Exact Structures and Purity

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There exists a fully faithful additive functor

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$$\operatorname{Mod} R \longrightarrow \operatorname{Add}(\operatorname{mod} R^{\operatorname{op}}, \operatorname{Ab}), \quad X \mapsto (-) \otimes_R X$$

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  $\Rightarrow (\mathcal{A}, \mathcal{E}_{\text{pure}})$ has enough injectives
- Ziegler spectrum of $\mathcal{A}$

$$\operatorname{Ind} \mathcal{A} = \{X \in (\mathcal{A}, \mathcal{E}_{\text{pure}}) \mid X \text{ is indecomposable injective}\}$$

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Fix $\mathcal{E}$ and corresponding $\mathcal{S}_{\mathcal{E}}, \mathcal{T}_{\mathcal{E}}, \mathcal{X}_{\mathcal{E}}, \mathcal{U}_{\mathcal{E}}$.

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Corollary

If $\mathcal{A}$ is abelian, then exact structures on $\operatorname{fp} \mathcal{A}$ are one to one with closed sets in $\operatorname{Ind} \mathcal{A}$ containing all indecomposable injectives in $\mathcal{A}$.

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For $X, Y \in \operatorname{mod} A$ there exists a functorial isomorphism

$$\operatorname{Ext}^1_{\mathcal{E}}(X, Y) \cong D \operatorname{Hom}_A(Y, \tau X)/\mathcal{I}_{\mathcal{E}}(Y, \tau X).$$

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Proof.

Use defect formula for an injective hull $Y \to Q$ in $(\operatorname{Mod} A, \bar{\mathcal{E}})$. □

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Example

Let $\mathcal{E} = \langle \text{almost split sequences} \rangle_{\text{mod A}}$ and $\text{rad}_A^{\omega} = \bigcap_{n=1}^{\infty} \text{rad}_A^{n}$.

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Let $\mathcal{E} = \langle \text{almost split sequences} \rangle_{\text{mod A}}$ and $\text{rad}_A^\omega = \bigcap_{n=1}^{\infty} \text{rad}_A^n$.
Then $\mathcal{I}_\mathcal{E} = \text{rad}_A^\omega + \langle \text{inj}\, A \rangle$ and

$$\begin{aligned} \text{Ext}^1_\mathcal{E}(X, Y) &\cong D\,\text{Hom}_A(Y, \tau X)/\mathcal{I}_\mathcal{E}(Y, \tau X) \\ &\cong \begin{cases} D\,\text{Hom}_A(Y, \tau X)/\text{rad}_A^\omega(Y, \tau X) & \text{if } Y \text{ no preinjective direct summand,} \\ & \end{cases} \end{aligned}$$

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