TILTING MODULES AND UNIVERSAL LOCALIZATION

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ABSTRACT. We show that every tilting module of projective dimension one over a ring R is associated in a natural way to the universal localization $R \to R_{\mathcal{U}}$ at a set $\mathcal{U}$ of finitely presented modules of projective dimension one. We then investigate tilting modules of the form $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$. Furthermore, we discuss the relationship between universal localization and the localization $R \to Q_{\mathcal{G}}$ given by a perfect Gabriel topology $\mathcal{G}$. Finally, we give some applications to Artin algebras and to Prüfer domains.

INTRODUCTION

Tilting modules of projective dimension one are often constructed via a localization. For example, if Σ is a left Ore set of regular elements in a ring R with the property that the localization $\Sigma^{-1}R$ is an R-module of projective dimension at most one, then $\Sigma^{-1}R \oplus \Sigma^{-1}R/R$ is a tilting right R-module, see [1, 20, 21]. More generally, it was recently shown in [2] that every injective homological ring epimorphism $R \to S$ such that S_R has projective dimension at most one gives rise to a tilting R-module $S \oplus S/R$.

Note, however, that in general not all tilting modules arise as above from an injective homological ring epimorphism. For example, if R is a commutative domain whose ring of fractions has projective dimension at least two, then the Fuchs' divisible module δ is a tilting R-module which is not of the form $S \oplus S/R$, cf. Example 3.10.

On the other hand, every tilting module T of projective dimension one is associated in a natural way to a reflective and coreflective subcategory of $\mathrm{Mod}R$, which is obtained as perpendicular category

$$\mathcal{X}_{T_1} = \{M_R \mid \mathrm{Hom}_R(T_1, M) = 0 = \mathrm{Ext}^1_R(T_1, M)\}$$

of a certain module T_1 in the additive closure $\mathrm{Add}\,T$. By a result of Gabriel and de la Peña [13, 1.2], the category $\mathcal{X}_{T_1}$ is then associated to a ring epimorphism $\lambda : R \to S$. We show that, choosing T_1 appropriately, one can find a set $\mathcal{U}$ of finitely presented modules of projective dimension one such that λ is the universal localization $R \to R_{\mathcal{U}}$ of R at $\mathcal{U}$ in the sense of Schofield [22]. More precisely, we prove the following result.

Theorem 2.2 *Let T be a tilting module of projective dimension one. Then there are an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ and a set $\mathcal{U}$ of finitely presented modules of projective dimension one such that*

(1) $T_0, T_1 \in \operatorname{Add} T$,

(2) $\operatorname{Gen} T = \{M \in \operatorname{Mod} R \mid \operatorname{Ext}^1_R(U, M) = 0 \text{ for every } U \in \mathcal{U}\}$,

(3) $\mathcal{X}_{T_1}$ *is equivalent to the category of right $R_{\mathcal{U}}$-modules via the restriction of scalars along $\lambda : R \to R_{\mathcal{U}}$.*

As a consequence, we see for instance that over an Artin algebra every finitely generated tilting module T which is of the form $S \oplus S/R$ for some injective homological ring epimorphism $\lambda : R \to S$ even arises from universal localization $R \to R_{\mathcal{U}}$ at a set of finitely presented modules (Corollary 2.7).

We also study tilting modules arising from perfect localization. In particular, we describe which tilting modules of the form $S \oplus S/R$ arise from the localization $R \to Q_{\mathcal{G}}$ given by a perfect Gabriel topology $\mathcal{G}$.

Theorem 3.9. *Let R be a ring and let T_R be a tilting module of projective dimension one. The following conditions are equivalent:*

(1) *There is a perfect Gabriel topology $\mathcal{G}$ such that R embeds in $Q_{\mathcal{G}}$ and $Q_{\mathcal{G}} \oplus Q_{\mathcal{G}}/R$ is a tilting module equivalent to T.*

(2) *There is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ such that $T_0, T_1 \in \operatorname{Add} T$, $\operatorname{Hom}_R(T_1, T_0) = 0$, and $\mathcal{X}_{T_1}$ is a Giraud subcategory of $\operatorname{Mod} R$.*

Observe that if R is semihereditary, then every perfect localization $R \to Q_{\mathcal{G}}$ arises from universal localization at a set of finitely presented modules (Proposition 4.3). Over a Prüfer domain, there is a converse result: every universal localization at a set of finitely presented cyclic modules can be viewed as the localization given by a perfect Gabriel topology (Proposition 4.8).

We apply these results to investigate tilting modules over Prüfer domains. Here the tilting classes are in one-one-correspondence with perfect Gabriel topologies, as shown by Bazzoni, Eklof and Trlifaj in [4]. More precisely, every tilting module T is associated to a perfect Gabriel topology $\mathcal{L}$ such that the tilting class $\operatorname{Gen} T$ coincides with the class of $\mathcal{L}$-divisible modules. Moreover, if the localization $Q_{\mathcal{L}}$ has projective dimension at most one over R, then it was shown by Salce [21] that T is equivalent to $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$. We recover Salce's result as a consequence of Theorem 3.9. Moreover, we obtain that over a Prüfer domain every tilting module of the form $S \oplus S/R$ arises from a universal localization $R \to R_{\mathcal{U}}$, as well as from a perfect localization $R \to Q_{\mathcal{G}}$, see Theorem 4.10.

1. Preliminaries

I. Notation. Let R be a ring, and let $\operatorname{Mod} R$ be the category of all right R-modules. By a *subcategory* of $\operatorname{Mod} R$ we always mean a full subcategory which is closed under isomorphic images and direct summands.

We denote by $\operatorname{mod} R$ the subcategory of modules possessing a projective resolution consisting of finitely generated modules.

Given a class of modules $\mathcal{C}$, we denote

$$\mathcal{C}^o = \{M \in \operatorname{Mod} R \mid \operatorname{Hom}_R(C, M) = 0 \text{ for all } C \in \mathcal{C}\},$$

$$\mathcal{C}^\perp = \{M \in \operatorname{Mod} R \mid \operatorname{Ext}_R^i(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and all } i > 0\}.$$

The classes ${}^o\mathcal{C}$, and ${}^\perp\mathcal{C}$ are defined similarly. The (right) *perpendicular category* of $\mathcal{C}$ is denoted by

$$\mathcal{X}_\mathcal{C} = \mathcal{C}^o \cap \mathcal{C}^\perp$$

Finally, we denote by $\operatorname{Add}\mathcal{C}$ the class consisting of all modules isomorphic to direct summands of direct sums of modules of $\mathcal{C}$ and by $\operatorname{Gen}\mathcal{C}$ the class of modules generated by modules of $\mathcal{C}$.

II. Reflections. We start by recalling the notion of a reflective subcategory.

Definition 1.1. Let M be a right R-module and $\mathcal{C}$ a subcategory of $\operatorname{Mod} R$.

A morphism $f \in \operatorname{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is said to be a $\mathcal{C}$-*preenvelope* of M provided the morphism of abelian groups $\operatorname{Hom}_R(f, C')\colon \operatorname{Hom}_R(C, C') \to \operatorname{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$, that is, for each morphism $f'\colon M \to C'$ there is a morphism $g\colon C \to C'$ such that the following diagram is commutative.

$$
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & C \\
 & \searrow{\scriptstyle f'} & \Big\downarrow{\scriptstyle g} \\
 & & C'
\end{array}
$$

Furthermore, a $\mathcal{C}$-preenvelope $f \in \operatorname{Hom}_R(M, C)$ is said to be a $\mathcal{C}$-*reflection* of M provided the morphism of abelian groups $\operatorname{Hom}_R(f, C')\colon \operatorname{Hom}_R(C, C') \to \operatorname{Hom}_R(M, C')$ is bijective for each $C' \in \mathcal{C}$, that is, the morphism $g\colon C \to C'$ in the diagram above is always uniquely determined. In this case f is also a $\mathcal{C}$-*envelope*, that is, a $\mathcal{C}$-preenvelope with the additional property that every $g \in \operatorname{End}_R(C)$ such that $f = gf$ is an automorphism.

Finally, $\mathcal{C}$ is said to be a *reflective* subcategory of $\operatorname{Mod} R$ if every R-module admits a $\mathcal{C}$-reflection. *Coreflective subcategories* are defined dually. A full subcategory $\mathcal{C}$ of $\operatorname{Mod} R$ which is both reflective and coreflective is called *bireflective*.

Remark 1.2. It is well known that a subcategory $\mathcal{C}$ is a reflective subcategory of $\operatorname{Mod} R$ if and only if the inclusion functor $\iota : \mathcal{C} \hookrightarrow \operatorname{Mod} R$ has a left adjoint functor $\ell : \operatorname{Mod} R \to \mathcal{C}$. In this case a $\mathcal{C}$-reflection of M is given as

$$\eta_M : M \to \iota\ell(M)$$

by the unit of the adjunction $\eta : 1_{\operatorname{Mod} R} \to \iota\ell$, see [23, Chapter X, §1].

Bireflective subcategories are closely related to ring epimorphisms.

Definition 1.3. A ring homomorphism $\lambda \colon R \to S$ is called a *ring epimorphism* if it is an epimorphism in the category of rings, that is, for every pair of morphisms of rings $\delta_i \colon S \to T$, $i = 1, 2$, the condition $\delta_1 \lambda = \delta_2 \lambda$ implies $\delta_1 = \delta_2$. Note that this holds true if and only if the restriction functor

$$\lambda_* \colon \operatorname{Mod} S \to \operatorname{Mod} R$$

induced by λ is full, see [23, Chapter XI, Proposition 1.2].

