On generalizations of an old theorem of Rickard's

Amnon Neeman

Università degli Studi di Milano

amnon.neeman@unimi.it

PATHs conference in Centraro

In honor of Manuel Saorin's 65$^{\text{th}}$ birthday

17 May 2024

Overview

1. Background on some classical derived categories
2. Rickard's 1989 theorem
3. Understanding the object $R \in \mathbf{D}(R\text{–}\mathrm{Mod})$
4. The generalized Rickard theorem
5. The relation with finitistic dimension

Reminder: decorated derived categories

Let R be an associative ring.

Reminder: decorated derived categories

Let R be an associative ring.

The derived categories of R come in many flavors, the following being a partial list.

$$\mathbf{D}(R\text{–Mod}), \quad \mathbf{D}^-(R\text{–Mod}), \quad \mathbf{D}^+(R\text{–Mod}), \quad \mathbf{D}^b(R\text{–Mod})$$
$$\mathbf{D}^-(R\text{–mod}) \quad \mathbf{D}^b(R\text{–proj}) \quad \mathbf{D}^b(R\text{–mod})$$

Reminder: decorated derived categories

Let R be an associative ring.

The derived categories of R come in many flavors, the following being a partial list.

$$\mathbf{D}(R\text{–Mod}), \quad \mathbf{D}^-(R\text{–Mod}), \quad \mathbf{D}^+(R\text{–Mod}), \quad \mathbf{D}^b(R\text{–Mod})$$
$$\mathbf{D}^-(R\text{–mod}) \quad \mathbf{D}^b(R\text{–proj}) \quad \mathbf{D}^b(R\text{–mod})$$

In all of these categories, the objects are cochain complexes of R-modules

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$$

Reminder: decorated derived categories

Let R be an associative ring.

The derived categories of R come in many flavors, the following being a partial list.

$$\mathbf{D}(R\text{–Mod}), \quad \mathbf{D}^-(R\text{–Mod}), \quad \mathbf{D}^+(R\text{–Mod}), \quad \mathbf{D}^b(R\text{–Mod})$$
$$\mathbf{D}^-(R\text{–mod}) \quad \mathbf{D}^b(R\text{–proj}) \quad \mathbf{D}^b(R\text{–mod})$$

In all of these categories, the objects are cochain complexes of R-modules

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$$

and the decorations explain what restrictions we place on the complexes.

Reminder: decorated derived categories, generalized to schemes

Let X be a scheme. One can form the derived categories

$$\begin{array}{cccc} \mathbf{D}_{\mathbf{qc}}(X), & \mathbf{D}^-_{\mathbf{qc}}(X), & \mathbf{D}^+_{\mathbf{qc}}(X), & \mathbf{D}^b_{\mathbf{qc}}(X), \\ \mathbf{D}^-_{\mathbf{coh}}(X), & \mathbf{D}^{\mathrm{perf}}(X), & \mathbf{D}^b_{\mathbf{coh}}(X) & \end{array}$$

Reminder: decorated derived categories, generalized to schemes

Let X be a scheme. One can form the derived categories

$$\mathbf{D}_{\mathbf{qc}}(X), \quad \mathbf{D}^-_{\mathbf{qc}}(X), \quad \mathbf{D}^+_{\mathbf{qc}}(X), \quad \mathbf{D}^b_{\mathbf{qc}}(X),$$
$$\mathbf{D}^-_{\mathbf{coh}}(X), \quad \mathbf{D}^{\mathrm{perf}}(X), \quad \mathbf{D}^b_{\mathbf{coh}}(X)$$

or refine to the relative version, where $Z \subset X$ is a closed subset

$$\mathbf{D}_{\mathbf{qc},Z}(X), \quad \mathbf{D}^-_{\mathbf{qc},Z}(X), \quad \mathbf{D}^+_{\mathbf{qc},Z}(X), \quad \mathbf{D}^b_{\mathbf{qc},Z}(X),$$
$$\mathbf{D}^-_{\mathbf{coh},Z}(X), \quad \mathbf{D}^{\mathrm{perf}}_Z(X), \quad \mathbf{D}^b_{\mathbf{coh},Z}(X)$$

Rickard's 1989 theorem

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings.

Rickard's 1989 theorem

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings.

Then the existence of a triangulated equivalence $\mathbf{D}_{?}^{\square}(R) \cong \mathbf{D}_{?}^{\square}(S)$ is independent of the decorations $?$ and $\square$.

Rickard's 1989 theorem

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings.

Then the existence of a triangulated equivalence $\mathbf{D}^{\square}_{?}(R) \cong \mathbf{D}^{\square}_{?}(S)$ is independent of the decorations $?$ and $\square$.

Before we spell this out, let us introduce a convention valid only for the next slide.

Rickard's 1989 theorem

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings.

Then the existence of a triangulated equivalence $\mathbf{D}_?^\square(R) \cong \mathbf{D}_?^\square(S)$ is independent of the decorations $?$ and $\square$.

Before we spell this out, let us introduce a convention valid only for the next slide.

If $\mathcal{S}$ and $\mathcal{T}$ are two triangulated categories, then $\mathcal{S} \cong \mathcal{T}$ will mean: there exists a triangulated equivalence between $\mathcal{S}$ and $\mathcal{T}$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^{-}(R\text{–Mod}) \cong \mathbf{D}^{-}(S\text{–Mod})$.
3. $\mathbf{D}^{+}(R\text{–Mod}) \cong \mathbf{D}^{+}(S\text{–Mod})$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.
3. $\mathbf{D}^+(R\text{–Mod}) \cong \mathbf{D}^+(S\text{–Mod})$.
4. $\mathbf{D}^b(R\text{–Mod}) \cong \mathbf{D}^b(S\text{–Mod})$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.
3. $\mathbf{D}^+(R\text{–Mod}) \cong \mathbf{D}^+(S\text{–Mod})$.
4. $\mathbf{D}^b(R\text{–Mod}) \cong \mathbf{D}^b(S\text{–Mod})$.
5. $\mathbf{D}^-(R\text{–mod}) \cong \mathbf{D}^-(S\text{–mod})$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^{-}(R\text{–Mod}) \cong \mathbf{D}^{-}(S\text{–Mod})$.
3. $\mathbf{D}^{+}(R\text{–Mod}) \cong \mathbf{D}^{+}(S\text{–Mod})$.
4. $\mathbf{D}^{b}(R\text{–Mod}) \cong \mathbf{D}^{b}(S\text{–Mod})$.
5. $\mathbf{D}^{-}(R\text{–mod}) \cong \mathbf{D}^{-}(S\text{–mod})$.
6. $\mathbf{D}^{b}(R\text{–proj}) \cong \mathbf{D}^{b}(S\text{–proj})$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.
3. $\mathbf{D}^+(R\text{–Mod}) \cong \mathbf{D}^+(S\text{–Mod})$.
4. $\mathbf{D}^b(R\text{–Mod}) \cong \mathbf{D}^b(S\text{–Mod})$.
5. $\mathbf{D}^-(R\text{–mod}) \cong \mathbf{D}^-(S\text{–mod})$.
6. $\mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj})$.
7. $\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod})$.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.
3. $\mathbf{D}^+(R\text{–Mod}) \cong \mathbf{D}^+(S\text{–Mod})$.
4. $\mathbf{D}^b(R\text{–Mod}) \cong \mathbf{D}^b(S\text{–Mod})$.
5. $\mathbf{D}^-(R\text{–mod}) \cong \mathbf{D}^-(S\text{–mod})$.
6. $\mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj})$.
7. $\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod})$.

