5.9. This implies that S is indeed left artinian, and so the finitely generated module ${}_SM$ is artinian and in particular coperfect. Moreover, M_R and all its direct summands are finitely generated endonoetherian modules and thus satisfy also the second statement in condition (4).
(4)⇒(3) follows immediately from Theorem 3.17.
(1)⇔(6) is proven with similar arguments as (1)⇔(3), taking into account that over a semilocal ring we can restrict to indecomposable finitely generated (or finitely presented) modules A_R and then apply Lemma 5.1 and Proposition 5.2 to verify that ${}_{\mathrm{End}_R\,M'}\mathrm{Hom}_R\,(A, M')$ is finitely generated if and only if A_R admits a map as stated. □

Observe that we cannot omit the second part of condition (5) in the above Corollary. In fact, $\mathcal{M}$ can have left almost split maps though M is not even product-rigid, see 6.14 and 6.15.

Moreover, if the M_i are not finitely generated, then (1) does no longer imply (5). Indeed, in [79, p. 410] Ringel gives an example of an indecomposable generic, i. e. endofinite and not finitely generated, module M such that $S = \mathrm{End}_R\,M$ is semiprimary but not artinian, and hence the category $\mathcal{M} = \{M\}$ does not have left almost split maps.

Following Abrams [1], we call a ring R **(right) π-homogeneous** if it has following property: For any indecomposable projective right R-module P, all indecomposable modules $A \in \mathrm{Prod}\,P$ are isomorphic to P. In [1], Abrams gives an example of a (right) F-ring, i. e. a ring with all direct products of free right R-modules being free, which is right artinian but not π-homogeneous. For this purpose, he proves that certain F-rings are π-homogeneous if and only if the category $\boldsymbol{Pr}$ of all indecomposable projective right R-modules has left almost split maps. Observe that any F-ring is product-complete and therefore Σ-pure-injective. On the other hand, a product-complete ring needs not be an F-ring, see [43]. Employing Theorem 5.12, we can now extend Abrams' result to Σ-pure-injective rings, and moreover, we obtain that a Σ-pure-injective ring is right π-homogeneous if and only if it is left artinian.

Corollary 5.13 (cp. [1, Proposition 2 and Theorem 3]) The following statements are equivalent for a Σ-pure-injective ring R.
(1) R is right π-homogeneous.
(2) All subcategories of Pr are product-rigid.
(3) Pr has left almost split maps.
(4) R is left artinian.
(5) The Jacobson radical $J(R)$ is a finitely generated left R-module.

Proof: First of all, note that every indecomposable projective module P is Σ-pure-injective by assumption, which implies that every indecomposable module $A \in \operatorname{Prod} P$ has a local endomorphism ring. But then R is π-homogeneous if and only if all indecomposable projective modules are product-rigid, or by Proposition 4.8, if and only if all indecomposable projective modules are product-complete. Using the fact that the class of product-complete modules is closed under finite coproducts [66] together with Lemma 4.7, we then conclude that condition (1) is equivalent to (2).
Next, we know from [69] that every Σ-pure-injective ring R is semiprimary. Thus there are local idempotents $e_1, \ldots, e_n \in R$ such that $\{e_1R, \ldots, e_nR\}$ is a complete irredundant set of representatives of the isomorphism classes of $\mathcal{M} = Pr$. Then $M = \coprod_{i=1}^n e_iR$ is endofinite if and only if so is R_R [41, 4.5], or in other words, if and only if R is left artinian. Moreover, M is a generator and is therefore finitely generated over $S = \operatorname{End}_R M$. Furthermore, M is Σ-pure-injective, and S is a semiprimary ring by [69]. So, Theorem 5.12 tells us that the conditions (2), (3) and (4) are equivalent. Finally, the equivalence of (4) and (5) follows from the fact that R is semiprimary. $\square$

6 APPLICATIONS AND EXAMPLES

The aim of this chapter is twofold. On one hand, we exhibit some applications of the results obtained in the previous chapters. On the other hand, we illustrate our investigations by studying some special cases, like hereditary artin algebras or pure-semisimple rings.

6.1 Tilting theory

We begin by applying some of our results to tilting theory. We consider here only classical, that is, finitely generated tilting or cotilting modules. An example of an infinite-dimensional tilting module will be given in section 6.3.

According to Colpi [33], a right module M_R over a ring R is said to be a $*$**-module** if it induces an equivalence of categories in the sense of Menini-Orsatti's Representation Theorem [73]. This means that if we put $S = \mathrm{End} M_R$ and $M^* = \mathrm{Hom}_R\,(M, W)$ for some injective cogenerator W_R, then the functors $F = \mathrm{Hom}_R\,(M,\ ) : \mathrm{Mod}\,R \to \mathrm{Mod}\,S$ and $G = \ \otimes_S M : \mathrm{Mod}\,S \to \mathrm{Mod}\,R$ induce mutually inverse equivalences between the category $\mathbf{Gen}\,\boldsymbol{M}$ of M-generated R-modules and the category $\mathrm{Cogen}\,M^*$ of M^*-cogenerated S-modules. It was established in [88] that $*$-modules are always finitely generated.

Proposition 6.1 Let R be right noetherian, and assume that M_R is a $*$-module. Then the following statements hold true.
(1) $\mathcal{M}$ is contravariantly finite in $\mathrm{mod}\,R$.
(2) S is right π-coherent, and $\mathcal{Y} = \{Y_S \in \mathrm{Cogen}\,M^*_S \mid Y \text{ is finitely generated}\}$ consists of finitely presented modules.
(3) $\mathrm{gen}\,M = \mathrm{pres}\,M \subset \mathrm{mod}\,R$, and F and G induce mutually inverse equivalences between $\mathrm{gen}\,M$ and $\mathcal{Y}$.
(4) $\mathcal{M}$ is covariantly finite in $\mathrm{mod}\,R$ if and only if S is left (π-)coherent and

${}_SM$ is finitely generated.
(5) All matrix subgroups of M_R are finitely generated over S if and only if S is left strongly coherent and ${}_SM$ is finitely generated.