Two ring epimorphisms $\lambda \colon R \to S$ and $\lambda' \colon R \to S'$ are said to be *equivalent* if there is a ring isomorphism $\varphi \colon S \to S'$ such that $\lambda' = \varphi \lambda$. The *epiclasses* of R are the equivalence classes with respect to the equivalence relation defined above.

Theorem 1.4. [13, 1.2], [14], [17, 1.6.3] *The following assertions are equivalent for a subcategory $\mathcal{X}$ of $\operatorname{Mod} R$.*

1. *$\mathcal{X}$ is a bireflective subcategory of $\operatorname{Mod} R$.*
2. *$\mathcal{X}$ is closed under isomorphic images, direct sums, direct products, kernels and cokernels.*
3. *There is a ring epimorphism $\lambda \colon R \to S$ such that $\mathcal{X}$ is the essential image of the restriction functor $\lambda_* \colon \operatorname{Mod} S \to \operatorname{Mod} R$.*

More precisely, there is a bijection between the epiclasses of the ring R and the bireflective subcategories of $\operatorname{Mod} R$. Moreover, the map $\lambda \colon R \to S$ in condition (3), viewed as an R-homomorphism, is an $\mathcal{X}$-reflection of R.

III. Universal localization. Next, let us recall Schofield's notion of universal localization.

Theorem 1.5 ([22, Theorem 4.1]). *Let Σ be a set of morphisms between finitely generated projective right R-modules. Then there are a ring R_Σ and a morphism of rings $\lambda \colon R \to R_\Sigma$ such that*

1. *λ is Σ-inverting, i.e. if $\alpha \colon P \to Q$ belongs to Σ, then $\alpha \otimes_R 1_{R_\Sigma} \colon P \otimes_R R_\Sigma \to Q \otimes_R R_\Sigma$ is an isomorphism of right R_Σ-modules, and*
2. *λ is universal Σ-inverting, i.e. if S is a ring such that there exists a Σ-inverting morphism $\psi \colon R \to S$, then there exists a unique morphism of rings $\bar{\psi} \colon R_\Sigma \to S$ such that $\bar{\psi} \lambda = \psi$.*

The morphism $\lambda \colon R \to R_\Sigma$ is a ring epimorphism with $\operatorname{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$. It is called the *universal localization of R at Σ.*

Let now $\mathcal{U}$ be a set of finitely presented right R-modules of projective dimension at most one. For each $U \in \mathcal{U}$, consider a morphism α_U between finitely generated projective right R-modules such that

$$0 \to P \overset{\alpha_U}{\to} Q \to U \to 0$$

We will denote by $\lambda_{\mathcal{U}} \colon R \to R_{\mathcal{U}}$ the universal localization of R at the set $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$. In fact, $R_{\mathcal{U}}$ does not depend on the class Σ chosen, cf. [8, Theorem 0.6.2], and we will also call it the *universal localization of R at $\mathcal{U}$.*

We now show that $\mathcal{X}_{\mathcal{U}}$ is the bireflective subcategory of $\operatorname{Mod} R$ corresponding to the ring epimorphism $\lambda_{\mathcal{U}}$.

Lemma 1.6. *Let $\mathcal{U}$ be a set of finitely presented right R-modules of projective dimension at most one, and let $R_{\mathcal{U}}$ be the universal localization of R at $\mathcal{U}$. Then every right $R_{\mathcal{U}}$-module belongs to $\mathcal{X}_{\mathcal{U}}$.*

Proof. For each $U \in \mathcal{U}$, consider a sequence $0 \to P \xrightarrow{\alpha_U} Q \to U \to 0$ as above. Since $\alpha_U \otimes R_{\mathcal{U}}$ is an isomorphism, we have $U \otimes R_{\mathcal{U}} = \mathrm{Tor}_R^1(U, R_{\mathcal{U}}) = 0$.

Let now M be a right $R_{\mathcal{U}}$-module. Note that the canonical map $\rho_M : M \to M \otimes_R R_{\mathcal{U}}$ is an $R_{\mathcal{U}}$-isomorphism, because $\lambda \colon R \to R_{\mathcal{U}}$ is a ring epimorphism. For any $f \in \mathrm{Hom}_R(U, M)$ we then have a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\;\;f\;\;} & M \\
\Big\downarrow{\scriptstyle\rho_U} & & \Big\downarrow{\scriptstyle\rho_M} \\
0 = U \otimes_R R_{\mathcal{U}} & \xrightarrow{f \otimes R_{\mathcal{U}}} & M \otimes_R R_{\mathcal{U}}
\end{array}
$$

where $\rho_M f = 0$ implies $f = 0$. Hence $M \in \mathcal{U}^0$.

Next, let $0 \to M \xrightarrow{f} N \to U \to 0$ be an exact sequence in $\mathrm{Mod}\,R$. Then we have the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\;\;f\;\;} & N & \longrightarrow & U & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle\rho_M} & & \Big\downarrow{\scriptstyle\rho_N} & & & & \\
0 = \mathrm{Tor}_R^1(U, R_{\mathcal{U}}) & \longrightarrow & M \otimes_R R_{\mathcal{U}} & \xrightarrow[f \otimes R_{\mathcal{U}}]{} & N \otimes_R R_{\mathcal{U}} & \longrightarrow & U \otimes_R R_{\mathcal{U}} = 0
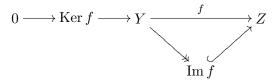\end{array}
$$

where $f \otimes R_{\mathcal{U}}$ is an isomorphism. This implies that $\rho_N f$ is an isomorphism as well. Hence f is a split monomorphism, and we deduce that $M \in \mathcal{U}^\perp$. ∎

Proposition 1.7. *Let $\mathcal{U}$ be a set of finitely presented right R-modules of projective dimension at most one. Then the following statements hold true.*
  (1) *The perpendicular category $\mathcal{X}_{\mathcal{U}}$ is bireflective.*
  (2) *$\mathcal{X}_{\mathcal{U}}$ coincides with the essential image of the restriction functor $\mathrm{Mod}\,R_{\mathcal{U}} \to \mathrm{Mod}\,R$ induced by the universal localization at $\mathcal{U}$.*

Proof. (1) Clearly, $\mathcal{X}_{\mathcal{U}}$ is closed under direct products, and $\mathcal{U}^0$ is closed under direct products and submodules, hence also under direct sums. Furthermore, the assumptions on $\mathcal{U}$ imply that $\mathcal{U}^\perp$ is closed under epimorphic images and direct sums. So, we deduce that $\mathcal{X}_{\mathcal{U}}$ is closed under direct sums.

We now verify that $\mathcal{X}_{\mathcal{U}}$ is closed under kernels. Consider

$$
\begin{array}{ccccc}
0 \longrightarrow \mathrm{Ker}\, f \longrightarrow & Y & \xrightarrow{\quad\quad f \quad\quad} & Z \\
& {\scriptstyle}\searrow & & \nearrow{\scriptstyle} \\
& & \mathrm{Im}\, f & &
\end{array}
$$

with $Y, Z \in \mathcal{X}_{\mathcal{U}}$. Since $\mathcal{U}^0$ is closed under submodules and $\mathcal{U}^\perp$ is closed under epimorphic images, we have $\mathrm{Im}\, f \in \mathcal{U}^0 \cap \mathcal{U}^\perp = \mathcal{X}_{\mathcal{U}}$. Now, for $U \in \mathcal{U}$, applying $\mathrm{Hom}_R(U, -)$ to the short exact sequence $0 \to \mathrm{Ker}\, f \to Y \to \mathrm{Im}\, f \to 0$, we get $\mathrm{Ext}_R^1(U, \mathrm{Ker}\, f) = 0$. This shows that $\mathrm{Ker}\, f \in \mathcal{X}_{\mathcal{U}}$.

The closure under cokernels is proved by similar arguments.

So, we conclude from Theorem 1.4 that $\mathcal{X}_{\mathcal{U}}$ is bireflective.

(2) We know from Theorem 1.4 that there is a ring epimorphism $\lambda : R \to S$ such that $\mathcal{X}_{\mathcal{U}}$ is the essential image of the restriction functor $\lambda_* : \operatorname{Mod} S \to \operatorname{Mod} R$ induced by λ. We claim that λ is equivalent to the universal localization $\lambda_{\mathcal{U}} : R \to R_{\mathcal{U}}$ at $\mathcal{U}$. First of all, we choose a set $\Sigma = \{\alpha_U \mid U \in \mathcal{U}\}$ where the

$$0 \to P_1 \stackrel{\alpha_U}{\to} P_0 \to U \to 0$$

are exact sequences with finitely generated projective modules P_0, P_1, and we claim that λ is Σ-inverting.