This can be found in Theorem 1.1 of:

Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure and Appl. Algebra **61** (1989), 303–317.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.
3. $\mathbf{D}^+(R\text{–Mod}) \cong \mathbf{D}^+(S\text{–Mod})$.
4. $\mathbf{D}^b(R\text{–Mod}) \cong \mathbf{D}^b(S\text{–Mod})$.
5. $\mathbf{D}^-(R\text{–mod}) \cong \mathbf{D}^-(S\text{–mod})$.
6. $\mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj})$.
7. $\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod})$.

This can be found in Theorem 1.1 of:

Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure and Appl. Algebra **61** (1989), 303–317.

Rickard's 1989 theorem, spelt out

Theorem (Rickard, 1989)

Let R and S be sufficiently nice rings. Then the following are equivalent:

1. $\mathbf{D}(R\text{–Mod}) \cong \mathbf{D}(S\text{–Mod})$.
2. $\mathbf{D}^-(R\text{–Mod}) \cong \mathbf{D}^-(S\text{–Mod})$.
3. $\mathbf{D}^+(R\text{–Mod}) \cong \mathbf{D}^+(S\text{–Mod})$.
4. $\mathbf{D}^b(R\text{–Mod}) \cong \mathbf{D}^b(S\text{–Mod})$.
5. $\mathbf{D}^-(R\text{–mod}) \cong \mathbf{D}^-(S\text{–mod})$.
6. $\mathbf{D}^b(R\text{–proj}) \cong \mathbf{D}^b(S\text{–proj})$.
7. $\mathbf{D}^b(R\text{–mod}) \cong \mathbf{D}^b(S\text{–mod})$.

Question (Krause 2018): Is there an algorithm to produce $\mathbf{D}^b(R\text{–mod})$ out of $\mathbf{D}^b(R\text{–proj})$?

Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}_c^b$ determine each other*, https://arxiv.org/abs/1806.06471.

Background: compact generation and t–structures

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

Background: compact generation and t–structures

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

Notation to remember: the subcategory of all compact objects will be denoted $\mathcal{T}^c \subset \mathcal{T}$.

Background: compact generation and t–structures

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

Notation to remember: the subcategory of all compact objects will be denoted $\mathcal{T}^c \subset \mathcal{T}$.

The compact objects in $\mathbf{D}(R\text{–}\mathrm{Mod})$ identify as

$$\mathbf{D}(R\text{–}\mathrm{Mod})^c =$$

Background: compact generation and t–structures

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

Notation to remember: the subcategory of all compact objects will be denoted $\mathcal{T}^c \subset \mathcal{T}$.

The compact objects in $\mathbf{D}(R\text{–}\mathrm{Mod})$ identify as

$$\mathbf{D}(R\text{–}\mathrm{Mod})^c = \mathbf{D}^b(R\text{–}\mathrm{proj}).$$

Background: compact generation and t–structures

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

Notation to remember: the subcategory of all compact objects will be denoted $\mathcal{T}^c \subset \mathcal{T}$.

The compact objects in $\mathbf{D}(R\text{–}\mathrm{Mod})$ identify as

$$\mathbf{D}(R\text{–}\mathrm{Mod})^c = \mathbf{D}^b(R\text{–}\mathrm{proj}).$$

The compact object $G \in \mathcal{T}$ generates $\mathcal{T}$ if every nonzero object $X \in \mathcal{T}$ admits a nonzero map $G[i] \longrightarrow X$, for some $i \in \mathbb{Z}$.

Background: compact generation and t–structures

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

Notation to remember: the subcategory of all compact objects will be denoted $\mathcal{T}^c \subset \mathcal{T}$.

The compact objects in $\mathbf{D}(R\text{–}\mathrm{Mod})$ identify as

$$\mathbf{D}(R\text{–}\mathrm{Mod})^c = \mathbf{D}^b(R\text{–}\mathrm{proj}).$$

The compact object $G \in \mathcal{T}$ generates $\mathcal{T}$ if every nonzero object $X \in \mathcal{T}$ admits a nonzero map $G[i] \longrightarrow X$, for some $i \in \mathbb{Z}$.

A compact generator $G \in \mathcal{T}$ is a compact object that generates.

Background: compact generation and t–structures

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

Notation to remember: the subcategory of all compact objects will be denoted $\mathcal{T}^c \subset \mathcal{T}$.

The compact objects in $\mathbf{D}(R\text{–}\mathrm{Mod})$ identify as

$$\mathbf{D}(R\text{–}\mathrm{Mod})^c = \mathbf{D}^b(R\text{–}\mathrm{proj}).$$

The compact object $G \in \mathcal{T}$ generates $\mathcal{T}$ if every nonzero object $X \in \mathcal{T}$ admits a nonzero map $G[i] \longrightarrow X$, for some $i \in \mathbb{Z}$.

A compact generator $G \in \mathcal{T}$ is a compact object that generates.

The object $R \in \mathbf{D}(R\text{–}\mathrm{Mod})$ is a compact generator.

Example (the standard *t–structure* on $\mathbf{D}(R\text{–}\mathrm{Mod})$)

We define two full subcategories of $\mathbf{D}(R\text{–}\mathrm{Mod})$:

- $\mathbf{D}(R\text{–}\mathrm{Mod})^{\leq 0} = \{A \in \mathbf{D}(R\text{–}\mathrm{Mod}) \mid H^i(A) = 0 \text{ for all } i > 0\}$
- $\mathbf{D}(R\text{–}\mathrm{Mod})^{\geq 0} = \{A \in \mathbf{D}(R\text{–}\mathrm{Mod}) \mid H^i(A) = 0 \text{ for all } i < 0\}$

Example (the standard t–structure on $\mathbf{D}(R\text{–Mod})$)

We define two full subcategories of $\mathbf{D}(R\text{–Mod})$:

- $\mathbf{D}(R\text{–Mod})^{\leq 0} \quad = \quad \{A \in \mathbf{D}(R\text{–Mod}) \mid H^i(A) = 0 \text{ for all } i > 0\}$
- $\mathbf{D}(R\text{–Mod})^{\geq 0} \quad = \quad \{A \in \mathbf{D}(R\text{–Mod}) \mid H^i(A) = 0 \text{ for all } i < 0\}$

Definition

A t–structure on a triangulated category $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying

Example (the standard *t*–structure on $\mathbf{D}(R\text{–}\mathrm{Mod})$)

We define two full subcategories of $\mathbf{D}(R\text{–}\mathrm{Mod})$:

- $\mathbf{D}(R\text{–}\mathrm{Mod})^{\leq 0} \quad = \quad \{A \in \mathbf{D}(R\text{–}\mathrm{Mod}) \mid H^i(A) = 0 \text{ for all } i > 0\}$
- $\mathbf{D}(R\text{–}\mathrm{Mod})^{\geq 0} \quad = \quad \{A \in \mathbf{D}(R\text{–}\mathrm{Mod}) \mid H^i(A) = 0 \text{ for all } i < 0\}$

Definition

A *t*–structure on a triangulated category $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying

- $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq 0}[1]$
- $\mathrm{Hom}\Big(\mathcal{T}^{\leq 0}[1]\ ,\ \mathcal{T}^{\geq 0}\Big) = 0$
- Every object $B \in \mathcal{T}$ admits a triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow$$

  with $A \in \mathcal{T}^{\leq 0}[1]$ and $C \in \mathcal{T}^{\geq 0}$.

