On the Ring Theory of dg-Rings

Alexander Zimmermann

In honor of Manolo Saorín's 65th birthday
May 15-17, 2024
Cetraro, Italy

Where did we meet first?

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- First probably in Cocoyoc 1994, then in Luminy 1999, then in Bandung 2011, setting a joint project following Yoshino's lecture;  then in Murcia, Aachen Amiens, ...

ICRA 1994

Bandung 2011

Abarán 2012

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DG-algebras

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- A a $\mathbb{Z}$-graded K-algebra
- $d : A \longrightarrow A$ K-linear of degree 1 with $d^2 = 0$
- with $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$ for all $a, b \in A$, $|a| :=$ degree of a.

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A dg-module (M, δ) over (A, d) is a

- $\mathbb{Z}$-graded A-module M with K-linear $\delta : M \to M$ of degree 1 and $\delta^2 = 0$
- satisfying $\delta(a \cdot m) = d(a) \cdot m + (-1)^{|a|} a \cdot \delta(m)$,
- likewise for right modules, bimodules.

A is a ring.
What are ring and module invariants in this setting ?

dg-simplicity versus dg-semisimplicity

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Theorem (Aldrich and Garcia-Rozas 2002)

Let (A, d) be a differential graded K-algebra. Then the category of dg-left modules is semisimple if and only if

- (A, d) *is acyclic and*
- $\ker(d)$ *is gr-semisimple.*

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$$(A, d) \text{ dg-simple } \not\Rightarrow (A, d) \text{ dg-semisimple}$$

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Let K be a field and let (A, d) be a finite dimensional dg-K-algebra which is simple as an algebra. Then there is a skew-field D and a bounded complex (C, δ) of finite dimensional D-modules such that

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Here: If $(M, \delta_M) =: M^\bullet$ and $(N, \delta_N) =: N^\bullet$ are dg-(A, d)-module, then

$$(Hom_A^\bullet(M^\bullet, N^\bullet))^{(n)} = \{f : M \longrightarrow N \mid \begin{array}{l} f(M_k) \subseteq N_{n+k}; \\ f(am) = (-1)^{|a||f|} af(m) \end{array} \}$$

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$$d_{\mathrm{Hom}}(f) = \delta_N \circ f - (-1)^{|f|} f \circ \delta_M$$

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- A is left Goldie if there is no infinite direct sum of left ideals, and A satisfies the ACC on left annihilators.

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Theorem (Goodearl and Stafford (2000))

Let A be an algebra graded by an abelian group, suppose that A is graded-prime left graded-Goldie ring. Then the left Ore localisation Q at the homogeneous regular elements is a graded-simple algebra.

Theorem

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Then

$$d(b,s) := (-1)^{|s|+1}(d(s),s)\cdot(b,s) + (-1)^{|s|}(d(b),s)$$

defines a differential graded structure on the left Ore localisation A_S, and the natural homomorphism is a dg ring homomorphism $\lambda : (A,d) \longrightarrow (A_S, d_S)$

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There is a more technical version for S being not necessarily regular.

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Get an injective (by dg-simplicity) dg ring homomorphism

$$(A_{S_{\ker(d)}}, d_{S_{\ker(d)}}) \longrightarrow (A_S, d_S)$$

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- $\ker(d_S) = K[X^2, X^{-2}]$ is graded-simple.
- Hence by the dg-Goldie Theorem (A_S, d_S) is dg-simple and by Aldrich and Garcia-Rozas (A_S, d_S) is dg-semisimple. $(H(A_S) = 0)$

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Hence converse of our dg-Goldie theorem is not true.

Happy birthday Manolo !