Take $U \in \mathcal{U}$ and set $\alpha = \alpha_U$. We have to show that $\alpha \otimes_R S$ is an isomorphism. For any S-module M we have $M_R \in \mathcal{X}_{\mathcal{U}}$, and thus we get the exact sequence

$$0 = \operatorname{Hom}_R(U, M) \to \operatorname{Hom}_R(P_0, M) \xrightarrow{\operatorname{Hom}_R(\alpha, M)} \operatorname{Hom}_R(P_1, M) \to \operatorname{Ext}_R^1(U, M) = 0$$

showing that $\operatorname{Hom}_R(\alpha, M)$ is an isomorphism. Moreover, since $M \cong \operatorname{Hom}_S(S, M)$ as R-modules, we have the following isomorphisms

$$0 = \operatorname{Hom}_R(U, M) \cong \operatorname{Hom}_R(U, \operatorname{Hom}_S(S, M)) \cong \operatorname{Hom}_S(U \otimes_R S, M).$$

In particular, choosing $M = U \otimes_R S$, we see $\operatorname{Hom}_S(U \otimes_R S, U \otimes_R S) = 0$, hence $U \otimes_R S = \operatorname{Coker}(\alpha \otimes_R S) = 0$.

Similarly, we see that

$$\operatorname{Hom}_S(\alpha \otimes_R S, M) : \operatorname{Hom}_S(P_0 \otimes_R S, M) \to \operatorname{Hom}_S(P_1 \otimes_R S, M)$$

is an isomorphism. In particular, choosing $M = P_1 \otimes_R S$, we obtain that $\operatorname{Hom}_S(\alpha \otimes_R S, P_1 \otimes_R S)$ is an isomorphism and hence $\alpha \otimes_R S$ is a split monomorphism. Thus $\alpha \otimes_R S$ is an isomorphism.

Now, by the definition of universal localization, there is a (unique) map ψ such that the following diagram commutes

$$
\begin{array}{ccc}
R & \xrightarrow{\lambda_{\mathcal{U}}} & R_{\mathcal{U}} \\
{\scriptstyle\lambda}\downarrow & \swarrow{\scriptstyle\psi} & \\
S & &
\end{array}
$$

Further, since $R_{\mathcal{U}} \in \mathcal{X}_{\mathcal{U}}$ by Lemma 1.6, and λ is an $\mathcal{X}_{\mathcal{U}}$-reflection by Theorem 1.4, there is a (unique) map φ such that the following diagram commutes

$$
\begin{array}{ccc}
R & \xrightarrow{\lambda_{\mathcal{U}}} & R_{\mathcal{U}} \\
{\scriptstyle\lambda}\downarrow & \nearrow{\scriptstyle\varphi} & \\
S & &
\end{array}
$$

Now $\psi\varphi\lambda = \psi\lambda_{\mathcal{U}} = \lambda$, hence $\psi\varphi = \operatorname{id}_R$. Moreover $\varphi\psi\lambda_{\mathcal{U}} = \varphi\lambda = \lambda_{\mathcal{U}}$ and this implies $\varphi\psi = \operatorname{id}_{R_{\mathcal{U}}}$. Hence we deduce that ψ and φ are isomorphisms, and the proof is complete. ∎

Remark 1.8. [7, 18] The map $\lambda : R \to R_{\mathcal{U}}$ can also be described as *ring of definable scalars* for $\tilde{\Sigma} = \{P_1 \oplus U \stackrel{(\alpha,0)}{\to} P_0 \mid \alpha \in \Sigma, \ U = \operatorname{Coker}\alpha\}$, or as *biendomorphism ring* of a module M which is constructed as follows: take

N as the direct product of a representative set of the indecomposable pure-injective modules in $\mathcal{X}_{\mathcal{U}}$, set $\kappa = \mathrm{card}\,N$, and $M = N^{\kappa}$. For details, see [18, 12.13, 12.16 and 11.7].

IV. Tilting modules. Finally, let us review the notion of a tilting module and its relationship with ring epimorphisms.

Definition 1.9. A module T is said to be a *tilting module (of projective dimension at most one)* if $\mathrm{Gen}\,T = T^{\perp}$, or equivalently, if the following conditions are satisfied:

(T1) proj.dim$(T) \leq 1$;

(T2) $\mathrm{Ext}^1_R(T, T^{(I)}) = 0$ for each set I; and

(T3) there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ where T_0, T_1 belong to Add$T$.

The class $T^{\perp}$ is then called a *tilting class*. We say that two tilting modules T and T' are *equivalent* if their tilting classes coincide.

Here is a typical pattern for constructing tilting modules.

Proposition 1.10. [2, 2.5] *Let* $\lambda : R \to S$ *be an injective ring epimorphism with* $\mathrm{Tor}^R_1(S, S) = 0$. *Then* $\mathrm{pd}\,S_R \leq 1$ *if and only if* $S \oplus S/R$ *is a tilting right R-module.*

The following statements, relying on Theorem 1.4 and results from [9], are shown in [2, proof of Theorem 2.10].

Lemma 1.11. [2] *Let T be a tilting module of projective dimension one, and let* $0 \to R \to T_0 \to T_1 \to 0$ *be an exact sequence with $T_0, T_1 \in \mathrm{Add}\,T$. Then*

(1) $\mathrm{Gen}\,T = \mathrm{Gen}\,T_0 = T_1^{\perp}$.

(2) $T_0 \oplus T_1$ *is a tilting module equivalent to T.*

(3) $\mathcal{X}_{T_1}$ *is a bireflective subcategory of $\mathrm{Mod}\,R$, so there is a ring epimorphism $\lambda : R \to S$ such that $\mathcal{X}_{T_1}$ coincides with the essential image of the restriction functor $\lambda_* : \mathrm{Mod}\,S \to \mathrm{Mod}\,R$ induced by λ.*

In fact, the observations above are used to prove the following result.

Theorem 1.12. [2, 2.10] *Let T_R be a tilting module of projective dimension one. The following assertions are equivalent:*

(1) *There is an injective ring epimorphism $\lambda : R \to S$ such that $\mathrm{Tor}^R_1(S, S) = 0$ and $S \oplus S/R$ is a tilting module equivalent to T_R.*

(2) *There is an exact sequence $0 \to R \xrightarrow{a} T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathrm{Add}\,T$ and $\mathrm{Hom}_R(T_1, T_0) = 0$.*

Moreover, under these conditions, $a : R \to T_0$ is a $T^{\perp}$-envelope of R, and $\mathcal{X}_{T_1}$ coincides with the essential image of the restriction functor $\lambda_* : \mathrm{Mod}\,S \to \mathrm{Mod}\,R$ induced by λ.

2. Tilting modules arising from universal localization

Aim of this section is to show that every tilting module of projective dimension one is associated in a natural way to a ring epimorphism which, moreover, can be interpreted as a universal localization at a set of finitely presented modules of projective dimension one.

The following result by Bazzoni and Herbera will play an important role.

Theorem 2.1. [5] *Let T be a tilting module of projective dimension one. Then there is a set $\mathcal{S}$ of modules in $\operatorname{mod} R$ of projective dimension one such that $T^\perp = \mathcal{S}^\perp$. More precisely, $\mathcal{S}$ can be chosen as a set of representatives of the isomorphism classes of non-projective modules from $^\perp(T^\perp) \cap \operatorname{mod} R$.*

Recall that, given a module M, an increasing chain of submodules $\mathcal{M} = (M_\alpha \mid \alpha \leq \sigma)$ of M, indexed by an ordinal σ, is called a *filtration* of M provided that $M_0 = 0$, $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $M_\sigma = M$. Moreover, if all consecutive factors $M_{\alpha+1}/M_\alpha$, $\alpha < \sigma$, belong to a given subcategory $\mathcal{C}$ of $\operatorname{Mod} R$, we say that M is $\mathcal{C}$–*filtered*.

Let us now fix a tilting module T of projective dimension one, and let $\mathcal{S}$ be a set of representatives of the isomorphism classes of the non-projective modules in $^\perp(T^\perp) \cap \operatorname{mod} R$. Then $T^\perp = \mathcal{S}^\perp$ by Theorem 2.1. Hence by [15, 3.2.1] there exists an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ where $T_0 \in T^\perp$ and T_1 is $\mathcal{S}$-filtered.

Theorem 2.2. *There exist an exact sequence*

$$0 \to R \to T_0 \to T_1 \to 0$$

and a set $\mathcal{U} \subseteq \operatorname{mod} R$ of modules of projective dimension one such that

   (1) *$T_0, T_1 \in \operatorname{Add} T$ and T_1 is $\mathcal{U}$-filtered.*
   (2) *$\operatorname{Gen} T = \mathcal{U}^\perp$.*
   (3) *$\mathcal{X}_{T_1} = \mathcal{X}_\mathcal{U}$ coincides with the essential image of the restriction functor $\operatorname{Mod} R_\mathcal{U} \to \operatorname{Mod} R$ induced by the universal localization at $\mathcal{U}$.*

Proof. (1) From the discussion above we know that there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ where $T_0 \in T^\perp$ and T_1 is $\mathcal{S}$-filtered.

The module T_1 then belongs to $^\perp(T^\perp)$ by [15, 3.1.2]. Since $T^\perp = \operatorname{Gen} T$ is closed under quotients, T_1 also belongs to $T^\perp$. So $T_1 \in T^\perp \cap {}^\perp(T^\perp) = \operatorname{Add} T$.

Moreover, $T_0 \in T^\perp \cap {}^\perp(T^\perp) = \operatorname{Add} T$, because R belongs to $^\perp(T^\perp)$ which is closed under extensions.