' '

Definition (equivalent t–structures)

Let $\mathcal{T}$ be any triangulated category, and let $\big(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0}\big)$ and $\big(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0}\big)$ be two t–structures on $\mathcal{T}$. We declare them equivalent if they are a finite distance from each other.

‘ ‘

Definition (equivalent t–structures)

Let $\mathcal{T}$ be any triangulated category, and let $\left(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0}\right)$ and $\left(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0}\right)$ be two t–structures on $\mathcal{T}$. We declare them equivalent if they are a finite distance from each other.

To spell it out: the two t–structures are equivalent if there exists an integer $A > 0$ with

$$\mathcal{T}_1^{\leq -A} \subset \mathcal{T}_2^{\leq 0} \subset \mathcal{T}_1^{\leq A}.$$

‘ ‘

Definition (equivalent t–structures)

Let $\mathcal{T}$ be any triangulated category, and let $\big(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0}\big)$ and $\big(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0}\big)$ be two t–structures on $\mathcal{T}$. We declare them equivalent if they are a finite distance from each other.

To spell it out: the two t–structures are equivalent if there exists an integer $A > 0$ with

$$\mathcal{T}_1^{\leq -A} \subset \mathcal{T}_2^{\leq 0} \subset \mathcal{T}_1^{\leq A}.$$

It is a formal consequence that, for the same integer $A > 0$, we have

$$\mathcal{T}_1^{\geq -A} \supset \mathcal{T}_2^{\geq 0} \supset \mathcal{T}_1^{\geq A}.$$

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ generated by G.

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\right)$ generated by G.

Among all t–structures $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ such that $G \in \mathcal{T}^{\leq 0}$, there exists a unique one with minimal $\mathcal{T}^{\leq 0}$.

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ generated by G.

Among all t–structures $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ such that $G \in \mathcal{T}^{\leq 0}$, there exists a unique one with minimal $\mathcal{T}^{\leq 0}$.

If G and H are two compact generators for $\mathcal{T}$, then the t–structures $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ and $\big(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0}\big)$ are equivalent.

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ generated by G.

Among all t–structures $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ such that $G \in \mathcal{T}^{\leq 0}$, there exists a unique one with minimal $\mathcal{T}^{\leq 0}$.

If G and H are two compact generators for $\mathcal{T}$, then the t–structures $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ and $\big(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0}\big)$ are equivalent.

We say that a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ is in the

preferred equivalence class

if it is equivalent to $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ for some compact generator G, hence for every compact generator.

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^{-} = \bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^{+} = \bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^{b} = \mathcal{T}^{-} \cap \mathcal{T}^{+}$$

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^- = \bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^+ = \bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^- = \bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^+ = \bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Now assume that $\mathcal{T}$ has coproducts and there exists a single compact generator G. Then there is a preferred equivalence class of t–structures, and a corresponding preferred $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^- = \bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^+ = \bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Now assume that $\mathcal{T}$ has coproducts and there exists a single compact generator G. Then there is a preferred equivalence class of t–structures, and a corresponding preferred $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

These are intrinsic, they're independent of any choice.

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^- = \bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^+ = \bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Now assume that $\mathcal{T}$ has coproducts and there exists a single compact generator G. Then there is a preferred equivalence class of t–structures, and a corresponding preferred $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

These are intrinsic, they're independent of any choice.

In the remainder of the slides we only consider the “preferred” $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ in the preferred equivalence class.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ in the preferred equivalence class.

We define:

$$\mathcal{T}_c^- \quad = \quad \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For every } n > 0 \text{ there exists a morphism} \\ \varphi : E \longrightarrow F \\ \text{with } E \text{ compact and such that,} \\ \text{in the triangle } E \longrightarrow F \longrightarrow D, \\ \text{we have } D \in \mathcal{T}^{\leq -n} \end{array} \right\}$$

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class.

We define:

$$\mathcal{T}_c^- = \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For every } n > 0 \text{ there exists a morphism} \\ \varphi : E \longrightarrow F \\ \text{with } E \text{ compact and such that,} \\ \text{in the triangle } E \longrightarrow F \longrightarrow D, \\ \text{we have } D \in \mathcal{T}^{\leq -n} \end{array} \right\}$$

We furthermore define $\mathcal{T}_c^b = \mathcal{T}^b \cap \mathcal{T}_c^-$.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ in the preferred equivalence class.

We define:

$$\mathcal{T}_c^- \quad = \quad \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For every } n > 0 \text{ there exists a morphism} \\ \varphi : E \longrightarrow F \\ \text{with } E \text{ compact and such that,} \\ \text{in the triangle } E \longrightarrow F \longrightarrow D, \\ \text{we have } D \in \mathcal{T}^{\leq -n} \end{array} \right\}$$

We furthermore define $\mathcal{T}_c^b = \mathcal{T}^b \cap \mathcal{T}_c^-$.

It's obvious that the category $\mathcal{T}_c^-$ is intrinsic. As $\mathcal{T}_c^-$ and $\mathcal{T}^b$ are both intrinsic, so is their intersection $\mathcal{T}_c^b$.

It can be computed that:

Example (The special case $\mathcal{T} = \mathbf{D}(R\text{–Mod})$, with R a coherent ring)

$$\begin{array}{lcllcl} \mathcal{T}^+ & = & \mathbf{D}^+(R\text{–Mod}), & \mathcal{T}^- & = & \mathbf{D}^-(R\text{–Mod}), \\ \mathcal{T}^b & = & \mathbf{D}^b(R\text{–Mod}), & \mathcal{T}^c & = & \mathbf{D}^b(R\text{–proj}), \\ \mathcal{T}_c^- & = & \mathbf{D}^-(R\text{–proj}), & \mathcal{T}_c^b & = & \mathbf{D}^b(R\text{–mod}) \end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a coherent scheme and $Z \subset X$ a closed subset)

$$\begin{array}{lcllcllcl} \mathcal{T}^+ & = & \mathbf{D}^+_{\mathbf{qc},Z}(X), & \mathcal{T}^- & = & \mathbf{D}^-_{\mathbf{qc},Z}(X), & \mathcal{T}^c & = & \mathbf{D}^{\mathrm{perf}}_Z(X), \\ \mathcal{T}^b & = & \mathbf{D}^b_{\mathbf{qc},Z}(X), & \mathcal{T}_c^- & = & \mathbf{D}^-_{\mathbf{coh},Z}(X), & \mathcal{T}_c^b & = & \mathbf{D}^b_{\mathbf{coh},Z}(X) \end{array}$$

It can be computed that:

