Commutative hearts

II

Continuity of mutation

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R commutative noetherian ring.

$(\mathcal{U}, \mathcal{V})$ intermediate, compactly generated t-structure in $\mathcal{D}(R)$

Recall from Jorge's talk: the heart $\mathcal{H} := \mathcal{U}[-1] \cap \mathcal{V}$

- is Grothendieck
- is seminoetherian
- has $G\mathrm{Spec}\,\mathcal{H} \overset{1:1}{\longleftrightarrow} \mathrm{Spec}\,R$
- has T_0 Alexandrov full support topology $\mathcal{O}_{\mathcal{H}}$.

Goal: compare the various full support topologies $\mathcal{O}_{\mathcal{H}}$.

Cosilting mutation

[Angeleri Hügel - Laking - Šťovíček - Vitória] defined

(left- and) right- mutation of pure-injective cosilting objects.

pure-injective cosilting objects

$\updownarrow$

compactly generated t-structures

right-mutation

$\updownarrow$

HRS-tilting at a hereditary torsion pair of finite type.

Thm [Hebek - P. - Vitória]

"Right-mutation refines the full support topology"

For a right-mutation $\mathcal{C} \rightsquigarrow \mathcal{C}'$,

$$\mathrm{id} : (\operatorname{Spec} R, \mathcal{O}_{\mathcal{H}_{\mathcal{C}}}) \longrightarrow (\operatorname{Spec} R, \mathcal{O}_{\mathcal{H}_{\mathcal{C}'}})$$

is open.

$\mathcal{H} := \mathcal{H}_c \qquad \mathcal{H}' := \mathcal{H}_{c'}$

<u>Rmk</u> $\theta_{\mathcal{H}}$, $\theta_{\mathcal{H}'}$ correspond to partial orders $\leq, \leq'$ on $\operatorname{Spec} R$.

$\theta_{\mathcal{H}'}$ is finer than $\theta_{\mathcal{H}}$ $\iff$ $\mathfrak{p} \leq' \mathfrak{q} \Rightarrow \mathfrak{p} \leq \mathfrak{q}$

How is $\leq$ defined?

Let $E_{\mathfrak{p}} \in \operatorname{gSpec} \mathcal{H}$ correspond to $\mathfrak{p} \in \operatorname{Spec} R$. Then

$$\mathfrak{p} \leq \mathfrak{q} \overset{\text{def}}{\iff} E_{\mathfrak{p}} \in \operatorname{Cogen} E_{\mathfrak{q}} \iff E_{\mathfrak{p}} \in \operatorname{Prod} E_{\mathfrak{q}}$$

But:
- difficult to compare $E_{\mathfrak{p}}$ and $E'_{\mathfrak{p}}$
- difficult to compare products in $\mathcal{H}$ and $\mathcal{H}'$.

Let $S_\mathfrak{p} := k(\mathfrak{p})[-n_\mathfrak{p}]$ be the Gabriel simple of $\mathcal{H}$ corresponding to $\mathfrak{p}$. We have $E_\mathfrak{p} = E(S_\mathfrak{p})$.

$\forall$ hereditary $(\mathcal{T},\mathcal{F})$, $S_\mathfrak{p} \in \mathcal{T} \cup \mathcal{F}$ [Hrbek Nakamura]

The $S_\mathfrak{p}$'s are hereditary-torsion-simple.

Hence: $\mathfrak{p} \le \mathfrak{q} \overset{\text{def}}{\iff} E_\mathfrak{p} \in \operatorname{Cogen} E_\mathfrak{q} \iff$

$\iff S_\mathfrak{p} \in \operatorname{Cogen} E_\mathfrak{q} \iff$

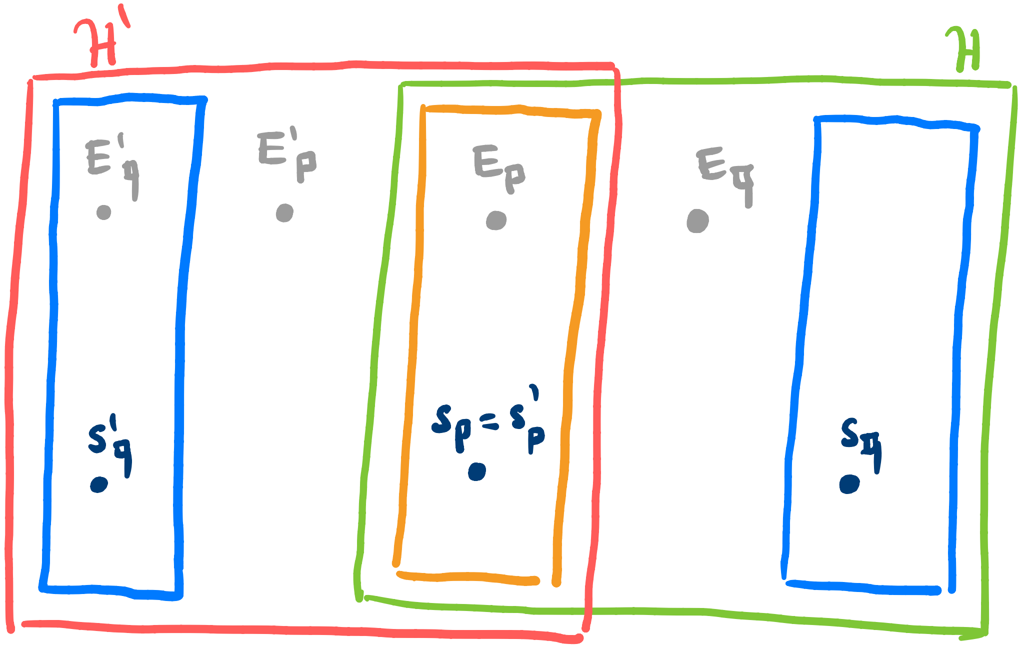
$\iff \neg\left(S_\mathfrak{p} \notin \operatorname{Cogen} E_\mathfrak{q} \right) \iff$ torsion simplicity w.r.t. $({}^{\perp}E_\mathfrak{q}, \operatorname{Cogen} E_\mathfrak{q})$