Take now an $\mathcal{S}$-filtration $(M_\alpha \mid \alpha \leq \sigma)$ of T_1 and set

$$\mathcal{U} = \{M_{\alpha+1}/M_\alpha \mid \alpha < \sigma\}.$$

Note that $\mathcal{U}$ consists of modules of projective dimension one by [15, 5.1.8], and T_1 is obviously $\mathcal{U}$-filtered.

(2) From $\mathcal{U} \subseteq {}^\perp(T^\perp)$ we infer $\operatorname{Gen} T = T^\perp \subseteq \mathcal{U}^\perp$.

For the reverse inclusion, recall from Lemma 1.11 that $\operatorname{Gen} T = T_1{}^\perp$. Since T_1 is $\mathcal{U}$-filtered and $\mathcal{U} \subseteq {}^\perp(\mathcal{U}^\perp)$, we deduce from [15, 3.1.2] that $T_1 \in {}^\perp(\mathcal{U}^\perp)$, hence $\mathcal{U}^\perp \subseteq T_1{}^\perp$.

(3) We start by showing $\mathcal{X}_{T_1} \subseteq \mathcal{X}_\mathcal{U}$. Let $X \in \mathcal{X}_{T_1}$. Then $X \in T_1{}^\perp = \operatorname{Gen} T = \mathcal{U}^\perp$. Assume $X \notin \mathcal{U}^0$. Then there exists $0 \neq f : U \to X$ for some

$U = M_{\alpha+1}/M_\alpha \in \mathcal{U}$. This implies that also $g : M_{\alpha+1} \twoheadrightarrow U \to X$ is different from zero.

Indeed, for all $\beta > \alpha$ there exists $0 \neq g_\beta : M_\beta \to X$, as we are going to show. For $\beta = \alpha + 1$, we take $g_\beta = g$. Given g_β, we consider

$$0 \longrightarrow M_\beta \longrightarrow M_{\beta+1} \longrightarrow M_{\beta+1}/M_\beta \longrightarrow 0$$

with maps g_β and $g_{\beta+1}$ to X

and we use that the map g_β extends to $g_{\beta+1}$ since $X \in \mathcal{U}^\perp$. Further, for a limit ordinal β, we have that the $g_\gamma : M_\gamma \to X$ with $\gamma < \beta$ form a direct system inducing a non-zero map $g_\beta : M_\beta = \cup_{\gamma<\beta} M_\gamma \to X$.

In particular, we obtain $\operatorname{Hom}_R(M_\sigma, X) \neq 0$. But $M_\sigma = T_1$, so $X \notin T_1^0$, a contradiction. Thus we conclude that $X \in \mathcal{U}^\perp \cap \mathcal{U}^0 = \mathcal{X}_{\mathcal{U}}$.

We now show $\mathcal{X}_{\mathcal{U}} \subseteq \mathcal{X}_{T_1}$. Let $X \in \mathcal{X}_{\mathcal{U}} = \mathcal{U}^\perp \cap \mathcal{U}^0$. We already know that X then belongs to $\operatorname{Gen} T = T_1{}^\perp$, so it remains to verify that $X \in T_1^0$. Since $T_1 = M_\sigma$, this will follow once we show that $\operatorname{Hom}_R(M_\beta, X) = 0$ for all $\beta \leq \sigma$.

The claim is clear for $\beta = 0$ since $M_0 = 0$, and for $\beta = 1$ since $M_1 \in \mathcal{U}$. If $\beta = \alpha + 1$ for some α, then we have an exact sequence $0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$ where $M_{\alpha+1}/M_\alpha \in \mathcal{U} \subseteq {}^0X$, and M_α belongs to ${}^0X$ by inductive assumption. Since the class ${}^0X$ is closed under extensions, we infer that also $M_{\alpha+1}$ belongs to ${}^0X$.

Finally, if β is a limit ordinal, then $M_\beta = \varinjlim_{\alpha<\beta} M_\alpha$, and again by inductive assumption $\operatorname{Hom}_R(M_\beta, X) \cong \varprojlim \operatorname{Hom}_R(M_\alpha, X) = 0$.

So the first claim is verified, and Proposition 1.7 completes the proof. $\blacksquare$

Definition 2.3. We will say that a tilting module T *arises from universal localization* if there is a set $\mathcal{U} \subset \operatorname{mod} R$ of modules of projective dimension at most one such that R embeds in $R_{\mathcal{U}}$ and $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting R-module equivalent to T.

Corollary 2.4. *Let $0 \to R \to T_0 \to T_1 \to 0$ and $\mathcal{U}$ be as in Theorem 2.2. If $\lambda_{\mathcal{U}}$ is a $\mathcal{U}^\perp$-preenvelope of R, then T arises from universal localization.*

Proof. Since $\mathcal{U}^\perp$ contains all injective modules, $\lambda_{\mathcal{U}}$ is injective. Moreover, if $\lambda_{\mathcal{U}}$ is a $\mathcal{U}^\perp$-preenvelope, then $\operatorname{Gen} R_{\mathcal{U}} = \mathcal{U}^\perp = \operatorname{Gen} T$, cf. [2, 3.12].

We claim that $\operatorname{pd} R_{\mathcal{U}} \leq 1$. By assumption, there are R-epimorphisms

$$f : T^{(I)} \twoheadrightarrow R_{\mathcal{U}} \qquad \text{and} \qquad g : R_{\mathcal{U}}{}^{(J)} \twoheadrightarrow T^{(I)}.$$

The composition $fg : R_{\mathcal{U}}{}^{(J)} \twoheadrightarrow R_{\mathcal{U}}$ is an R-epimorphism, and also an $R_{\mathcal{U}}$-epimorphism since $\operatorname{Mod} R_{\mathcal{U}}$ is a full subcategory of $\operatorname{Mod} R$. Therefore fg is a split $R_{\mathcal{U}}$-epimorphism, so there is $h \in \operatorname{Hom}_{R_{\mathcal{U}}}(R_{\mathcal{U}}, R_{\mathcal{U}}{}^{(J)}) = \operatorname{Hom}_R(R_{\mathcal{U}}, R_{\mathcal{U}}{}^{(J)})$ such that $fgh = id_{R_{\mathcal{U}}}$. Thus f is a split R-epimorphism, showing that $\operatorname{pd} R_{\mathcal{U}} \leq \operatorname{pd} T^{(I)} \leq 1$.

Since $\operatorname{Tor}_1^R(R_{\mathcal{U}}, R_{\mathcal{U}}) = 0$, we conclude from Proposition 1.10 that $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting module equivalent to T. $\blacksquare$

Corollary 2.5. *Assume that there is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ such that $T_0, T_1 \in \operatorname{Add} T$, T_1 is $\mathcal{S}$-filtered, and $\operatorname{Hom}_R(T_1, T_0) = 0$. Then T arises from universal localization.*

Proof. By Theorem 1.12 there is an injective ring epimorphism $\lambda : R \to S$ such that the R-module $S \oplus S/R$ is a tilting module equivalent to T, and moreover, $\mathcal{X}_{T_1}$ coincides with the essential image of the restriction functor $\lambda_* : \operatorname{Mod} S \to \operatorname{Mod} R$ induced by λ. On the other hand, we have seen in Theorem 2.2 that $\mathcal{X}_{T_1}$ coincides with the essential image of the restriction functor $\operatorname{Mod} R_{\mathcal{U}} \to \operatorname{Mod} R$ induced by the universal localization at a set $\mathcal{U}$ of finitely presented modules of projective dimension one. Then it follows from Theorem 1.4 that $\lambda : R \to S$ and $\lambda_{\mathcal{U}} : R \to R_{\mathcal{U}}$ are in the same epiclass. So, $\lambda_{\mathcal{U}}$ is injective and $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting module equivalent to T. ■

Example 2.6. Assume that $T \in \operatorname{mod} R$ and that the category $T^{\perp} \cap \operatorname{mod} R$ is covariantly finite in $\operatorname{mod} R$, that is, every module in $\operatorname{mod} R$ has a $T^{\perp} \cap \operatorname{mod} R$-preenvelope. Assume further that there is an exact sequence $0 \to R \xrightarrow{a} T_0 \to T_1 \to 0$ such that $T_0, T_1 \in \operatorname{Add} T$, and $\operatorname{Hom}_R(T_1, T_0) = 0$. Then T arises from universal localization.
In fact, this will follow immediately from Corollary 2.5 once we prove that T_1 belongs to $\operatorname{mod} R$ (and is therefore trivially $\mathcal{S}$-filtered).
Let us start by considering a $T^{\perp} \cap \operatorname{mod} R$-preenvelope $f : R \to B$. We claim that f is even a $T^{\perp}$-preenvelope. Indeed, if $h : R \to X$ with $X \in T^{\perp} = \operatorname{Gen} T$, then there exists an epimorphism $g : T^{(\alpha)} \twoheadrightarrow X$, and h factors through g via a homomorphism $h' : R \to T^{(\alpha)}$. Since the image of h' is contained in a finite subsum $T^{(\alpha_0)}$ of $T^{(\alpha)}$, we can even factor $h = g' h''$ where $h'' : R \to T^{(\alpha_0)}$ and $g : T^{(\alpha_0)} \twoheadrightarrow X$. Now $T^{(\alpha_0)} \in T^{\perp} \cap \operatorname{mod} R$, so there is a map $\tilde{h} : B \to T^{(\alpha_0)}$ such that $h'' = \tilde{h}f$, hence $h = g'\tilde{h}f$. This proves our claim.
On the other hand, we know from Theorem 1.12 that a is a $T^{\perp}$-envelope. Thus T_0 is isomorphic to a direct summand of B, and since $B \in \operatorname{mod} R$, we infer that T_0, T_1 belong to $\operatorname{mod} R$.