Example (The special case $\mathcal{T} = \mathbf{D}(R\text{–Mod})$, with R a ring)

$$\begin{array}{lllllll} \mathcal{T}^+ & = & \mathbf{D}^+(R\text{–Mod}), & & \mathcal{T}^- & = & \mathbf{D}^-(R\text{–Mod}), \\ \mathcal{T}^b & = & \mathbf{D}^b(R\text{–Mod}), & & \mathcal{T}^c & = & \mathbf{D}^b(R\text{–proj}), \\ \mathcal{T}_c^- & = & \mathbf{D}^-(R\text{–proj}), & & \mathcal{T}_c^b & = & \mathbf{D}^b(R\text{–mod}) \end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a scheme and $Z \subset X$ a closed subset)

$$\begin{array}{lllllllllll} \mathcal{T}^+ & = & \mathbf{D}^+_{\mathbf{qc},Z}(X), & & \mathcal{T}^- & = & \mathbf{D}^-_{\mathbf{qc},Z}(X), & & \mathcal{T}^c & = & \mathbf{D}^{\mathrm{perf}}_Z(X), \\ \mathcal{T}^b & = & \mathbf{D}^b_{\mathbf{qc},Z}(X), & & \mathcal{T}_c^- & = & \mathbf{D}^-_{\mathbf{coh},Z}(X), & & \mathcal{T}_c^b & = & \mathbf{D}^b_{\mathbf{coh},Z}(X) \end{array}$$

It can be computed that:

Example (The special case $\mathcal{T} = \mathbf{D}(R\text{–Mod})$, with R a ring)

$$\begin{array}{lcl lcl} \mathcal{T}^+ &=& \mathbf{D}^+(R\text{–Mod}), & \mathcal{T}^- &=& \mathbf{D}^-(R\text{–Mod}), \\ \mathcal{T}^b &=& \mathbf{D}^b(R\text{–Mod}), & \mathcal{T}^c &=& \mathbf{D}^b(R\text{–proj}), \\ \mathcal{T}_c^- &=& \mathbf{D}^-(R\text{–proj}), & \mathcal{T}_c^b &=& \mathbf{D}^b(R\text{–mod}) \end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a scheme and $Z \subset X$ a closed subset)

$$\begin{array}{lcl lcl lcl} \mathcal{T}^+ &=& \mathbf{D}^+_{\mathbf{qc},Z}(X), & \mathcal{T}^- &=& \mathbf{D}^-_{\mathbf{qc},Z}(X), & \mathcal{T}^c &=& \mathbf{D}^{\mathrm{perf}}_Z(X), \\ \mathcal{T}^b &=& \mathbf{D}^b_{\mathbf{qc},Z}(X), & \mathcal{T}_c^- &=& \mathbf{D}^-_{\mathbf{coh},Z}(X), & \mathcal{T}_c^b &=& \mathbf{D}^b_{\mathbf{coh},Z}(X) \end{array}$$

For R not coherent, $\mathbf{D}^b(R\text{–mod})$ should be replaced by $\mathbf{K}^{-,b}(R\text{–proj})$.

It can be computed that:

Example (The special case $\mathcal{T} = \mathbf{D}(R\text{–Mod})$, with R a ring)

$$\begin{array}{rclrcl} \mathcal{T}^+ & = & \mathbf{D}^+(R\text{–Mod}), & \mathcal{T}^- & = & \mathbf{D}^-(R\text{–Mod}), \\ \mathcal{T}^b & = & \mathbf{D}^b(R\text{–Mod}), & \mathcal{T}^c & = & \mathbf{D}^b(R\text{–proj}), \\ \mathcal{T}^-_c & = & \mathbf{D}^-(R\text{–proj}), & \mathcal{T}^b_c & = & \mathbf{D}^b(R\text{–mod}) \end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathbf{qc},Z}(X)$, with X a scheme and $Z \subset X$ a closed subset)

$$\begin{array}{rclrclrcl} \mathcal{T}^+ & = & \mathbf{D}^+_{\mathbf{qc},Z}(X), & \mathcal{T}^- & = & \mathbf{D}^-_{\mathbf{qc},Z}(X), & \mathcal{T}^c & = & \mathbf{D}^{\mathrm{perf}}_Z(X), \\ \mathcal{T}^b & = & \mathbf{D}^b_{\mathbf{qc},Z}(X), & \mathcal{T}^-_c & = & \mathbf{D}^-_{\mathbf{coh},Z}(X), & \mathcal{T}^b_c & = & \mathbf{D}^b_{\mathbf{coh},Z}(X) \end{array}$$

For R not coherent, $\mathbf{D}^b(R\text{–mod})$ should be replaced by $\mathbf{K}^{-,b}(R\text{–proj})$. The objects are the bounded-above cochain complexes of finitely-generated projective modules, and the b in the superscript means that all but finitely many cohomology groups vanish.

Projective resolutions

Suppose we are given an object $F^* \in \mathbf{D}(R\text{–}\mathrm{Mod})$, meaning a cochain complex

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$$

Assume $F^* \in \mathbf{D}(R\text{–}\mathrm{Mod})^{\leq 0}$, meaning

$$H^i(F^*) = 0 \qquad \text{for all } i > 0.$$

Then F^* has a projective resolution. We can produce a cochain map

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & F^2 & \longrightarrow & \cdots \end{array}$$

inducing an isomorphism in cohomology, and so that the P^i are projective.

Projective resolutions—a different perspective

We have found in $\mathbf{D}(R\text{–}\mathrm{Mod})$ an isomorphism $P^* \longrightarrow F^*$. Now consider

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

This gives in $\mathbf{D}(R\text{–}\mathrm{Mod})$ triangles

$$E_n^* \longrightarrow F^* \longrightarrow D_n^* \longrightarrow$$

with $D_n^* \in \mathbf{D}(R\text{–}\mathrm{Mod})^{\leq -n-1}$ and E_n^* not too complicated.

The black box construction of $\overline{\langle G\rangle}^{[-A,A]}$, of $\overline{\langle G\rangle}_A^{[-A,A]}$, of $\overline{\langle G\rangle}^{(-\infty,A]}$ and of $\langle G\rangle_A$

Let $\mathcal{T}$ be a triangulated category. Let $G \in \mathcal{T}$ be an object, and let $A > 0$ be an integer. I ask the audience to accept, as a black box, that there are sensible constructions of the following four full subcategories of $\mathcal{T}$:

1.

2.

3.

The black box construction of $\overline{\langle G\rangle}^{[-A,A]}$, of $\overline{\langle G\rangle}_A^{[-A,A]}$, of $\overline{\langle G\rangle}^{(-\infty,A]}$ and of $\langle G\rangle_A$

Let $\mathcal{T}$ be a triangulated category. Let $G \in \mathcal{T}$ be an object, and let $A > 0$ be an integer. I ask the audience to accept, as a black box, that there are sensible constructions of the following four full subcategories of $\mathcal{T}$:

1. $\langle G\rangle_A$. This is classical, it consists of the objects of $\mathcal{T}$ obtainable from G using no more than A extensions.
2.
3.