Proof: (1) M_R is finitely generated and hence noetherian. By [90, 27.3] we then know that the category $\sigma[M]$ of all M-subgenerated modules is locally noetherian. From [35, 3.2 and 6.1] it then follows that $F(X)_S$ is finitely generated for all $X \in \operatorname{gen} M$. Since R is right noetherian, for every finitely generated module X_R we have $\tau_M(X) \in \operatorname{gen} M$, and $F(X)_S \cong F(\tau_M(X))_S$ is therefore finitely generated. By Corollary 1.5 this means that $\mathcal{M}$ is contravariantly finite in $\operatorname{mod} R$.
(2) follows from (1) and Corollary 3.24.
(3) Since M_R is a $*$-module, we have $GF(X) \cong X$ for all $X \in \operatorname{Gen} M$. So, the functor G maps a projective presentation of $F(X)$ to an M-presentation of X. Now, if $X \in \operatorname{gen} M$, then we know by the above that $F(X)_S$ is finitely presented. This yields an M-presentation $M^m \to M^n \to X \to 0$.
(4) If $\mathcal{M}$ is covariantly finite in $\operatorname{mod} R$, then we know from Theorem 3.17 and Proposition 3.26 that ${}_SM$ and ${}_SS$ are π-coherent. Conversely, we claim that under the stated assumptions, all finite matrix subgroups of M_R are finitely generated over S. Covariantly finiteness will then follow from Corollary 3.3.
Recall that we have used in the proof of 3.27 a result of Zimmermann [101] asserting that every matrix subgroup $\mathrm{H}_{Y,y}(S_S)$ of S_S is isomorphic as a left S-module to the S-submodule $\mathrm{H}_{G(Y),\underline{y}}(M_R)$ of M^n defined in 3.2, where $\underline{y} = (y \otimes m_1, \ldots, y \otimes m_n)$ and $m_1, \ldots, m_n$ is a generating set of M_R. For a $*$-module which is finitely generated over its endomorphism ring, there is also a sort of converse relationship. More precisely, every matrix subgroup $\mathrm{H}_{A,a}(M)$ of M_R is an S-epimorphic image of an S-submodule of S^n of the form $\mathrm{H}_{Y,\underline{y}}(S_S)$ for some $n \in I\!N$, $Y \in \operatorname{Mod} S$ and $\underline{y} \in Y^n$. In fact, since ${}_SM$ is finitely generated, we know from [8, Proposition 1.2] that every module A_R has a $\operatorname{Gen} M$-preenvelope $f : A \to X$, where X_R can be chosen in $\operatorname{gen} M$ if A_R is finitely generated. Obviously, $\mathrm{H}_{A,a}(M)$ then equals $\mathrm{H}_{X,x}(M)$ with $x = f(a)$. Further, the module $X \in \operatorname{Gen} M$ has the form $X \cong G(Y_S)$ with $Y = F(X) \in \operatorname{Mod} S$, and the map $\rho_X : Y \otimes_S M = GF(X) \to X$ given by $\alpha \otimes m \mapsto \alpha(m)$ is an isomorphism. So, there are elements $\underline{y} = (y_1, \ldots, y_n) \in Y^n$ and $\underline{m} = (m_1, \ldots, m_n) \in M^n$ such that $x = \rho_X(\sum_{j=1}^n y_j \otimes m_j)$. We then obtain the desired result according to the following commutative diagram with S-linear maps

$$\begin{array}{ccccc}
\mathrm{Hom}_S\,(Y,S) & \simeq & \mathrm{Hom}_R\,(Y \otimes_S M, M) & \simeq & \mathrm{Hom}_R\,(X,M) \\
\downarrow \varepsilon_{\underline{y}} & & & & \varepsilon_x \downarrow \\
\mathrm{H}_{Y,\underline{y}}(S_S) & & & & \mathrm{H}_{X,x}(M) \\
\cap & & & & \cap \\
S^n & \simeq & \mathrm{Hom}_R\,(M^n, M) & \xrightarrow{\quad \varepsilon_{\underline{m}} \quad} & {}_S M
\end{array}$$

where $\varepsilon_{\underline{y}}$ and ε_x are defined as in Proposition 3.2. Now, if we start with a finitely generated module A_R, then $X_R \in \operatorname{gen} M$ and Y_S is therefore finitely presented. Moreover, since S is left coherent, we know from Corollary 3.5 that all finite matrix subgroups of S_S are finitely generated left ideals. Hence we deduce from Proposition 3.2 that $\mathrm{H}_{Y,\underline{y}}(S_S)$ and $\mathrm{H}_{A,a}(M)$ are finitely generated over S, and our claim is proven.
(5) The only-if part follows from Proposition 3.27, the if part is shown as in (4). $\square$

We now turn to tilting modules. A module M_R is called a **(classical) tilting module** if it satisfies

(i) $\mathrm{pd}(M_R) \leq 1$;

(ii) $\mathrm{Ext}^1_R(M,M) = 0$;

(iii) there is no nonzero module X such that $\mathrm{Hom}_R(M,X) = \mathrm{Ext}^1_R(M,X) = 0$;

(iv) M_R is finitely presented;

or equivalently, if M_R is a faithful $*$-module such that ${}_SM$ is finitely generated [34]. Note that ${}_SM_R$ is then faithfully balanced and ${}_SM$ is a tilting module, too. Condition (iii) can also be replaced by the more usual condition

(iii') There is an exact sequence $0 \longrightarrow R \longrightarrow M_0 \longrightarrow M_1 \longrightarrow 0$ with M_0 and M_1 in $\operatorname{add} M$.