In particular, we deduce the following result from [3].

Corollary 2.7. *Let R be an Artin algebra, and let T be a finitely generated tilting right R-module of projective dimension one. The following assertions are equivalent:*

(1) *There is a set of finitely generated modules $\mathcal{U}$ of projective dimension one such that $R_{\mathcal{U}} \oplus R_{\mathcal{U}}/R$ is a tilting module equivalent to T_R.*

(2) *There is an exact sequence $0 \to R \xrightarrow{a} T_0 \to T_1 \to 0$ with $T_0, T_1 \in \operatorname{Add} T$ and $\operatorname{Hom}_R(T_1, T_0) = 0$.*

In the last section, we will see that a similar result holds true over Prüfer domains.

3. Tilting modules arising from perfect localization

In this section we investigate tilting modules arising from perfect localization. We start by recalling some basic notions and results. For details we refer to [12, 19, 23].

Definition 3.1. (1) A (full) subcategory $\mathcal{X}$ of $\operatorname{Mod} R$ is called a *Giraud subcategory* if the canonical inclusion $\iota : \mathcal{X} \to \operatorname{Mod} R$ has a left adjoint $\ell : \operatorname{Mod} R \to \mathcal{X}$ which is an exact functor. The composition functor $L = \iota \circ \ell$ is then called *localization functor*.

(2) A non-empty set of right R-ideals $\mathcal{G}$ is said to be a *Gabriel topology* on R if satisfies the following conditions:

(a) If $I \in \mathcal{G}$ and a right ideal K contains I, then K belongs to $\mathcal{G}$.
(b) If I and K belong to $\mathcal{G}$ then also $I \cap K$ belongs to $\mathcal{G}$.
(c) If $I \in \mathcal{G}$ and $x \in R$ then $(I : x) = \{r \in R | \ xr \in I\}$ belongs to $\mathcal{G}$.
(d) If K is a right ideal and if there is some $I \in \mathcal{G}$ such that $(K : x) \in \mathcal{G}$ for any $x \in I$, then also K belongs to $\mathcal{G}$.

Further, a Gabriel topology $\mathcal{G}$ is *of finite type* if it has a *basis* of finitely generated ideals, that is, every $I \in \mathcal{G}$ contains a finitely generated right ideal $I' \in \mathcal{G}$.

(3) Let $\mathcal{G}$ be a Gabriel topology on R. A right R-module C is said to be $\mathcal{G}$-*closed* if for any short exact sequence $0 \to I \to R \to R/I \to 0$ with $I \in \mathcal{G}$ the morphism of abelian groups $\operatorname{Hom}_R(R, C) \to \operatorname{Hom}_R(I, C)$ is bijective. A left R-module $_RX$ is said to be $\mathcal{G}$-*divisible* if $IX = X$ for all $I \in \mathcal{G}$.

(4) A pair of subcategories $(\mathcal{T}, \mathcal{F})$ is said to be a *torsion pair* if $\mathcal{T} = {}^o\mathcal{F}$ and $\mathcal{T}^o = \mathcal{F}$. In this case, $\mathcal{T}$ is a *torsion class*, that is, it is closed under epimorphic images, extensions, and direct sums. If, in addition, $\mathcal{T}$ is closed under submodules, then $(\mathcal{T}, \mathcal{F})$ is called a *hereditary* torsion pair.

Theorem 3.2. [23, VI, 5.1 and X, 2.1] *There are bijective correspondences between the hereditary torsion pairs in* $\operatorname{Mod} R$*, the Gabriel topologies on* R*, and the Giraud subcategories of* $\operatorname{Mod} R$*.*

More precisely, under these bijections, a hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ *in* $\operatorname{Mod} R$ *corresponds to the Gabriel topology*

$$\mathcal{G} = \{I \leq R \mid R/I \in \mathcal{T}\}$$

as well as to the Giraud subcategory $\mathcal{X}_\mathcal{T}$*. Conversely, a Giraud subcategory* $\mathcal{X}$ *with localization functor* L *is associated to the hereditary torsion pair with torsion class*

$$\mathcal{T} := \{M \in \operatorname{Mod} R | \ L(M) = 0\}$$

Finally, if $\mathcal{G}$ *is a Gabriel topology, then the category* $\mathcal{X}(\mathcal{G})$ *of all* $\mathcal{G}$*-closed modules is the corresponding Giraud subcategory.*

Let now $\mathcal{G}$ be a Gabriel topology. Consider the adjoint pair (ℓ, ι) corresponding to the Giraud subcategory $\mathcal{X}(\mathcal{G})$ of all $\mathcal{G}$-closed modules, and the localization functor $L = \iota \circ \ell$. Recall from Remark 1.2 that the unit of the adjunction $\eta_M : M \to L(M)$ defines an $\mathcal{X}(\mathcal{G})$-reflection. In particular, $\eta_R : R \to L(R)$ induces a ring structure on $Q_\mathcal{G} = L(R)$, and we obtain a ring homomorphism $\lambda_\mathcal{G} : R \to Q_\mathcal{G}$.

Theorem 3.3 ([23, XI, 3.4]). *Let* $\mathcal{G}$ *be a Gabriel topology on* R*, and let* $\mathcal{X}(\mathcal{G})$ *be the corresponding Giraud subcategory of all* $\mathcal{G}$*-closed modules. The following assertions are equivalent.*

(1) $\mathcal{X}(\mathcal{G})$ *is a coreflective subcategory of* $\operatorname{Mod} R$*.*

(2) $\mathcal{X}(\mathcal{G})$ *coincides with the essential image of the restriction functor* $\operatorname{Mod} Q_{\mathcal{G}} \to \operatorname{Mod} R$ *induced by* $\lambda_{\mathcal{G}}$.

(3) *The left R-module $Q_{\mathcal{G}}$ is $\mathcal{G}$-divisible.*

Definition 3.4. A Gabriel topology $\mathcal{G}$ that satisfies the equivalent conditions of Theorem 3.3 is called a *perfect Gabriel topology*.

Remark 3.5. Let $\mathcal{G}$ be a Gabriel topology on R, and let $(\mathcal{T}, \mathcal{F})$ be the corresponding hereditary torsion pair. The torsion class $\mathcal{T}$ consists of all modules X such that every $x \in X$ has annihilator $\operatorname{ann}_R(x) \in \mathcal{G}$, and the torsion-free class $\mathcal{F}$ is given by the modules M for which the $\mathcal{X}(G)$-reflection $\eta_M : M \to L(M)$ is injective [23, VI, 5.1, and X, 1.5].

Assume now that $\mathcal{G}$ is perfect. Then $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$ is a ring epimorphism, $Q_{\mathcal{G}}$ is a flat left R-module, and $\mathcal{G}$ is a Gabriel topology of finite type [23, XI, 3.4]. Moreover, $\mathcal{F} = \mathcal{C}^o$ where $\mathcal{C}$ is the class of all finitely presented cyclic modules in $\mathcal{T}$. This follows easily from [23, VI, 3.6] by using that $\mathcal{G}$ has finite type.

We now fix a tilting module T of projective dimension one together with an exact sequence

$$0 \to R \to T_0 \to T_1 \to 0$$

where $T_i \in \operatorname{Add} T$. We know from Lemma 1.11 that $\mathcal{X}_{T_1}$ is a bireflective subcategory of $\operatorname{Mod} R$. We denote by ℓ the left adjoint of the inclusion functor $\iota : \mathcal{X}_{T_1} \hookrightarrow \operatorname{Mod} R$. In [9], the functor ℓ is constructed explicitly by using Bongartz preenvelopes. More precisely, if M_R is a right R-module, and c is the minimal number of generators of $\operatorname{Ext}_R^1(T_1, M)$ as a module over $\operatorname{End}_R(T_1)$, then there exists an exact sequence

$$0 \to M \xrightarrow{i} M_0 \to T_1^{(c)} \to 0$$

with $M_0 \in T_1^{\perp}$. In particular, i is a special $\operatorname{Gen} T$-preenvelope of M, called the *Bongartz preenvelope* of M. It is shown in [9, 1.3] that $\ell(M)$ can be computed as

$$\ell(M) := M_0 / \operatorname{tr}_{T_1} M_0$$

where $\operatorname{tr}_{T_1} M_0 = \Sigma \{\operatorname{Im} f | \ f \in \operatorname{Hom}_R(T_1, M_0)\}$ denotes the T_1-trace of M_0. We use this description in order to determine the kernel of the functor ℓ.

Lemma 3.6. *For each $M \in \operatorname{Mod} R$ fix a Bongartz preenvelope M_0. Then the following statements are equivalent.*

(1) $M \in {}^0\mathcal{X}_{T_1}$

(2) $\ell(M) = 0$

(3) $M_0 \in \operatorname{Gen} T_1$

Moreover, these conditions are satisfied whenever $M \in \operatorname{Gen} T_1$.

Proof. (1)$\Rightarrow$ (2): Recall that $\eta_M : M \to \iota\ell(M)$ is the $\mathcal{X}_{T_1}$-reflection of M, which is uniquely determined up to isomorphism. So, if $M \in {}^0\mathcal{X}_{T_1}$, we must have $\ell(M) = 0$.