The black box construction of $\overline{\langle G\rangle}^{[-A,A]}$, of $\overline{\langle G\rangle}_A^{[-A,A]}$, of $\overline{\langle G\rangle}^{(-\infty,A]}$ and of $\langle G\rangle_A$

Let $\mathcal{T}$ be a triangulated category. Let $G \in \mathcal{T}$ be an object, and let $A > 0$ be an integer. I ask the audience to accept, as a black box, that there are sensible constructions of the following four full subcategories of $\mathcal{T}$:

1. $\langle G\rangle_A$. This is classical, it consists of the objects of $\mathcal{T}$ obtainable from G using no more than A extensions.
2. Assuming $\mathcal{T}$ has coproducts: $\overline{\langle G\rangle}^{(-\infty,A]}$. Also classical, the bound is on the allowed suspensions.
3.

The black box construction of $\overline{\langle G\rangle}^{[-A,A]}$, of $\overline{\langle G\rangle}_A^{[-A,A]}$, of $\overline{\langle G\rangle}^{(-\infty,A]}$ and of $\langle G\rangle_A$

Let $\mathcal{T}$ be a triangulated category. Let $G \in \mathcal{T}$ be an object, and let $A > 0$ be an integer. I ask the audience to accept, as a black box, that there are sensible constructions of the following four full subcategories of $\mathcal{T}$:

1. $\langle G\rangle_A$. This is classical, it consists of the objects of $\mathcal{T}$ obtainable from G using no more than A extensions.
2. Assuming $\mathcal{T}$ has coproducts: $\overline{\langle G\rangle}^{(-\infty,A]}$. Also classical, the bound is on the allowed suspensions. The variant $\overline{\langle G\rangle}^{[-A,A]}$ is new but similar.
3.

The black box construction of $\overline{\langle G\rangle}^{[-A,A]}$, of $\overline{\langle G\rangle}_A^{[-A,A]}$, of $\overline{\langle G\rangle}^{(-\infty,A]}$ and of $\langle G\rangle_A$

Let $\mathcal{T}$ be a triangulated category. Let $G\in\mathcal{T}$ be an object, and let $A>0$ be an integer. I ask the audience to accept, as a black box, that there are sensible constructions of the following four full subcategories of $\mathcal{T}$:

1. $\langle G\rangle_A$. This is classical, it consists of the objects of $\mathcal{T}$ obtainable from G using no more than A extensions.
2. Assuming $\mathcal{T}$ has coproducts: $\overline{\langle G\rangle}^{(-\infty,A]}$. Also classical, the bound is on the allowed suspensions. The variant $\overline{\langle G\rangle}^{[-A,A]}$ is new but similar.
3. Also assumes $\mathcal{T}$ has coproducts: $\overline{\langle G\rangle}_A^{[-A,A]}$. This is new, both the allowed suspensions and the number of extensions allowed are bounded.

Definition (formal definition of weak approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if

Definition (formal definition of weak approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, and an integer $A > 0$ such that

Definition (formal definition of weak approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, and an integer $A > 0$ such that

- $G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.

Definition (formal definition of weak approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, and an integer $A > 0$ such that

- $G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.

  This means: $\mathrm{Hom}\big(G\,,\,\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}\big) = 0$.

Definition (formal definition of weak approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, and an integer $A > 0$ such that

- $G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.

  This means: $\mathrm{Hom}\big(G\,,\,\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}\big) = 0$.

  Equivalently: $\mathrm{Hom}(G, G[n]) = 0$ for $n \gg 0$.

Definition (formal definition of weak approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is weakly approximable if

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in the preferred equivalence class, and an integer $A > 0$ such that

- $G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.

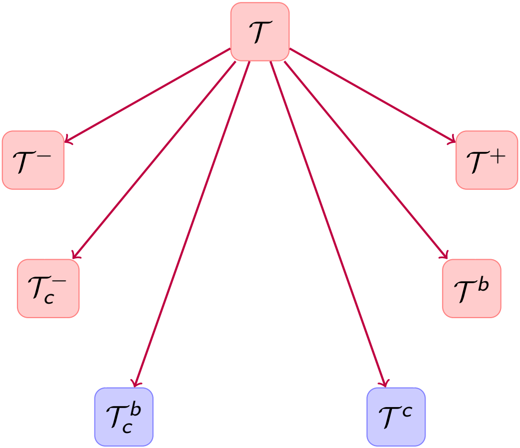
  This means: $\mathrm{Hom}\big(G\,,\,\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}\big) = 0$.

  Equivalently: $\mathrm{Hom}(G, G[n]) = 0$ for $n \gg 0$.

- For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and with $E \in \overline{\langle G \rangle}^{[-A,A]}$.

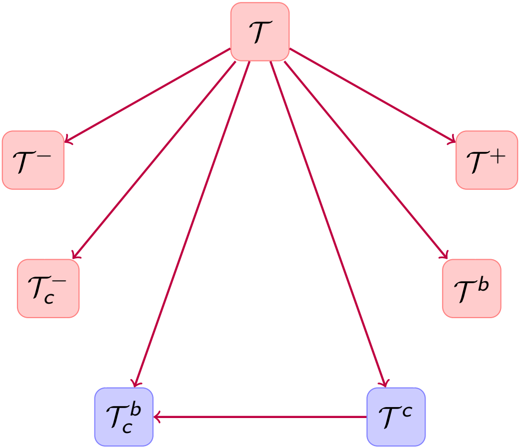
purely triangulated results

Let $\mathcal{T}$ be a weakly approximable triangulated category.



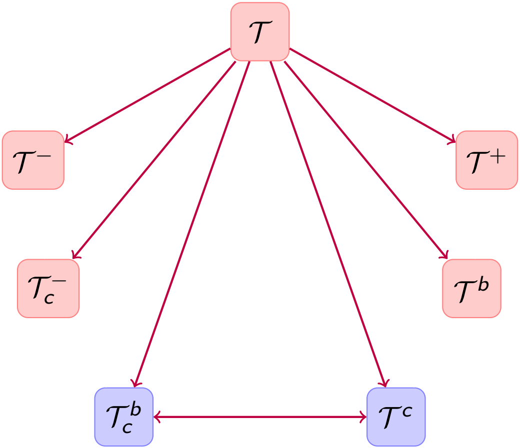
purely triangulated results

Let $\mathcal{T}$ be a weakly approximable triangulated category.



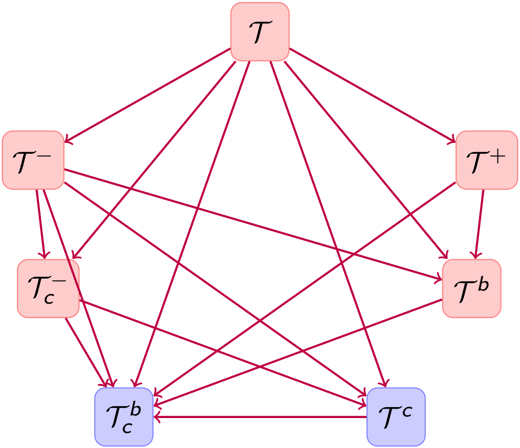
purely triangulated results

Let $\mathcal{T}$ be a coherent, weakly approximable triangulated category.