If M_R is a tilting module, then we have a further pair of mutually inverse equivalences which is given by the derived functors $F' = \mathrm{Ext}^1_R\,(M,\ )$: $\mathrm{Mod}\,R \to \mathrm{Mod}\,S$ and $G' = \mathrm{Tor}^S\,(M,\ )$: $\mathrm{Mod}\,S \to \mathrm{Mod}\,R$. This was first proven for finitely generated modules over artin algebras by Brenner and Butler [25] and later extended to the category of all modules over an arbitrary ring by Colby and Fuller [31]. More precisely, the pair $(\operatorname{Gen} M, \operatorname{Ker} F)$ is a torsion theory in $\mathrm{Mod}\,R$, the pair $(\operatorname{Ker} G, \operatorname{Cogen} M^*)$ is a torsion theory in $\mathrm{Mod}\,S$, and we have equivalences $\operatorname{Gen} M \overset{F,G}{\longleftrightarrow} \operatorname{Cogen} M^*$, and $\operatorname{Ker} F \overset{F',G'}{\longleftrightarrow} \operatorname{Ker} G$. In the artin algebra case, moreover, these equivalences can be restricted to the categories of finitely presented modules.

We are now going to see that the above equivalences can be restricted to the categories of finitely presented modules also when R is right noetherian. Observe that S then need not be right noetherian, as shown by an example of Tachikawa and Valenta [89, §1]. However, we know from Proposition 6.1 that S is right coherent and M^*_S is π-coherent, and this will be enough for our purpose. But let us first comment on the assumption that R is noetherian.

Remark 6.2 The following statements are equivalent for a tilting module M_R.
(1) R is right noetherian.
(2) M_R is noetherian.
(3) Every finitely generated module has an add$\mathcal{M}$-precover, and M^*_S is π-coherent.
(4) F carries finitely generated modules to finitely presented modules.

Proof: (1)⇔(2) is clear since M_R is finitely generated and R is finitely M-cogenerated.
(1)⇒(3) is shown in Proposition 6.1.
(3)⇒(4) : From Corollary 1.5, we know that F carries finitely generated modules to finitely generated modules. But since M^*_S is π-coherent, this implies (4).
(4)⇒(2) : Let $0 \longrightarrow A \longrightarrow M \longrightarrow L \longrightarrow 0$ be an exact sequence. Take a generating set $f_1, \ldots, f_n$ of the finitely presented module $Y_S = F(L)$, construct the coproduct map $f : M^n \to L$ and set $K = \operatorname{Ker} f$. Of course, f is an add M-precover of $L \in \operatorname{gen} M$, and so f and $F(f)$ are epimorphisms. From the long exact sequence $0 \to F(K) \to F(M^n) \xrightarrow{F(f)} Y \to F'(K) \to F'(M^n) = 0$, we then deduce that K lies in $\operatorname{Ker} F' = \operatorname{Gen} M$, see [31]. In the commutative diagram

$$\begin{array}{ccccccccc}
 & & GF(K) & \longrightarrow & GF(M^n) & \longrightarrow & G(Y) & \longrightarrow & 0 \\
 & & \downarrow\rho_K & & \downarrow\rho_{M^n} & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M^n & \xrightarrow{f} & L & \longrightarrow & 0
\end{array}$$

we thus have that ρ_K and ρ_{M^n} are isomorphisms, and we conclude that $L \simeq G(Y_S)$ is finitely presented since so are Y_S and M_R. But this means that A_R is finitely generated, and the proof is complete. □

Let us now show the Brenner-Butler Theorem for the noetherian setting.

Theorem 6.3 (TILTING THEOREM) Let R be a right noetherian ring and M_R a tilting module. Denote $\mathcal{T} = \operatorname{gen} M$, $\mathcal{F} = \operatorname{Ker} F \cap \operatorname{mod} R$, $\mathcal{X} = \operatorname{Ker} G \cap \operatorname{mod} S$, and $\mathcal{Y} = \operatorname{Cogen} M_S^* \cap \operatorname{mod} S$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\operatorname{mod} R$, $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in $\operatorname{mod} S$, and there are equivalences $\mathcal{T} \overset{F,G}{\longleftrightarrow} \mathcal{Y}$ and $\mathcal{F} \overset{F',G'}{\longleftrightarrow} \mathcal{X}$.