(2)$\Rightarrow$ (3) is clear.

(3)$\Rightarrow$ (1): Assume that $M_0 \in \operatorname{Gen} T_1$ and $M \notin {}^0\mathcal{X}_{T_1}$. Then there exists a map $0 \neq f \in \operatorname{Hom}_R(M, X)$ for some $X \in \mathcal{X}_{T_1} \subseteq T_1^\perp$. Then

$$0 \longrightarrow M \stackrel{i}{\longrightarrow} M_0 \longrightarrow T_1^{(c)} \longrightarrow 0$$

with vertical map $f : M \to X$ and diagonal map h.

there is a map $0 \neq h \in \operatorname{Hom}_R(M_0, X)$. Since $M_0 \in \operatorname{Gen} T_1$ there exists a map $0 \neq h' \in \operatorname{Hom}_R(T_1, X)$. But $X \in \mathcal{X}_{T_1} \subseteq T_1^0$, a contradiction.

Finally, if $M \in \operatorname{Gen} T_1 \subseteq \operatorname{Gen} T = T_1^\perp$, then $\operatorname{Ext}_R^1(T_1^{(c)}, M) = 0$, so $M_0 \cong M \oplus T_1^{(c)} \in \operatorname{Gen} T_1$. $\blacksquare$

Proposition 3.7. *Let $\lambda : R \to S$ be a ring epimorphism such that the essential image of the restriction functor $\lambda_* : \operatorname{Mod} S \to \operatorname{Mod} R$ coincides with $\mathcal{X}_{T_1}$. Then the following assertions are equivalent:*

   (1) ${}_R S$ *is a flat left R-module.*
   (2) $\mathcal{X}_{T_1}$ *is a Giraud subcategory of $\operatorname{Mod} R$.*
   (3) *All submodules of modules in $\operatorname{Gen} T_1$ belong to ${}^0\mathcal{X}_{T_1}$.*
   (4) $({}^0\mathcal{X}_{T_1}, ({}^0\mathcal{X}_{T_1})^0)$ *is a hereditary torsion pair.*
   (5) *There is a perfect Gabriel topology $\mathcal{G}$ such that $\lambda : R \to S$ is equivalent to $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$.*

Proof. The equivalence of $(1) - (3)$ is proved in [9, 2.1].
$(2) \Rightarrow (5)$: By Theorem 3.2 we have that the Giraud subcategory $\mathcal{X}_{T_1}$ is the category $\mathcal{X}(\mathcal{G})$ of $\mathcal{G}$-closed modules for some Gabriel topology $\mathcal{G}$. Since $\mathcal{X}_{T_1}$ is a coreflective subcategory of $\operatorname{Mod} R$ by Lemma 1.11, we infer from Theorem 3.3 that $\mathcal{G}$ is a perfect Gabriel topology. Then λ and $\lambda_{\mathcal{G}}$ are in the same epiclass by Theorem 1.4.
$(5) \Rightarrow (4)$: Since λ and $\lambda_{\mathcal{G}}$ are in the same epiclass, the perpendicular category $\mathcal{X}_{T_1}$ and the category $\mathcal{X}(\mathcal{G})$ of all $\mathcal{G}$-closed modules coincide. In particular, $\mathcal{X}_{T_1}$ is a Giraud subcategory, and combining Theorem 3.2 and Lemma 3.6 we know that the corresponding hereditary torsion pair is $({}^0\mathcal{X}_{T_1}, ({}^0\mathcal{X}_{T_1})^0)$.
$(4) \Rightarrow (3)$: Since $({}^0\mathcal{X}_{T_1}, ({}^0\mathcal{X}_{T_1})^0)$ is a hereditary torsion pair, ${}^0\mathcal{X}_{T_1}$ is closed under submodules. Thus (3) is a consequence of Lemma 3.6. $\blacksquare$

Definition 3.8. We will say that a tilting module *arises from perfect localization* if there is a perfect Gabriel topology $\mathcal{G}$ such that R embeds in $Q_{\mathcal{G}}$ and $Q_{\mathcal{G}} \oplus Q_{\mathcal{G}}/R$ is a tilting module equivalent to T.

Theorem 3.9. *Let T_R be a tilting module of projective dimension one. The following conditions are equivalent.*

   (1) *There is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ such that $T_i \in \operatorname{Add} T$, $\operatorname{Hom}_R(T_1, T_0) = 0$ and $\mathcal{X}_{T_1}$ Giraud subcategory of $\operatorname{Mod} R$.*
   (2) *T arises from perfect localization.*

Proof. $(1) \Rightarrow (2)$: By Theorem 1.12 there is an injective ring epimorphism $\lambda : R \to S$ such that $S \oplus S/R$ is a tilting module equivalent to T, and $\mathcal{X}_{T_1}$ coincides with the essential image of the restriction functor

$\lambda_* : \mathrm{Mod}\, S \to \mathrm{Mod}\, R$ induced by λ. Now, since $\mathcal{X}_{T_1}$ is a Giraud sub-category, we infer from Proposition 3.7 that there exists a perfect Gabriel topology $\mathcal{G}$ such that $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$ and λ are in the same epiclass. So, $\lambda_{\mathcal{G}}$ is injective, and $Q_{\mathcal{G}} \oplus Q_{\mathcal{G}}/R$ is a tilting module equivalent to T.

$(2) \Rightarrow (1)$: Let $\mathcal{G}$ be a perfect Gabriel topology such that R embeds in $Q_{\mathcal{G}}$ and $Q_{\mathcal{G}} \oplus Q_{\mathcal{G}}/R$ is a tilting module equivalent to T. Then the sequence $0 \to R \to Q_{\mathcal{G}} \to Q_{\mathcal{G}}/R \to 0$ has the stated properties. In fact, if $T_1 = Q_{\mathcal{G}}/R$, then we know from Theorem 1.12 that $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$ induces an equivalence between $\mathcal{X}_{T_1}$ and $\mathrm{Mod}\, Q_{\mathcal{G}}$. Then $\mathcal{X}_{T_1}$ coincides with $\mathcal{X}(\mathcal{G})$ by Theorem 3.3, and it is therefore a Giraud subcategory. ■

Example 3.10. Exact sequences $0 \to R \to T_0 \to T_1 \to 0$ such that $T_i \in \mathrm{Add}\, T$ and $\mathcal{X}_{T_1}$ is a Giraud subcategory of $\mathrm{Mod}\, R$ may exist even when T is a tilting module which is not of the form $S \oplus S/R$.

Let R be a commutative domain, and Q its quotient field. Denote by $\mathcal{D}$ the class of all divisible modules. It was shown by Facchini that there is a tilting module of projective dimension one generating $\mathcal{D}$, namely the Fuchs' divisible module δ, cf. [11, §VII.1]. Recall further that $\mathcal{D} = \mathcal{U}^\perp$ where $\mathcal{U} = \{R/rR \mid r \in R\}$ denotes a set of representatives of all cyclically presented modules. Moreover, the module $T_1 = \delta/R$ in the exact sequence $0 \to R \to \delta \to \delta/R \to 0$ is $\mathcal{U}$-filtered, and the perpendicular category $\mathcal{X}_{T_1} = \mathcal{X}_{\mathcal{U}}$ is the class of all divisible torsion-free modules.

Note that the universal localization of R at $\mathcal{U}$ is exactly Q, see [2, 3.7]. So the $\mathcal{X}_{T_1}$-reflection of R is given by the injective flat epimorphism $\lambda : R \to Q$, and $\mathcal{X}_{T_1}$ is a Giraud subcategory of Mod-R. On the other hand, δ has not the form described in Theorem 1.12, unless $\mathrm{pd}\, Q_R \leq 1$, that is, R is a Matlis domain, see [2, 2.11 (4)].

4. Tilting modules over semihereditary rings

As we have seen in Remark 3.5, the hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ corresponding to a perfect Gabriel topology $\mathcal{G}$ is always generated by some set of finitely presented modules $\mathcal{C}$. If the ring R is right coherent, we have a further useful information.

Proposition 4.1. [16, 2.8],[18] *Let R be a right coherent ring, and let $\mathcal{G}$ be a perfect Gabriel topology on R with associated hereditary torsion pair $(\mathcal{T}, \mathcal{F})$. Let $\mathcal{S}$ be the class of all finitely presented modules from $\mathcal{T}$. Then $\mathcal{F} = \mathcal{S}^o$ and $\mathcal{T} = \varinjlim \mathcal{S}$.*

We will use this result for comparing perfect localization with universal localization.