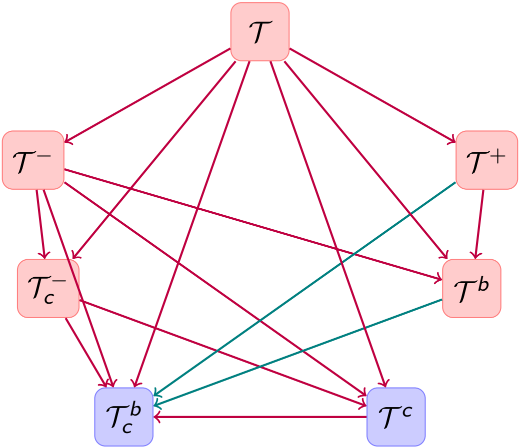
purely triangulated results

Let $\mathcal{T}$ be a weakly approximable triangulated category.



purely triangulated results

Let $\mathcal{T}$ be a weakly approximable triangulated category.



How does one recognize the subcategory $\mathcal{T}_c^b$ in $\mathcal{T}^b$ or in $\mathcal{T}^+$?

How does one recognize the subcategory $\mathcal{T}_c^b$ in $\mathcal{T}^b$ or in $\mathcal{T}^+$?

In a special case: how does one recognize $\mathbf{D}^b(R\text{–mod})$ in $\mathbf{D}^b(R\text{–Mod})$ or in $D^+(R\text{–Mod})$?

How does one recognize the subcategory $\mathcal{T}_c^b$ in $\mathcal{T}^b$ or in $\mathcal{T}^+$?

In a special case: how does one recognize $\mathbf{D}^b(R\text{–}\mathrm{mod})$ in $\mathbf{D}^b(R\text{–}\mathrm{Mod})$ or in $D^+(R\text{–}\mathrm{Mod})$?

The objects $C \in \mathcal{T}_c^b$ are all compact, meaning $\mathrm{Hom}(C, -)$ commutes with the coproducts that exist in $\mathcal{T}^+$ or in $\mathcal{T}^b$.

How does one recognize the subcategory $\mathcal{T}_c^b$ in $\mathcal{T}^b$ or in $\mathcal{T}^+$?

In a special case: how does one recognize $\mathbf{D}^b(R\text{–mod})$ in $\mathbf{D}^b(R\text{–Mod})$ or in $D^+(R\text{–Mod})$?

The objects $C \in \mathcal{T}_c^b$ are all compact, meaning $\mathrm{Hom}(C, -)$ commutes with the coproducts that exist in $\mathcal{T}^+$ or in $\mathcal{T}^b$.

But also: for every object $C \in \mathcal{T}_c^b$

How does one recognize the subcategory $\mathcal{T}_c^b$ in $\mathcal{T}^b$ or in $\mathcal{T}^+$?

In a special case: how does one recognize $\mathbf{D}^b(R\text{–}\mathrm{mod})$ in $\mathbf{D}^b(R\text{–}\mathrm{Mod})$ or in $D^+(R\text{–}\mathrm{Mod})$?

The objects $C \in \mathcal{T}_c^b$ are all compact, meaning $\mathrm{Hom}(C, -)$ commutes with the coproducts that exist in $\mathcal{T}^+$ or in $\mathcal{T}^b$.

But also: for every object $C \in \mathcal{T}_c^b$, for every t–structure in the preferred equivalence class such that $\mathcal{T}^\heartsuit$ is a Grothendieck abelian category

How does one recognize the subcategory $\mathcal{T}_c^b$ in $\mathcal{T}^b$ or in $\mathcal{T}^+$?

In a special case: how does one recognize $\mathbf{D}^b(R\text{–mod})$ in $\mathbf{D}^b(R\text{–Mod})$ or in $D^+(R\text{–Mod})$?

The objects $C \in \mathcal{T}_c^b$ are all compact, meaning $\mathrm{Hom}(C, -)$ commutes with the coproducts that exist in $\mathcal{T}^+$ or in $\mathcal{T}^b$.

But also: for every object $C \in \mathcal{T}_c^b$, for every t–structure in the preferred equivalence class such that $\mathcal{T}^\heartsuit$ is a Grothendieck abelian category, and for every filtered functor $F : I \longrightarrow \mathcal{T}^\heartsuit$

How does one recognize the subcategory $\mathcal{T}_c^b$ in $\mathcal{T}^b$ or in $\mathcal{T}^+$?

In a special case: how does one recognize $\mathbf{D}^b(R\text{–mod})$ in $\mathbf{D}^b(R\text{–Mod})$ or in $D^+(R\text{–Mod})$?

The objects $C \in \mathcal{T}_c^b$ are all compact, meaning $\mathrm{Hom}(C,-)$ commutes with the coproducts that exist in $\mathcal{T}^+$ or in $\mathcal{T}^b$.

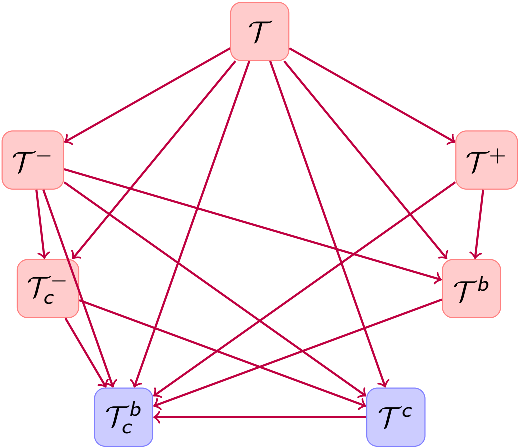
But also: for every object $C \in \mathcal{T}_c^b$, for every t–structure in the preferred equivalence class such that $\mathcal{T}^\heartsuit$ is a Grothendieck abelian category, and for every filtered functor $F : I \longrightarrow \mathcal{T}^\heartsuit$, the natural map

$$\mathrm{Colim}\left(\mathrm{Hom}\big(C, F(-)\big)\right) \longrightarrow \mathrm{Hom}\Big(C, \mathrm{Colim}\, F(-)\Big)$$

is an isomorphism.

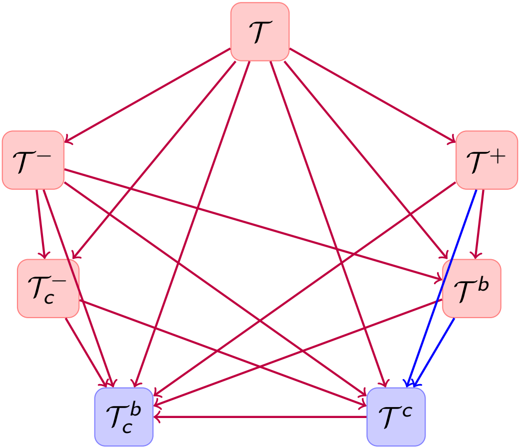
purely triangulated results

Let $\mathcal{T}$ be a weakly approximable triangulated category.