Proof: We will explain the crucial points and only give a brief outline of the rest of the proof. We start with a module $Z \in \operatorname{mod} S$, denote by $t(Z)$ the trace of the torsion class $\operatorname{Ker} G$ in Z and consider the canonical exact sequence $0 \longrightarrow t(Z) \overset{\alpha}{\longrightarrow} Z \longrightarrow Z/t(Z) \longrightarrow 0$. Recall that S is right coherent and that the finitely generated modules in $\operatorname{Cogen} M_S^*$ are finitely presented by Proposition 6.1. Hence Z_S is coherent and $Z/t(Z) \in \operatorname{mod} S$. Thus $t(Z)$ is finitely generated and even finitely presented, that is, $t(Z) \in \mathcal{X}$.
As a first consequence, we obtain that every $Z \in \operatorname{mod} S$ with $\operatorname{Hom}_S(X, Z) = 0$ for all $X \in \mathcal{X}$ has $\alpha = 0$ and therefore belongs to $\mathcal{Y}$, from which we immediately conclude that $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in $\operatorname{mod} S$.
Next, we show that the functor G' carries finitely presented modules to finitely presented modules. In fact, if $Z \in \operatorname{mod} S$, then $G'(Z)_R \simeq G'(t(Z))_R$. Moreover, since the modules in $\operatorname{Ker} F$ are generated by $R/\tau_M(R)$ where $\tau_M(R)$ denotes the trace of M in R, we have $\operatorname{Ker} G = \operatorname{Gen} X$ with $X = F'(R/\tau_M(R))$. So, we have an exact sequence $0 \longrightarrow K \longrightarrow X^n \longrightarrow t(Z) \longrightarrow 0$ where $n \in I\!N$ and K_S is finitely generated. This yields an exact sequence $G'(X^n) \to G'(t(Z)) \to G(K) \to G(X^n) = 0$. Since $G'(X^n)$ and $G(K)$ are finitely generated, we conclude that $G'(Z)_R$ is finitely generated.
Let us now turn to F'. It is well known that $F'(R)_S$ is finitely presented, which can easily be verified by employing condition (ii) and (iii'). Moreover, by condition (i), any exact sequence $0 \longrightarrow K \longrightarrow R^m \longrightarrow X \longrightarrow 0$ yields an exact sequence $F'(K)_S \to F'(R^m)_S \to F'(X)_S \to 0$. So, F' carries finitely generated modules to finitely generated modules. Further, if X is finitely presented, then K_R and therefore also $F'(K)_S$ are finitely generated, which shows that $F'(X)_S$ is finitely presented as well.
Finally, the noetherian assumption ensures that $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\operatorname{mod} R$. By the above considerations, the proof is now complete. $\square$

In the artin algebra case, the usual duality $D : \operatorname{mod} R \to R\operatorname{mod}$ provides also a dual version of the Tilting Theorem, relating the dual notion of a cotilting module to Morita duality, just as tilting modules are related to Morita equivalence.

In the general case, however, the situation for cotilting modules seems to be more complex. The first work aiming to a generalization of the cotilting modules considered in the representation theory of artin algebras, was done by Colby in [32]. He extended the notion of cotilting to modules which are finitely generated over a noetherian ring and also noetherian over their endomorphism ring, and proved a Cotilting Theorem for this setting.

Another generalization has later been studied by Colpi, D'Este, Tonolo and Trlifaj in [37] and [39]. Their definition of a cotilting module is obtained by dualizing the definition of a (possibly not finitely generated) tilting module given in [40]. In recent work of Colpi, Fuller and Tonolo [36], [38] [86] it is shown that also this kind of cotilting theory extends Morita duality. Observe, however, that even in the noetherian situation, this concept of cotilting is quite different from the one investigated by Colby in [32], see [4, Examples 2.1 and 2.4].

As an attempt to find a bridge between these two approaches, we have discussed in [4] a notion of cotilting module endowed with certain finiteness conditions. We call a module M_R over an arbitrary ring R a **finitely cotilting module** if it satisfies the following conditions:

(i) $\mathrm{id}(M_R) \leq 1$;

(ii) $\mathrm{Ext}^1_R(M, M) = 0$;

(iii) there is no nonzero module X such that $\mathrm{Hom}_R(X, M) = \mathrm{Ext}^1_R(X, M) = 0$;

(iv) M_R is finitely generated;

(v) the functor $\Delta = \mathrm{Hom}_R(\ , M) : \mathrm{Mod}\, R \to S\,\mathrm{Mod}$ carries finitely generated modules to finitely generated modules.

Our concept coincides with Colby's definition in [32] when we restrict our attention to Morita rings [4, Theorem 3.3]. On the other hand, it coincides with the definition given by Colpi, D'Este and Tonolo [37] in case the module is finitely generated and product complete [4, Proposition 2.2]. In particular, all three notions coincide with the usual cotilting modules when we are dealing with finitely generated modules over artin algebras.

Condition (v) in the above definition means by Theorem 3.17 and Corollary 1.5 that ${}_SM$ is π-coherent, or in other words, that $\Delta : \mathrm{Mod}\, R \to S\,\mathrm{Mod}$ carries finitely generated modules to finitely presented modules. So, according to Remark 6.2, finitely cotilting modules can be viewed as the dual counterparts of noetherian tilting modules, though they need not be noetherian, neither over the ground ring nor over the endomorphism ring [4, Example 3.6].

This is essentially the reason why Colby's Cotilting Theorem [32, 2.4] extends to our setting. We consider the functors $\Delta = \operatorname{Hom}_R(\ ,M) : \operatorname{Mod} R \to S\operatorname{Mod}$ and $\Gamma = \operatorname{Ext}^1_R(\ ,M) : \operatorname{Mod} R \to S\operatorname{Mod}$, and denote by Δ and Γ also the corresponding functors $S\operatorname{Mod} \to \operatorname{Mod} R$ induced by ${}_SM$.

Theorem 6.4 (Cotilting Theorem [4, 4.4]) Let R be an arbitrary ring and ${}_SM_R$ a faithfully balanced bimodule which is a finitely cotilting module on both sides. Define subcategories of $\operatorname{Mod} R$, respectively of $S\operatorname{Mod}$, by setting $\mathcal{Y} = \{Y \in \operatorname{Cogen} M \mid Y \text{ is finitely generated}\}$, $\mathcal{X} = \{X \in \operatorname{Ker}\Delta \mid X \text{ is finitely generated}\}$, and $\mathcal{X}' = \{X \in \operatorname{Ker}\Delta \mid X \text{ is finitely presented}\}$. Then $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in the category of all finitely generated right R-modules, respectively left S-modules. Moreover, $\mathcal{Y}$ consists of finitely presented modules, and there are dualities $\Delta : \mathcal{Y} \longleftrightarrow \mathcal{Y}$ and $\Gamma : \mathcal{X}' \longleftrightarrow \mathcal{X}'$.

