

Exact Structures and Purity

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There exists a fully faithful additive functor

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- Ziegler spectrum of \mathcal{A}

$$\text{Ind } \mathcal{A} = \{X \in (\mathcal{A}, \mathcal{E}_{\text{pure}}) \mid X \text{ is indecomposable injective}\}$$

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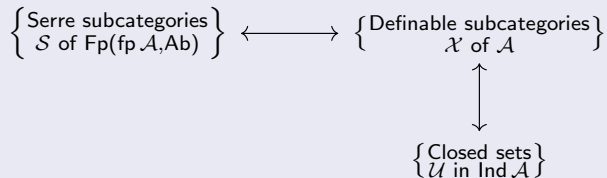
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$$\left\{ \begin{array}{l} \text{Serre subcategories} \\ \mathcal{S} \text{ of } \text{Fp}(\text{fp } \mathcal{A}, \text{Ab}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Definable subcategories} \\ \mathcal{X} \text{ of } \mathcal{A} \end{array} \right\}$$

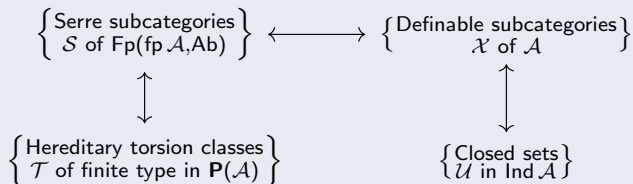
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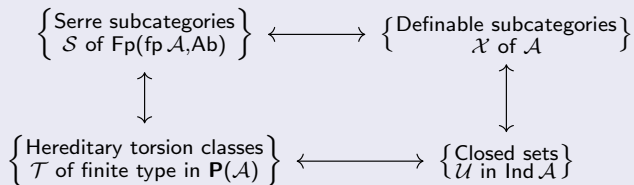
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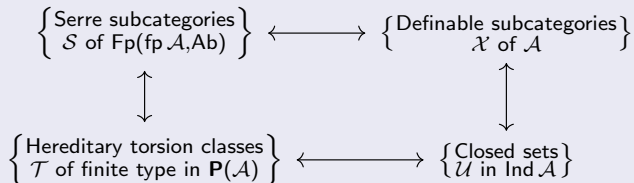
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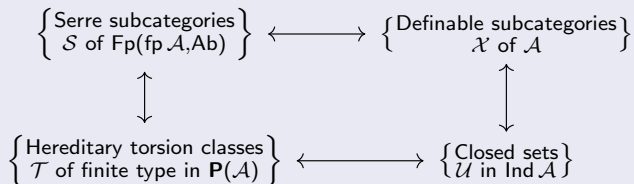


and an injective assignment [Enomoto]

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Fix \mathcal{E} and corresponding $\mathcal{S}_{\mathcal{E}}, \mathcal{T}_{\mathcal{E}}, \mathcal{X}_{\mathcal{E}}, \mathcal{U}_{\mathcal{E}}$.

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Corollary

If \mathcal{A} is abelian, then exact structures on $\text{fp } \mathcal{A}$ are one to one with closed sets in $\text{Ind } \mathcal{A}$ containing all indecomposable injectives in \mathcal{A} .

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Theorem

For $X, Y \in \text{mod } A$ there exists a functorial isomorphism

$$\text{Ext}_{\mathcal{E}}^1(X, Y) \cong D \text{Hom}_A(Y, \tau X) / \mathcal{I}_{\mathcal{E}}(Y, \tau X).$$

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Proof.

Use defect formula for an injective hull $Y \rightarrow Q$ in $(\text{Mod } A, \bar{\mathcal{E}})$. \square

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Example

Let $\mathcal{E} = \langle \text{almost split sequences} \rangle_{\text{mod } A}$ and $\text{rad}_A^\omega = \bigcap_{n=1}^{\infty} \text{rad}_A^n$.

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