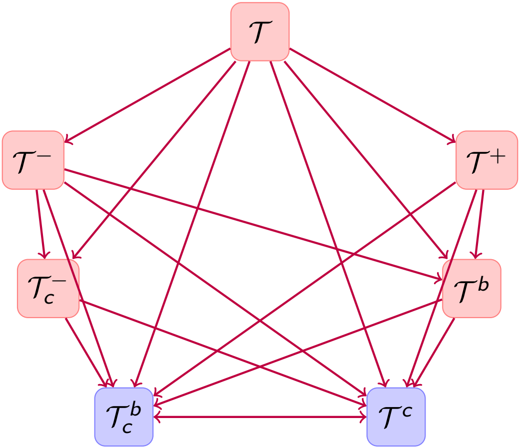
purely triangulated results

Let $\mathcal{T}$ be a weakly approximable triangulated category.



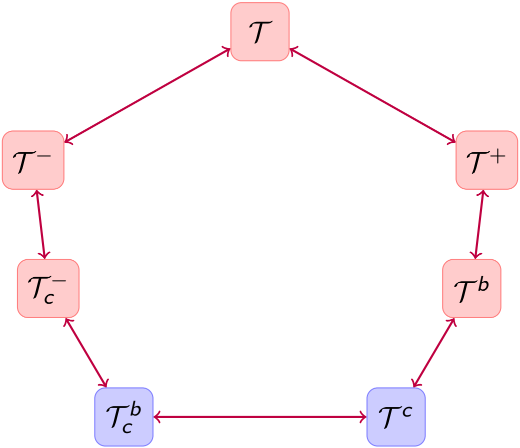
purely triangulated results

Let $\mathcal{T}$ be a coherent, weakly approximable triangulated category.



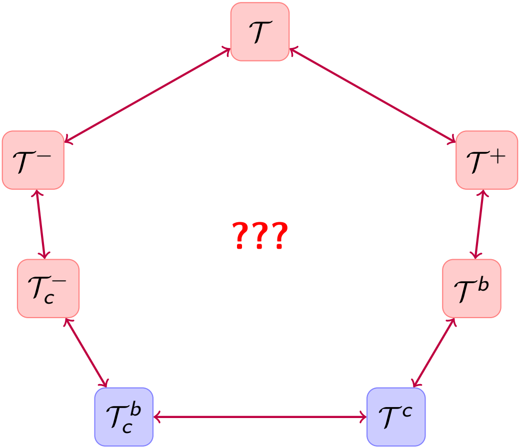
triangulated results in combination with enhancement techniques

Let $\mathcal{T}$ be a coherent, weakly approximable triangulated category. If either $\mathcal{T}^c$ or $\mathcal{T}_c^b$ has a unique enhancement:



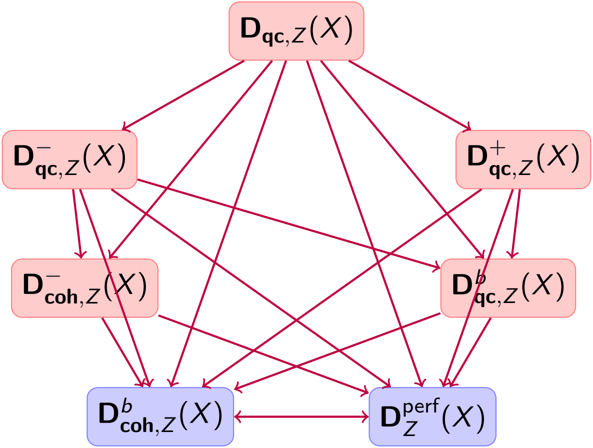
Work in progress: triangulated results in combination with enhancement techniques

Let $\mathcal{T}$ be a weakly approximable triangulated category. If either $\mathcal{T}^c$ or $\mathcal{T}_c^b$ has a unique enhancement:



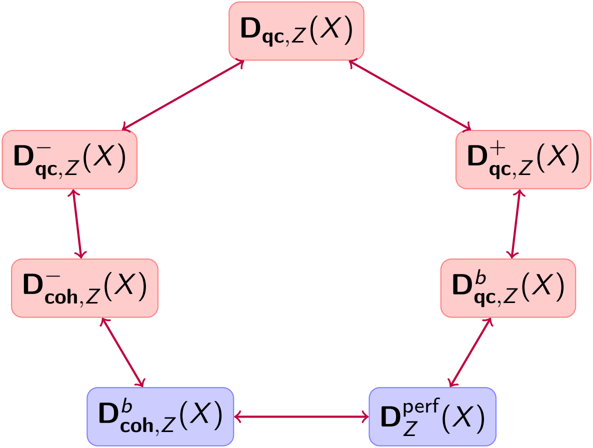
purely triangulated results

Let X be a   scheme, and let $Z \subset X$ be a closed subset.



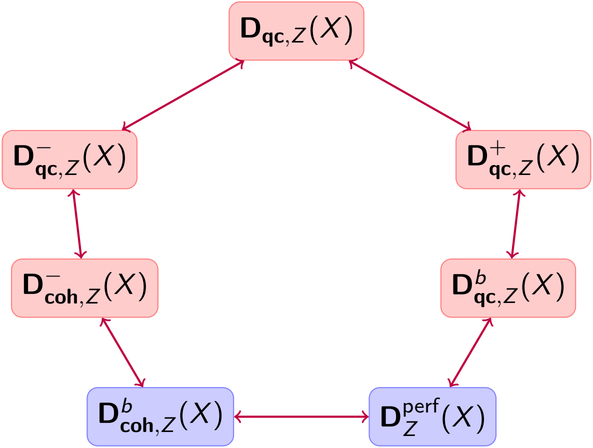
triangulated results in combination with enhancement techniques

Let X be a noetherian scheme, and let $Z \subset X$ be a closed subset.



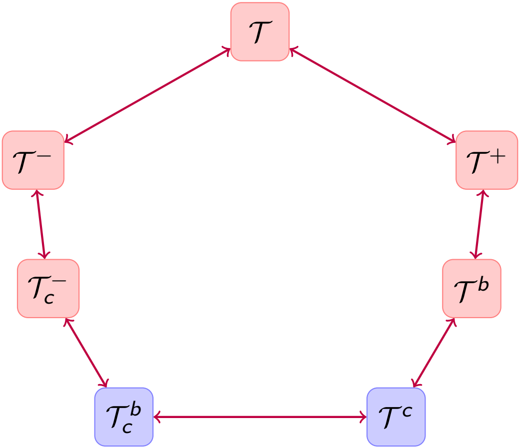
Work in progress: triangulated results in combination with enhancement techniques

Let X be a scheme, and let $Z \subset X$ be a closed subset.