Remark 6.5 [4, 4.5] In the situation of Theorem 6.4, we have that $\mathcal{X} = \mathcal{X}'$ if and only if Γ carries finitely generated modules to finitely generated modules, which is further equivalent to R being right noetherian, resp. S being left noetherian.

We remark that this approach to cotilting theory has then been further considered in [87].

6.2 Cofinendo modules

In this section, we describe the modules M such that every right R-module has a $\operatorname{Prod} M$-precover. We prove that every Σ-pure-injective module M has this property (Proposition 6.10). Moreover, we compare the class of all modules with this property with the class of product-complete modules. The relationship between these two classes resembles the relationship between noetherian and artinian rings.

Let us first observe that Rada's and Saorín's result 1.2(2) has the following consequence.

Proposition 6.6 Let A_R be a module and $J = \operatorname{Hom}_R(A, M)$. Then A has an $\operatorname{Add} M$-preenvelope if and only if the left $\operatorname{End}_R M^{(J)}$-module $\operatorname{Hom}_R(A, M^{(J)})$ is finitely generated.

Proof: Let $a : A \to X$ be an $\mathrm{Add}\, M$-preenvelope. From 1.2(2) we know that a factors through a map $a' : A \to M^{(J)}$. But then also a' is an $\mathrm{Add}\, M$-preenvelope, and in particular, the map $\mathrm{Hom}_R\,(a', M^{(J)}) : \mathrm{End}_R\, M^{(J)} \to \mathrm{Hom}_R\,(A, M^{(J)})$ is surjective. For the converse implication, let $c_1, \dots, c_n$ be a generating set of $\mathrm{Hom}_R\,(A, M^{(J)})$ over $\mathrm{End}_R\, M^{(J)}$. Using again 1.2(2), it is easy to see that the product map $c : A \to M^{(J)\,n}$ induced by the c_k is an $\mathrm{Add}\, M$-preenvelope. □

We will now give a dual version of Proposition 6.6.

Proposition 6.7 A module C_R has a $\mathrm{Prod}\, M$-precover if and only if there is a cardinal β such that both of the following conditions are satisfied.
(i) The right $\mathrm{End}_R\, M^\beta$-module $\mathrm{Hom}_R\,(M^\beta, C)$ is finitely generated.
(ii) For every cardinal α, all maps $M^\alpha \to C$ factor through some coproduct of copies of M^β.

Proof: Let $b : Y \to C$ be a $\mathrm{Prod}\, M$-precover. Then there is a cardinal β and a split epimorphism $p : M^\beta \to Y$, and for any cardinal α, all maps $M^\alpha \to C$ factor through b and thus through bp. So, condition (ii) is satisfied. Moreover, if we take $\alpha = \beta$, we see that the map $\mathrm{Hom}_R\,(M^\beta, bp) : \mathrm{End}_R\, M^\beta \to \mathrm{Hom}_R\,(M^\beta, C)$ is an epimorphism, which shows (i).
The converse implication is proven with arguments dual to those in 6.6. □

Given a module C_R, let us say that M_R is ***C*-cofinendo** if the above condition (ii) is satisfied, that is, if there is a cardinal β such that for every cardinal α, all maps $M^\alpha \to C$ factor through some coproduct of copies of M^β. Observe that if C is an injective cogenerator, then the definition of C-cofinendo coincides with the notion of cofinendo introduced in [8]. We showed in [8, 1.6] that a module M is cofinendo if and only if the injective cogenerator C has a $\mathrm{Prod}\, M$-precover. We now give a natural extension of this result.

Corollary 6.8 Let C_R be pure-injective. Then C has a $\mathrm{Prod}\, M$-precover if and only if M is C-cofinendo.

Proof: By 6.7, we have only to prove the if-part. We use the same arguments as in [8, 1.6]. Assume that M is C-cofinendo, and consider the coproduct map $b' : M' = M^{\beta\,(I)} \to C$ induced by all maps in $I = \mathrm{Hom}_R\,(M^\beta, C)$. Observe that M' is a pure submodule of $Y = M^{\beta\, I} \in \mathrm{Prod}\, M$, and since C is pure-injective, b' can be lifted to a map $b : Y \to C$. Take now a map $h : X \to C$

where $X \in \operatorname{Prod} M$. Then there is a split epimorphism $p : M^\alpha \to X$, and by assumption, the map $hp : M^\alpha \to C$, and therefore also the map h, factor through b', hence through b. Thus b is a $\operatorname{Prod} M$-precover. □

We are now ready to characterize when $\operatorname{Prod} M$ is a precover class.

Corollary 6.9 $\operatorname{Prod} M$ is a precover class if and only if M is C-cofinendo for every module C, and $\operatorname{Prod} M$ is closed under coproducts.

Proof: By 6.7 and 1.1(II), we have only to prove the if-part. We know that for every module C there is a cardinal β such that every map $Y \to C$ where $Y \in \operatorname{Prod} M$ factors through $\operatorname{Add} M^\beta$. But then, taking the sets $\mathcal{X}_C = \{M^\beta\} \subset \operatorname{Prod} M$, we see that $\operatorname{Prod} M$ is locally finally small. Since $\operatorname{Prod} M$ is also closed under coproducts, we obtain our claim from 1.1(II). □

Proposition 6.10 If M is Σ-pure-injective, then $\operatorname{Prod} M$ is a precover class.

