On the construction of realization functors via stable derivators

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Abstract

Let $\mathbf{T}$ be a triangulated category and consider a t-structure $\mathbf{t} = (\mathbb{D}^{\leqslant 0}, \mathbb{D}^{\geqslant 0})$ in $\mathbf{T}$, whose heart is the Abelian category $\mathcal{A} = \mathbb{D}^{\leqslant 0} \cap \mathbb{D}^{\geqslant 0}$. Denote by $\mathbf{D}(\mathcal{A})$ (resp., $\mathbf{D}^b(\mathcal{A})$, $\mathbf{D}^+(\mathcal{A})$, $\mathbf{D}^-(\mathcal{A})$) the unbounded (resp., bounded, left-bounded, right-bounded) derived category of $\mathcal{A}$. It is a classical problem to construct a "canonical" functor $\mathbf{D}^*(\mathcal{A}) \to \mathbf{T}$ (with $* = \text{blank}, b, +, -$), called a *realization functor*, that extends the inclusion $\mathcal{A} \to \mathbf{T}$. In this seminar, we will expose a partial solution to this problem in the case $\mathbf{T}$ is the underlying category of a stable derivator (e.g., this includes the case of $\mathbf{T}$ the homotopy category of a stable model category).

After recalling the definition and some basic facts about stable derivators, we will give two constructions of realization functors $\mathbf{D}^b(\mathcal{A}) \to \mathbf{T}$. The first approach will be elementary and it will consist in showing that, in the context of stable derivators, there are functorial choices of cones for a suitable class of morphisms in $\mathbf{T}$, similarly to what happens in the context of the "new triangulated categories" studied in [Nee91]. The second approach will consist in adapting the ideas of [BBD82, Bei87], realizing that, when $\mathbf{T}$ is the underlying category of a stable derivator, there is a canonical f-category over it. In fact, in this generality, many of the proofs in [BBD82] become way simpler and somehow more natural.

Finally, we will introduce the notions of left and right-completeness of t-structures and use them to construct a canonical realization functor $\mathbf{D}^+(\mathcal{A}) \to \mathbf{T}$ that extends the bounded realization functor obtained via f-categories. We will also give a criterion for this new functor to be fully faithful.

References

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