Lemma 4.2. *Let R be right coherent, and let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion pair. Let $\mathcal{S}$ be the class of all finitely presented modules from $\mathcal{T}$, and assume that $\mathcal{T} = \varinjlim \mathcal{S}$. Denote further by $\mathcal{C}$ the class of all cyclic modules in $\mathcal{S}$. If $\mathcal{U} \subseteq \mathcal{S}$ satisfies $\mathcal{X}_{\mathcal{U}} \subseteq \mathcal{F} \cap \mathcal{C}^\perp$, then $\mathcal{X}_{\mathcal{U}} = \mathcal{X}_{\mathcal{T}}$.*

Proof. The inclusion " $\supseteq$ " follows immediately from the fact that $\mathcal{U} \subseteq \mathcal{T}$. For the reverse inclusion, let $M \in \mathcal{X}_{\mathcal{U}}$. Then $M \in \mathcal{F}$, so we know from

[16, p. 518-519] that there is an exact sequence

$$0 \to M \overset{f_M}{\to} E_M \overset{g_M}{\to} C_M \to 0$$

where $E_M \in \mathcal{X}_{\mathcal{T}}$ and $C_M \in \mathcal{T}$. Then $C_M = \varinjlim S_i$ for some direct system (S_i) in $\mathcal{S}$. We claim that all $\mathrm{Hom}_R(S_i, C_M) = 0$. In fact, if Y is a cyclic submodule of S_i, then also Y belongs to $\mathcal{S}$, hence to $\mathcal{C}$, and therefore $\mathrm{Ext}^1_R(S_i, M) = 0$. This shows that every map $h \in \mathrm{Hom}_R(S_i, C_M)$ factors through g_M, thus $h = g_M h'$ with $h' \in \mathrm{Hom}_R(S_i, E_M)$, and since $E_M \in \mathcal{X}_{\mathcal{T}}$, we deduce $h' = h = 0$.

So we conclude that $C_M = 0$ and $M \cong E_M \in \mathcal{X}_{\mathcal{T}}$. ∎

Recall that a ring R is said to be right *semihereditary* if every finitely generated right ideal is projective. Then, by a classical result of Kaplansky, all finitely generated submodules of a right projective module are projective, hence all finitely presented modules have projective dimension at most one.

Proposition 4.3. *Let R be a right semihereditary ring. Let $\mathcal{G}$ be a perfect Gabriel topology on R, and let $(\mathcal{T}, \mathcal{F})$ be the hereditary torsion pair associated to $\mathcal{G}$. Then the ring epimorphism $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$ is equivalent to the universal localization at the set $\mathcal{U}$ of all finitely presented modules from $\mathcal{T}$.*

Proof. By Proposition 4.1 we have $\mathcal{F} = \mathcal{U}^o$, and $\mathcal{T} = \varinjlim \mathcal{U}$. Denote by $\mathcal{C}$ the class of all cyclic modules in $\mathcal{U}$. Of course $\mathcal{X}_{\mathcal{U}} \subseteq \mathcal{F} \cap \mathcal{C}^\perp$, hence $\mathcal{X}_{\mathcal{U}} = \mathcal{X}_{\mathcal{T}}$ by Lemma 4.2.

Note that the Giraud subcategory $\mathcal{X}_{\mathcal{T}}$ coincides with the category of $\mathcal{G}$-closed modules, see Theorem 3.2. Thus we infer from Theorem 3.3 that $\mathcal{X}_{\mathcal{T}}$ is the essential image of the restriction functor $\mathrm{Mod}\, Q_{\mathcal{G}} \to \mathrm{Mod}\, R$ induced by $\lambda_{\mathcal{G}}$. So, it follows from Proposition 1.7 and Theorem 1.4 that $\lambda_{\mathcal{G}}$ and $\lambda_{\mathcal{U}}$ are in the same epiclass. ∎

Corollary 4.4. *Over a semihereditary ring, every tilting module arising from perfect localization also arises from universal localization.*

Example 4.5. The converse implication in 4.3 or 4.4 does not hold true. Indeed, [9, 2.2] provides an example of a finitely generated tilting module T over a finite dimensional hereditary algebra R admitting an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ such that $T_i \in \mathrm{Add}\, T$ and $\mathrm{Hom}_R(T_1, T_0) = 0$, but $\mathcal{X}_{T_1}$ is not a Giraud subcategory of $\mathrm{Mod}\, R$, see also [2, 2.11]. Note that T arises from universal localization at T_1, cf. Example 2.6.

Let us now focus on the case where R is a Prüfer domain, that is, a commutative semihereditary domain. First of all, we recall the classification of tilting modules due to Bazzoni, Eklof and Trlifaj [4].

Theorem 4.6. [15, 6.2.15] *Let R be a Prüfer domain. There is a bijective correspondence between Gabriel topologies of finite type and tilting classes.*

The correspondence associates to a Gabriel topology of finite type $\mathcal{L}$ the tilting class of all $\mathcal{L}$-divisible modules. Conversely, if T is a tilting module, then the non-zero finitely generated ideals I such that $R/I \in {}^\perp(T^\perp)$ form a basis of the corresponding Gabriel topology.

Over a Prüfer domain, every Gabriel topology of finite type is perfect.

Lemma 4.7. *Let R be a Prüfer domain. Let further $\mathcal{L}$ be a Gabriel topology of finite type, and let $(\mathcal{T}, \mathcal{F})$ be the corresponding hereditary torsion pair. The following statements hold true.*

(1) $\mathcal{L}$ is a perfect Gabriel topology, $\lambda_{\mathcal{L}} : R \to Q_{\mathcal{L}}$ is an injective ring epimorphism, and $Q_{\mathcal{L}}/R \in \mathcal{T}$.

(2) If $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$ is a tilting module, then the tilting class $\operatorname{Gen} Q_{\mathcal{L}}$ coincides with the class of $\mathcal{L}$-divisible modules.

Proof.   (1) Either use [23, XI, 3.5], or proceed as follows. By Remark 3.5 the torsion class $\mathcal{T}$ consists of all modules X such that every $x \in X$ has annihilator $\operatorname{ann}_R(x) \in \mathcal{L}$. In particular $\operatorname{ann}_R(x) \neq 0$, hence $\mathcal{T}$ is contained in the class of all torsion modules. Thus $R \in (\mathcal{T})^0 = \mathcal{F}$, which shows that the $\mathcal{X}(\mathcal{L})$-reflection $\lambda_{\mathcal{L}} : R \to Q_{\mathcal{L}}$ is injective.

Moreover, from the construction of the $\mathcal{X}(\mathcal{L})$-reflection in [23, IX, 2.2] or [16, p. 518-519] we know that $Q_{\mathcal{L}} = \{x \in E(R)| \ (R : x) \in \mathcal{L}\}$ where $E(R)$ denotes the injective envelope of R, and $Q_{\mathcal{L}}/R \in \mathcal{T}$. In particular, $Q_{\mathcal{L}}$ is an overring of R, hence $Q_{\mathcal{L}}$ is flat and $\mathcal{L}$-divisible by [10, 1.1.1, 5.1.15, 5.1.11]. So $\mathcal{L}$ is a perfect Gabriel topology.

(2) If $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$ is a tilting module, then $Q_{\mathcal{L}}/R$ has projective dimension at most one, and therefore it has a filtration where the consecutive factors are finitely presented cyclic, see [11, VI, 6.5]. Denoting by $\mathcal{C}$ the class of all cyclic finitely presented modules from $\mathcal{T}$, we infer that $Q_{\mathcal{L}}/R$ is $\mathcal{C}$-filtered. Moreover, $Q_{\mathcal{L}}$ is $\mathcal{L}$-closed and therefore obviously contained in $\mathcal{C}^\perp$. Then we deduce as in [2, 3.12] that $\operatorname{Gen} Q_{\mathcal{L}} = \mathcal{C}^\perp$.

So, it remains to verify that $\mathcal{C}^\perp$ is the class of $\mathcal{L}$-divisibles. Now, let M be an $\mathcal{L}$-divisible module, and let $C = R/I \in \mathcal{C}$. Then $I \in \mathcal{L}$ is finitely generated, and M is I-divisible, so we infer from [15, 6.2.7] that $\operatorname{Ext}^1_R(R/I, X) = 0$. Conversely, if $M \in \mathcal{C}^\perp$ and $J \in \mathcal{L}$, then J contains a finitely generated ideal $I \in \mathcal{L}$. Since $R/I \in \mathcal{C}$, we infer again from [15, 6.2.7] that M is I-divisible, which implies that M is also J-divisible. ∎

Next, we prove a converse of Proposition 4.3.

Proposition 4.8. *Let R be a Prüfer domain, and let $\mathcal{U}$ be a set of finitely presented cyclic modules. Let further $\mathcal{L}$ be the Gabriel topology having as basis the set $\mathcal{B}$ of all non-zero finitely generated ideals I such that $R/I \in \mathcal{U}$. Then the universal localization at $\mathcal{U}$ is equivalent to $\lambda_{\mathcal{L}} : R \to Q_{\mathcal{L}}$.*

Proof.   The Gabriel topology $\mathcal{L}$ is obviously of finite type, hence perfect by Lemma 4.7. Let $(\mathcal{T}, \mathcal{F})$ be the hereditary torsion pair associated to $\mathcal{L}$, and let $\mathcal{S}$ be the class of all finitely presented modules from $\mathcal{T}$. By Proposition 4.1 we have $\mathcal{F} = \mathcal{S}^o$, and $\mathcal{T} = \varinjlim \mathcal{S}$.

We verify that $\mathcal{U}$ satisfies the assumptions of Lemma 4.2.

(i) $\mathcal{U} \subseteq \mathcal{S}$. In fact, $\mathcal{U}$ consists of modules of the form R/I where the ideal I belongs to the basis $\mathcal{B}$, so the annihilator of any element of R/I belongs to $\mathcal{L}$ since it contains I. From Remark 3.5 we infer $\mathcal{U} \subseteq \mathcal{T}$, hence $\mathcal{U} \subseteq \mathcal{S}$.