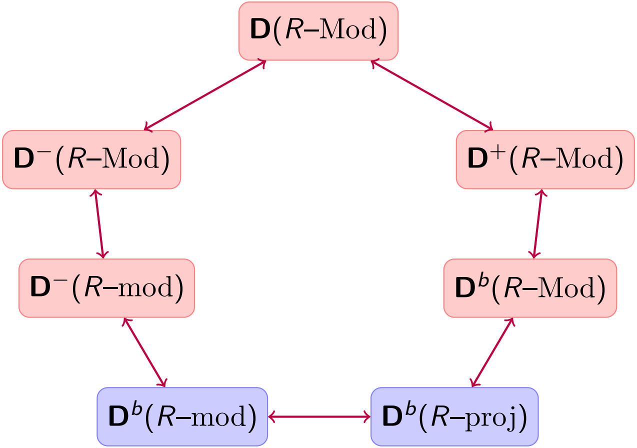
Example not from representation theory or algebraic geometry

If $\mathcal{T}$ is the homotopy category of spectra:



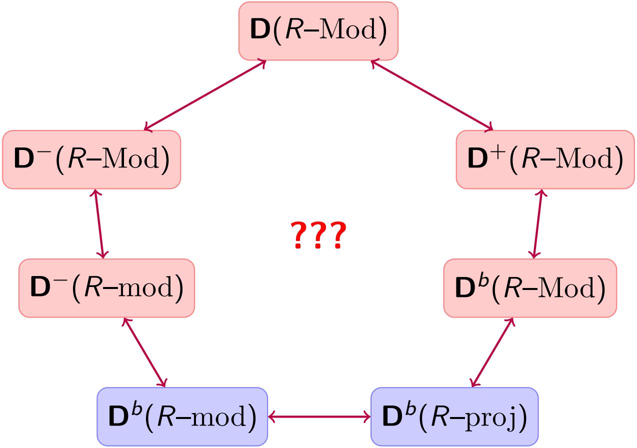
Back to Rickard's old result

If R is a coherent ring, not necessarily commutative:



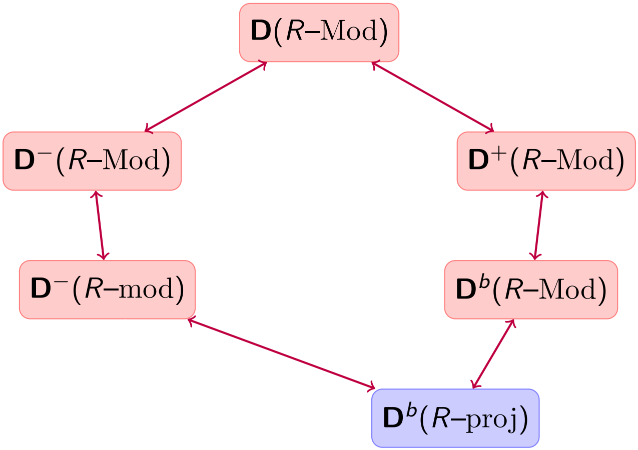
Back to Rickard's old result

If R is a ring, not necessarily commutative:



Back to Rickard's old result

If R is a ring, not necessarily commutative:



- Amnon Neeman, *Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^b_{\mathbf{coh}}(X)$*, Ann. of Math. (2) **193** (2021), no. 3, 689–732.
- Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, arXiv:1804.02240.
- Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, Forum Math. Sigma **11** (2023), Paper No. e110, 18 pages.
- Amnon Neeman, *The category $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ as functors on $\mathcal{T}^b_c$*, arXiv:1806.05777.
- Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}^b_c$ determine each other*, arXiv:1806.06471.
- Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, To appear in Acta Mathematica, arXiv:2202.08861.
- Alberto Canonaco, Amnon Neeman and Paolo Stellari (with appendix by Christian Haesemeyer), *The passage among the subcategories of weakly approximable triangulated categories*, arXiv:2402.04605.

- Amnon Neeman, *Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^{b}_{\mathbf{coh}}(X)$*, Ann. of Math. (2) **193** (2021), no. 3, 689–732.
- Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, arXiv:1804.02240.
- Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, Forum Math. Sigma **11** (2023), Paper No. e110, 18 pages.
- Amnon Neeman, *The category $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ as functors on $\mathcal{T}^b_c$*, arXiv:1806.05777.
- Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}^b_c$ determine each other*, arXiv:1806.06471.
- Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, To appear in Acta Mathematica, arXiv:2202.08861.
- Alberto Canonaco, Amnon Neeman and Paolo Stellari (with appendix by Christian Haesemeyer), *The passage among the subcategories of weakly approximable triangulated categories*, arXiv:2402.04605.

- Amnon Neeman, *Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^{b}_{\mathbf{coh}}(X)$*, Ann. of Math. (2) **193** (2021), no. 3, 689–732.
- Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, arXiv:1804.02240.
- Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, Forum Math. Sigma **11** (2023), Paper No. e110, 18 pages.
- Amnon Neeman, *The category $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ as functors on $\mathcal{T}^b_c$*, arXiv:1806.05777.
- Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}^b_c$ determine each other*, arXiv:1806.06471.
- Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, To appear in Acta Mathematica, arXiv:2202.08861.
- Alberto Canonaco, Amnon Neeman and Paolo Stellari (with appendix by Christian Haesemeyer), *The passage among the subcategories of weakly approximable triangulated categories*, arXiv:2402.04605.

- Amnon Neeman, *Strong generators in* $\mathbf{D}^{\mathrm{perf}}(X)$ *and* $\mathbf{D}^b_{\mathbf{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.
- Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, arXiv:1804.02240.
- Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, Forum Math. Sigma **11** (2023), Paper No. e110, 18 pages.
- Amnon Neeman, *The category* $[\mathcal{T}^c]^{\mathrm{op}}$ *as functors on* $\mathcal{T}^b_c$, arXiv:1806.05777.
- Amnon Neeman, *The categories* $\mathcal{T}^c$ *and* $\mathcal{T}^b_c$ *determine each other*, arXiv:1806.06471.
- Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, To appear in Acta Mathematica, arXiv:2202.08861.
- Alberto Canonaco, Amnon Neeman and Paolo Stellari (with appendix by Christian Haesemeyer), *The passage among the subcategories of weakly approximable triangulated categories*, arXiv:2402.04605.

- Amnon Neeman, *Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^b_{\mathbf{coh}}(X)$*, Ann. of Math. (2) **193** (2021), no. 3, 689–732.
- Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, arXiv:1804.02240.
- Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, Forum Math. Sigma **11** (2023), Paper No. e110, 18 pages.
- Amnon Neeman, *The category $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ as functors on $\mathcal{T}^b_c$*, arXiv:1806.05777.
- Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}^b_c$ determine each other*, arXiv:1806.06471.
- Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, To appear in Acta Mathematica, arXiv:2202.08861.
- Alberto Canonaco, Amnon Neeman and Paolo Stellari (with appendix by Christian Haesemeyer), *The passage among the subcategories of weakly approximable triangulated categories*, arXiv:2402.04605.

Published articles by others, building the theory further

- Mikhail V. Bondarko and S. V. Vostokov, *On weakly negative subcategories, weight structures, and (weakly) approximable triangulated categories*, Lobachevskii J. Math. **41** (2020), 151–159.
- Martin Kalck, Nebojsa Pavic, and Evgeny Shinder, *Obstructions to semiorthogonal decompositions for singular threefolds I: K-theory*, Mosc. Math. J. **21** (2021), no. 3, 567–592.
- Joseph Karmazyn, Alexander Kuznetsov, and Evgeny Shinder, *Derived categories of singular surfaces*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 2, 461–526.
- Yongliang Sun and Yaohua Zhang, *Ladders and completion of triangulated categories*, Theory Appl. Categ. **37** (2021), Paper No. 4, 95–106.

Preprints by others, building the theory further

- Rudradip Biswas, Hongxing Chen, Kabeer Manali Rahul, Chris J. