Direct-sum decompositions controlled by a number of permutations

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Consider the following general statement $\mathrm{S}(n,\mathscr{C})$, where $\mathscr{C}$ is a class of right modules over some ring and $n \geq 1$ is an integer:

($\mathrm{S}(n,\mathscr{C})$) There exist equivalence relations $\{\equiv_i \mid 1 \leq i \leq n\}$ on $\mathscr{C}$ such that, given modules $X_1, \dots, X_r, X'_1, \dots, X'_{r'}$ in $\mathscr{C}$, we have that $X_1 \oplus \cdots \oplus X_r \cong X'_1 \oplus \cdots \oplus X'_{r'}$ if and only if $r = r'$ and $X_j \equiv_i X'_{\sigma_i(j)}$ for $1 \leq i \leq n$ and $1 \leq j \leq r$, for suitable permutations $\sigma_1, \dots, \sigma_n$ of $\{1, \dots, r\}$.

It has long been known (Krull-Schmidt-Remak-Azumaya Theorem) that $\mathrm{S}(1,\mathscr{C})$ holds when $\mathscr{C}$ is any class of modules with local endomorphism ring, and $\equiv_1$ is "is isomorphic to". (Even when the number of summands is infinite.)

The question had been asked whether such uniqueness result holds for the class of indecomposable Artinian modules, and for the class $\mathscr{U}$ of uniserial modules. The answer is negative in both cases [7, 8]. Nevertheless, it is interesting that in the latter case, $\mathrm{S}(2,\mathscr{U})$ holds, and this is the first occurrence of such behaviour. More generally, $\mathrm{S}(2,\mathscr{B})$ holds when $\mathscr{B}$ is the class of biuniform modules [6]. Several classes $\mathscr{C}$ of modules have been discovered in 2008 such that $\mathrm{S}(2,\mathscr{C})$ holds [1, 3, 4].

For any fixed integer $n \geq 1$, any quiver Q on n vertices and any ring R, there exists a class $\mathscr{C}$ of representations of Q over R (= modules over $R[Q]$) such that $\mathrm{S}(n,\mathscr{C})$ holds [5]. These classes of modules contain as subclasses the aforementioned classes $\mathscr{C}'$ for which $\mathrm{S}(2,\mathscr{C}')$ holds.

A module of *finite type* is one whose endomorphism ring has finitely many maximal right ideals, all of which two-sided. Making use of a general framework for the study of categories of modules of finite type [2], a general setting has been found to produce more classes of modules $\mathscr{C}$ for which $\mathrm{S}(n,\mathscr{C})$ holds, for every fixed $n \geq 1$. Moreover, the validity of $\mathrm{S}(n,\mathscr{C})$ for a class of modules $\mathscr{C}$ of finite type $\leq n$ is closely related to its associated hypergraph $H(\mathscr{C})$ (defined in [2]).

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