Proof: By a result of Jensen and Gruson [62], we know that there is a cardinal κ such that every product of copies of M is a direct sum of modules of cardinality at most κ. Of course, the isomorphism classes of all κ-generated modules lying in $\operatorname{Prod} M$ form a set $\mathcal{K}$. Let K be the direct sum of all modules in $\mathcal{K}$, and P the direct product of all modules in $\mathcal{K}$. By the above we then have $\operatorname{Prod} M \subset \operatorname{Add} K$. Moreover, since $\operatorname{Prod} M$ is closed under products and consists of Σ-pure-injective modules, $P \in \operatorname{Prod} M$ is Σ-pure-injective. Hence the pure submodule K of P is also Σ-pure-injective and therefore a direct summand of P. This proves $K \in \operatorname{Prod} M$, so $\operatorname{Prod} K \subset \operatorname{Prod} M$, and further, by Σ-pure-injectivity, $\operatorname{Add} K \subset \operatorname{Prod} K$. We then conclude that $\operatorname{Prod} M = \operatorname{Add} K$. This gives the claim, since all classes of the form $\operatorname{Add} K$ are precover classes by 1.2 (1). □

Modules such that $\operatorname{Prod} M$ is a precover class can be seen as the duals of product-complete modules. We will now see that the relationship between these two classes of modules resembles the relationship between noetherian and artinian rings. Indeed, if M is product-complete, then M is Σ-pure-injective and $\operatorname{Prod} M = \operatorname{Add} M$ is a cover class by Proposition 2.2. However, the converse is far from being true.

Proposition 6.11 Let E be a minimal injective cogenerator of $\mathrm{Mod}\, R$.
(1) E_R is product-complete if and only if R is right artinian.
(2) $\mathrm{Prod}\, E$ is a cover class if and only if R is right noetherian.

Proof: (2) is shown in [48, 2.1]. Let us prove (1). By the above observation we know that $\mathrm{Prod}\, E$ is a cover class, hence R is right noetherian. So, E is the coproduct of the injective envelopes $E(S_i)$ of a complete irredundant set of simple right modules $\{S_i \mid i \in I\}$. Moreover, every indecomposable injective module is an object of $\mathrm{Prod}\, E$ having a local endomorphism ring, and therefore is isomorphic to some $E(S_i)$ since E is product-rigid by 4.8. This shows that R is right artinian. Conversely, assume that R is right artinian. Then every product of copies of E is injective, hence a coproduct of injective envelopes of simple right modules, in other words, a coproduct of indecomposable summands of E. By 4.6 the module E is then product-complete. $\square$

Let us close this section with another example of a module M such that $\mathrm{Prod}\, M$ is a cover class. It generalizes an example considered in [82, 3.4], [1], and [37, 5.3], [43].

Example 6.12 Let F, G be two skew-fields and ${}_F V_G$ a bimodule which is infinite-dimensional over F. Then $R = \begin{pmatrix} F & {}_F V_G \\ 0 & G \end{pmatrix}$ is a hereditary semiprimary ring. In particular, R is left coherent and right perfect, and thus R_R is product-complete and all projectives are π-projective by [28]. However, $P = e_1 R$ is not finitely generated over $\mathrm{End}_R P$ and hence not product-rigid by 4.9. This implies that $\mathrm{Prod}\, P$ contains an indecomposable module not isomorphic to P, and since the class $\mathrm{Prod}\, P$ is contained in the class Pr of all projective modules, we conclude that $\mathrm{Prod}\, P$ must contain $e_2 R$.
We now claim that $\mathrm{Prod}\, P = Pr$. In particular, $\mathrm{Prod}\, P$ will then be a cover class. We have only to verify the inclusion $\mathrm{Prod}\, P \supset Pr$. By the above considerations we already know that all indecomposable projectives are in $\mathrm{Prod}\, P$. Moreover, R is semiperfect, and so every projective module X has the form $X = \coprod_{j \in J} X_j$ for some indecomposable projective modules X_j. But then $X \simeq P_1 \oplus P_2$ where P_k is a coproduct of copies of $e_k R$, $k = 1, 2$. Since R_R is product-complete, R_R and its direct summands $e_k R$ are Σ-pure-injective, and so is X. This shows that X is a direct summand in $\prod_{j \in J} X_j \in \mathrm{Prod}\, P$, and the claim is proven. $\square$

6.3 Hereditary artin algebras

Throughout this section, we assume that R is an artin algebra with the usual duality $D : \mathrm{mod}\, R \to R\,\mathrm{mod}$. We denote by $\mathcal{P} = \{P_j \mid j \in J\}$, respectively $\mathcal{I} = \{I_k \mid k \in K\}$, a complete irredundant set of representatives of the isomorphism classes of the indecomposable preprojective right modules, respectively of the indecomposable preinjective left modules, and we set $P_R = \coprod_{j\in J} P_j$, and ${}_RI = \coprod_{k\in K} I_k$. Observe that $D(P_R) \cong \prod_{j\in J} D(P_j) \cong \prod_{k\in K} I_k$.

Let us start with the following observation.

Proposition 6.13 The following statements are equivalent.
(1) R is of finite representation type.
(2) ${}_RI$ is endonoetherian.
(3) ${}_RI$ is product-complete.
(4) P_R is product-complete.
(5) P_R is Σ-pure-injective.
(6) $\mathrm{Add}\,\mathcal{P}$ is a cover class.

Proof: Of course, if R is of finite representation type, then every module is endofinite [97, Theorem 6] and hence endonoetherian and product-complete. So, (1) implies any of the other conditions. Moreover, (4) is equivalent to (3) by Theorem 4.12, and, arguing as in [97, Corollary 12], we see that (5)⇔(2). Now, we know from section 4.2 that (4) implies (5). Further, it follows from Theorem 2.13 and Corollary 2.7 that (5) implies (6) and that (6) just means that the family $(P_j)_{j\in J}$ is locally T-nilpotent, hence T-nilpotent. But the latter implies by [14, 5.6] that R is of finite representation type. □

From now on, let R be a finite-dimensional hereditary algebra over a field.