(ii) If $J \in \mathcal{L}$ and $X \in \mathcal{X}_{\mathcal{U}}$, then $\operatorname{Hom}_R(R/J, X) = 0$. This is because J contains an ideal I such that $R/I \in \mathcal{U}$ and therefore $\operatorname{Hom}_R(R/I, X) = 0$.

(iii) $\mathcal{X}_{\mathcal{U}} \subseteq \mathcal{F}$. Indeed, if $X \in \mathcal{X}_{\mathcal{U}}$ and Y is a cyclic module in $\mathcal{T}$, then $Y = R/J$ with $J \in \mathcal{L}$, hence $\operatorname{Hom}_R(Y, X) = 0$ by (ii). But this implies $\operatorname{Hom}_R(M, X) = 0$ for all $M \in \mathcal{T}$, that is, $X \in \mathcal{F}$.

(iv) Let $\mathcal{C}$ be the class of all cyclic modules in $\mathcal{S}$, and let $X \in \mathcal{X}_{\mathcal{U}}$. We verify that $X \in \mathcal{C}^{\perp}$. If $C \in \mathcal{C}$, then $C = R/J$ for some finitely generated ideal $J \in \mathcal{L}$, and J must contain an element from the basis $\mathcal{B}$, that is, a non-zero finitely generated ideal I such that $R/I \in \mathcal{U}$. Then $\operatorname{Ext}_R^1(R/I, X) = 0$, which means by [15, 6.2.7] that X is I-divisible. But then X is also J-divisible, and again by [15, 6.2.7] we infer $\operatorname{Ext}_R^1(R/J, X) = 0$.

Now Lemma 4.2 yields that $\mathcal{X}_{\mathcal{U}} = \mathcal{X}_{\mathcal{T}}$, and we complete the proof as in Proposition 4.3. ∎

Corollary 4.9. *Let R be a Prüfer domain. Let T be a tilting module of projective dimension one, and let $0 \to R \to T_0 \to T_1 \to 0$ be an exact sequence where $T_0, T_1 \in \operatorname{Add} T$. Then $\mathcal{X}_{T_1}$ is a Giraud subcategory of $\operatorname{Mod} R$.*

Proof. By [15, 6.2.10] there is a class $\mathcal{U}$ of finitely presented cyclic modules in $^{\perp}(T^{\perp})$ such that T_1 is $\mathcal{U}$-filtered. By Theorem 2.2 it follows that $\mathcal{X}_{T_1}$ is the essential image of the restriction functor induced by the universal localization $\lambda_{\mathcal{U}}$. But $\lambda_{\mathcal{U}}$ is equivalent to a perfect localization by Proposition 4.8. So, Theorem 3.3 yields that $\mathcal{X}_{T_1}$ is a Giraud subcategory. ∎

If $\mathcal{L}$ is a Gabriel topology of finite type such that the localization $Q_{\mathcal{L}}$ has projective dimension at most one over R, then it was shown by Salce [21] that the corresponding tilting module T is equivalent to $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$. We recover Salce's result as a consequence of Theorem 3.9. Moreover, we obtain that every tilting module of the form $S \oplus S/R$ studied in Theorem 1.12 arises from perfect localization and from universal localization.

Theorem 4.10. *Let R be a Prüfer domain. Let T be a tilting module, and let $\mathcal{L}$ be the associated Gabriel topology of finite type. The following statements are equivalent.*

(1) $\operatorname{pd} Q_{\mathcal{L}} \leq 1$.
(2) *T arises from perfect localization.*
(3) *T arises from universal localization.*
(4) *There is an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ where $T_0, T_1 \in \operatorname{Add} T$ and $\operatorname{Hom}_R(T_1, T_0) = 0$.*

Moreover, under these conditions, T is equivalent to $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$.

Proof. First of all, recall that the tilting class $\operatorname{Gen} T$ is the class of all $\mathcal{L}$-divisible modules.

(1) $\Rightarrow$ (2): We know from Lemma 4.7(1) and Remark 3.5 that $\lambda_{\mathcal{L}} : R \to Q_{\mathcal{L}}$ is an injective ring epimorphism, and that $Q_{\mathcal{L}}$ is a flat R-module. If $\operatorname{pd} Q_{\mathcal{L}} \leq 1$, then it follows from Proposition 1.10 that $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$ is a tilting module. Since its tilting class $\operatorname{Gen} Q_{\mathcal{L}}$ coincides with the class of $\mathcal{L}$-divisible modules by Lemma 4.7(2), we conclude that $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$ is equivalent to T.

(2) $\Rightarrow$ (3) follows immediately from Corollary 4.4.

(3) $\Rightarrow$ (4) holds true by Theorem 1.12.

(4) $\Rightarrow$ (2) follows by combining Theorem 3.9 and Corollary 4.9.

$(2) \Rightarrow (1)$: Let $\mathcal{G}$ be a perfect Gabriel topology such that $\lambda_{\mathcal{G}} : R \to Q_{\mathcal{G}}$ is injective and $Q_{\mathcal{G}} \oplus Q_{\mathcal{G}}/R$ is a tilting module whose tilting class $\mathrm{Gen}\, Q_{\mathcal{G}}$ coincides with $\mathrm{Gen}\, T$. On the other hand, $\mathrm{Gen}\, Q_{\mathcal{G}}$ coincides with the class of $\mathcal{G}$-divisible modules by Lemma 4.7(2), and $\mathrm{Gen} T$ coincides with the class of $\mathcal{L}$-divisible modules. So, we infer by Theorem 4.6 that the Gabriel topologies $\mathcal{G}$ and $\mathcal{L}$ coincide. Hence T is equivalent to $Q_{\mathcal{L}} \oplus Q_{\mathcal{L}}/R$, and $\mathrm{pd}\, Q_{\mathcal{L}} \leq 1$. ∎

References

[1] L. Angeleri Hügel, D. Herbera, J. Trlifaj. *Divisibles modules and localization*. J. Algebra **294** (2005), 519-551.

[2] L. Angeleri Hügel, J. Sánchez Serdà. *Tilting modules arising from ring epimorphisms*. To appear in Algebras and Representation Theory.

[3] M. Auslander, I. Reiten. *Applications of contravariantly finite subcategories*. Adv. Math. **86** (1991), 111-152.

[4] S. Bazzoni, P. C. Eklof and J. Trlifaj. *Tilting cotorsion pairs*. Bull. London Math. Soc. **37** (2005), 683-696.

[5] S. Bazzoni and D. Herbera. *One dimensional Tilting modules are of finite type*. Algebr. Represent. Theory, **11** (2008), 43–61.

[6] S. Bazzoni and J. Šťovíček. *All Tilting modules are of finite type*. Proc. Amer. Math. Soc. **135** (2007), 3771–3781.

[7] K. Burke and M. Prest. *Rings of definable scalars and biendomorphism rings*. Model Theory of Groups and Automorphisms. London Math. Soc. Lecture Note Ser. Vol. **244** (1997), 188-201.

[8] P. M. Cohn. *Free Rings and Their Relations*. Second ed. London Mathematical Society Monographs, vol. **19**. Academic Press Inc. London (1985).

[9] R. Colpi, A. Tonolo and J. Trlifaj. *Perpendicular categories of infinite dimensional partial tilting modules and transfers of tilting torsion classes*. J. Pure Apll. Algebra **211** (2007), 223-234.

[10] M. Fontana, J. A. Huckaba, I. J. Papick. *Prüfer Domains*. Pure and applied mathematics. New York Marcel Dekker (1997).

[11] L. Fuchs and L. Salce. *Modules over non-Noetherian domains*. Mathematical surveys and monographs. Providence, Rhode Island AMS (2001).

[12] P. Gabriel. *Des catégories abéliennes*. Bulletin de la S. M. F. tome 90 (1962), 323-448.

[13] P. Gabriel, J. A. de la Peña. *Quotients of representation-finite algebras*. Comm. Algebra, **15** (1987), 279-307.

[14] W. Geigle, H. Lenzing. *Perpendicular categories with applications to representations and sheaves*. J. Algebra **144** (1991), 273-343.

[15] R. Göbel and J. Trlifaj. *Approximations and Endomorphism Algebras of Modules*. De Gruyter expositions in Mathematics. **41**.

[16] I. Herzog. *The Ziegler spectrum of a locally coherent Grothendieck category*. Proc. London Math. Soc. (3) **74** (1997), 503-558.

[17] O. Iyama. *Rejective subcategories of artin algebras and orders*. Preprint (2003).

[18] H. Krause. *The Spectrum of a Module Category*. Habilitationsschrift, Universität Bielefeld (1998), Mem. Amer. Math. Soc. **149** (2001).

[19] N. Popescu. *Abelian Categories with Applications to Rings and Modules*. Academic Press London (1973).

[20] L. Salce. *Tilting modules over valuation domains*. Forum. Math. **16** (2004), 539-552.

[21] L. Salce. *$\mathcal{F}$-divisible modules and tilting modules over Prüfer domains*. J. Pure Appl. Algebra. **199** (2005), n. 1-3, 245-259.

[22] A. H. Schofield, *Representation of rings over skew fields*. London Mathematical Society Lecture Note Series, vol. 2, Cambridge University Press, Cambridge, (1985).

[23] B. Stenstrøm. *Rings of quotients*. Springer-Verlag (1975).

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