$\iff \neg\left(\operatorname{Hom}(S_\mathfrak{p}, E_\mathfrak{q}) = 0\right) \iff \operatorname{Hom}(S_\mathfrak{p}, E(S_\mathfrak{q})) \neq 0 \iff$

$\iff \exists\ 0 \neq x \subseteq S_\mathfrak{p}$ s.t. $\operatorname{Hom}(x, S_\mathfrak{q}) \neq 0$.

$\mathcal{H} := \mathcal{H}_c$, $\quad (\mathcal{T}, \mathcal{F})$ hereditary torsion pair of finite type in $\mathcal{H}$

$\mathcal{H}' := \mathcal{H}_{c'} = \mathcal{F} * \mathcal{T}[-1]$



$$s'_p = \begin{cases} s_p & \text{if } \quad s_p \in \mathcal{F} \\ s_p[-1] & \text{if } \quad s_p \in \mathcal{T} \end{cases}$$

Claim

$$p \le' q \quad \text{i.e.} \quad \exists \; 0 \ne x' \overset{\text{in } \mathcal{H}'}{\subseteq} s'_p \;\; \text{st.} \;\; \mathrm{Hom}(x', s'_q) \ne 0$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$p \le q \qquad\qquad \exists \; 0 \ne x \overset{\text{in } \mathcal{H}}{\subseteq} s_p \;\; \text{st.} \;\; \mathrm{Hom}(x, s_q) \ne 0$$

s_p	s_q	Assumption:
T	T	$\exists \; 0 \ne x' \subseteq s_p[-1]$ st. $\mathrm{Hom}(x', s_q[-1]) \ne 0$
T	F	$\exists \; 0 \ne x' \subseteq s_p[-1]$ st. $\mathrm{Hom}(x', s_q) \ne 0$
F	T	$\exists \; 0 \ne x' \subseteq s_p$ st. $\mathrm{Hom}(x', s_q[-1]) \ne 0$
F	F	$\exists \; 0 \ne x' \subseteq s_p$ st. $\mathrm{Hom}(x', s_q) \ne 0$

Idea of proof: Case $\mathcal{F}$, $\mathcal{F}$

Assume

$$\begin{tikzcd}
 & 0 & & & \\
 & s_q \arrow[u] & & & \\
0 \arrow[r] & x' \arrow[u, "\neq 0"'] \arrow[r] & s_p \arrow[r] & y' \arrow[r] & 0 \\
 & z' \arrow[u] & & & \\
 & 0 \arrow[u] & & &
\end{tikzcd}$$

after localizing at q

in $\mathcal{H}'$

$$\Rightarrow \quad \begin{tikzcd}
 & 0 \arrow[r] & H^0 x' \arrow[r, "\subseteq \text{ in } \mathcal{H}"] \arrow[d, equal] & s_p \arrow[r] & H^0 y' \arrow[r] & H^1 x' \arrow[r] & 0 \\
0 \arrow[r] & H^0 z' \arrow[r] & H^0 x' \arrow[r, "\text{cannot be } 0"'] & s_q \arrow[r] & H^1 z' \arrow[r] & H^1 x' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{H}$

Thm In the notation above, $U := \operatorname{supp} \mathcal{C}$. Then:

- if $p \in U$, $q \in U$: $\quad p \leq' q \iff p \leq q$.
- if $p \in U^c$, $q \in U^c$: $\quad p \leq' q \iff p \leq q$
- (if $p \in U$, $q \in U^c$: neither of $p \leq q$, $p \leq' q$ holds)
- if $p \in U^c$, $q \in U$: $\quad p \leq' q \implies p \leq q$.

That is: ϑ_H and $\vartheta_{H'}$ induce the same subspace topologies on U and U^c, and $\vartheta_{H'}$ is finer than ϑ_H.

Thank you!

¡Matemáaaticooo… Manóolo!