Proposition 6.14 (1) $\mathcal{P}$ is covariantly finite in $\mathrm{mod}\, R$ and has right and left almost split maps.
(2) P_R is product-rigid.
(3) $\mathcal{I}$ is contravariantly finite in $\mathrm{mod}\, R$ and has right and left almost split maps.
(4) I is coperfect over its endomorphism ring.

Proof: (1) and (3) are well-known.
For (2), it is enough to show that every module A_R with a nonzero map $A \to P$ has a preprojective summand. This is proven as in [3, 4.1, step (4)]

using the fact that the almost split sequences ending at the indecomposable preprojective modules actually are almost split sequences in $\operatorname{Mod} R$.
To verify (4), we consider a chain $X_1 \to X_2 \to X_3 \to \ldots$ of non-zero non-isomorphisms between indecomposable preinjective modules and let $\mathcal{I}_0, \ldots, \mathcal{I}_\infty$ be the preinjective partition [16]. Now, if X_1 belongs to the layer $\mathcal{I}_n$, then $X_2, X_3, \ldots$ belong to the finite subcategory $\mathcal{I}_0 \cup \ldots \cup \mathcal{I}_n$, see [3, 4.1], [9]. So, we conclude from the Lemma of Harada and Sai that the family $(I_k)_{k\in K}$ is T-nilpotent, and Corollary 2.7 completes the proof. □

Proposition 6.15 (1) The following statements are equivalent.

(a) I is finitely generated over its endomorphism ring.

(b) $\mathcal{I}$ is covariantly finite in $\operatorname{mod} R$.

(c) $\mathcal{P}$ is contravariantly finite in $\operatorname{mod} R$.

(d) R is of finite representation type.

(2) The following statements are equivalent.

(a) ${}_R I$ is Σ-pure-injective.

(b) P_R is endonoetherian.

(c) R is tame.

In particular, ${}_R I$ is not product-rigid in general, though its dual always is. Moreover, in general, $\operatorname{End}_R P$ is not left noetherian.

Proof: (1) (a)⇔(b) follows from [16, 4.5] since $\operatorname{add}\mathcal{I}$ is closed under factors, see also [8, 1.2].
(b)⇔(c) and (d)⇒(a) are clear.
(b)⇒(d): Let $\mathcal{I}_0, \ldots, \mathcal{I}_\infty$ be the preinjective partition, and assume $A \in \mathcal{I}_\infty$. Then for each $n \in I\!N$ there is a non-split monomorphism $f_n : A \to Y_n$ where $Y_n \in \operatorname{add}\mathcal{I}_n$. On the other hand, by assumption there is an $\operatorname{add}\mathcal{I}$-preenvelope $a : A \to B$, and there is an $m \in I\!N$ such that all indecomposable summands of B lie in $\bigcup_{l\le m}\mathcal{I}_l$. But since all f_n factor through a and any indecomposable module Y with a nonzero map $B \to Y$ lies in $\bigcup_{l\le m}\mathcal{I}_l$ (see [3], [9]), we obtain a contradiction. So, $\operatorname{mod} R$ consists of preinjectives, and this means by [16, 6.1] that R has finite representation type.
(2) (a)⇔(b) is shown as in 6.13, and (a)⇔(c) was proven by Lenzing in [70, 4.6].
In particular, we infer from Propositions 4.8 and 6.13 that I is not product-rigid whenever R is representation-infinite tame. On the other hand, $P_R = \coprod_{k\in K} D(I_k)$ is always product-rigid by 6.14, and then we know from Lemma

4.7 that $D(I) \cong \prod_{k \in K} D(I_k)$ is product-rigid as well. Finally, when R is wild, then P is not endonoetherian, and since P is a generator and is therefore finitely generated over $\mathrm{End}_R P$, we infer that $\mathrm{End}_R P$ is not left noetherian. □

As mentioned on page 56, the direct sum of all Prüfer groups and $\mathbb{Q}$ is a product-complete $\mathbb{Z}$-module, while the single Prüfer groups are not even product-rigid. We now exhibit an analogous example over R. Consider the direct sum T of a complete irredundant set of representatives of the isomorphism classes of the Prüfer modules and the (unique) generic R-module Q, see [79]. As pointed out by Ringel, T is an infinite-dimensional tilting module in the sense of [40], see [7, 1.4].

Proposition 6.16 T is product-complete, while the Prüfer modules are not even product-rigid.

Proof: First of all, the Prüfer modules are infinite-dimensional indecomposable Σ-pure-injective but not endofinite, thus not product-rigid, see also [64]. Furthermore, since T is a tilting module, $\mathrm{Gen}\, T$ is closed under products, see [40] or [8]. Observe that T is a divisible module and that the class of all divisible modules is closed under coproducts and epimorphic images by [79, 4.6]. So, we infer that every product of copies of T is divisible and therefore, by [79, 4.7], a coproduct of indecomposable preinjective modules, Prüfer modules and copies of Q. Moreover, T (as every module over an artin algebra) is finitely product-rigid. Hence we conclude that $\mathrm{Prod}\, T \subset \mathrm{Add}\, T$, which means that T is product-complete by Theorem 4.6. □

6.4 Pure-semisimple rings

We now illustrate some of the notions discussed in the previous chapters in case that R is a left pure-semisimple ring. Recall that a ring R is said to be **left pure-semisimple** if every left R-module is a direct sum of finitely presented modules, or equivalently, if every left R-module is pure-injective. It is well known that a ring has finite representation type if and only if it is left and right pure semisimple, and it is conjectured that onesided pure semisimplicity is sufficient, i. e. that every left (or right) pure semisimple ring has finite representation type.

Proposition 6.17 The following hold true for a left pure-semisimple ring R.
(1) Every class $\mathcal{M} \subset \mathrm{mod}\, R$ is covariantly finite in $\mathrm{mod}\, R$, product-rigid, and

has (generalized) left almost split maps.
(2) A left module ${}_RC$ is product-complete if and only if it is finitely product-rigid.
(3) By [95], [45] we know that Rmod has a preinjective partition. Let us denote by $\mathcal{I} = \{I_k \mid k \in K\}$ a complete irredundant set of representatives of the isomorphism classes of the indecomposable preinjective left modules and set $I = \coprod_{k\in K} I_k$. Then R is of finite representation type if and only if ${}_RI$ is endocoherent.

Proof: (1) Since all pure-projective right R-modules are endonoetherian by [97, Theorem 9], the claim follows immediately from Corollaries 3.3 and 4.2 and Proposition 5.4.
(2) By Proposition 4.8 we have only to prove the if-part. Now, every product of copies of ${}_RC$ is a direct sum ${}_RC^I = \coprod_{k\in K} N_k$ of finitely presented modules N_k with local endomorphism ring, and since C is finitely product-rigid, each N_k is isomorphic to an indecomposable direct summand of C. Then C is product-complete by Theorem 4.6.
(3) By [95, Corollary B] we know that R is of finite representation type if and only if $\mathcal{I}$ is finitely product-rigid. By Theorem 3.17 the latter condition is verified whenever ${}_RI$ is endocoherent. Conversely, if R is of finite type, then I is a finitely generated endofinite module [97, Theorem 6], hence endocoherent. $\square$

Let us now turn to some characterizations of pure-semisimplicity. It is well known that in the case $\mathcal{M} = \operatorname{mod} R$, closure under direct limits of the category $\operatorname{Add}\mathcal{M}$ means exactly that R is right pure-semisimple. From Theorem 2.13 we then obtain a bunch of equivalent conditions which we collect in the following corollary. Besides from (5) which seems to be new, these characterizations were already given by Azumaya, Facchini and Simson [20, Proposition 5], [19, Theorem 6], [83, Theorem 6.3].

Corollary 6.18 The following statements are equivalent.
(1) R is right pure-semisimple.
(2) Every right R-module has a pure-projective cover.
(3) Every right R-module is locally pure-projective.
(4) Every right R-module is a Mittag-Leffler module.
(5) Every pure-projective right R-module has a semiregular endomorphism ring.
(6) Every finitely presented right R-module has a decomposition in modules

with local endomorphism ring, and every family of indecomposable finitely presented right R-modules is T-nilpotent.
(7) Every pure submodule (or every locally split submodule, or every local direct summand) of a pure-projective right R-module is a direct summand. □

We close the paper with a further result on pure-semisimple rings, which will appear in [5]. Namely, by applying some results from section 5.1, we obtain a new proof for a characterization of pure-semisimplicity in terms of generalized right almost split morphisms given by Brune in [26] and later improved by Zimmermann in [100]. Moreover, we give a dual characterization in terms of generalized left almost split morphisms.

Theorem 6.19 (1) (cp. [26, §3, Corollary 1] and [100, p. 372]) A right artinian ring R is right pure-semisimple if and only if every finitely presented indecomposable right R-module has generalized right almost split morphisms in $\operatorname{mod} R$.
(2) A semilocal ring R is left pure-semisimple if and only if every finitely presented indecomposable right R-module has generalized left almost split morphisms in $\operatorname{mod} R$.

Proof: (1) Take $\mathcal{M} = \operatorname{mod} R$ and let W_R be a minimal injective cogenerator of $\operatorname{Mod} R$ with $E = \operatorname{End}_R W$. As observed by Zimmermann in [100, p. 372], we can assume that R is a right Morita ring. In fact, by [100, Theorem 4], this property follows from the existence of right almost split morphisms in $\operatorname{mod} R$ for each simple non-projective module C_R, and is therefore satisfied whenever every finitely presented indecomposable right R-module has generalized right almost split morphisms in $\operatorname{mod} R$ or when R is right pure-semisimple. So, the module W_R induces a Morita-duality $\operatorname{mod} R \to E \operatorname{mod}$, and we have that $M^* = {}_E \operatorname{Hom}_R (M, W)_S$ is an E-S-bimodule with $E \cong \operatorname{Hom}_R (\operatorname{Hom}_R (M, W) \otimes_S M, W) \cong \operatorname{End}_S M^*$ and $S \cong (\operatorname{End}_E M^*)^{op}$. Moreover, the ${}_E\operatorname{Hom}_R (M_i, W)$ form a complete irredundant set of representatives of the isomorphism classes of the finitely presented left E-modules. Further, we know from [61, Theorem 7] or [84] that R is right pure-semisimple if and only if so is E, which means by [97, Theorem 6] that every pure-projective left E-module is endonoetherian, or in other words, that the left E-module $\coprod_{i\in I} \operatorname{Hom}_R (M_i, W)$ is endonoetherian. But by Proposition 5.4 the latter is equivalent to $\prod_{i\in I} \operatorname{Hom}_R (M_i, W) \cong {}_E M^*$ being endonoetherian. So, we see

that R is right pure-semisimple if and only if M_S^* is noetherian, and by Proposition 5.7 the proof is complete.
(2) Take M as in (1). By 5.4 we know that every finitely presented indecomposable module A_R has generalized left almost split morphisms in $\operatorname{mod} R$ if and only if M is endonoetherian. But the latter means that every pure-projective module is endonoetherian, which by [97, Theorem 6] is equivalent to R being left pure-semisimple. $\square$

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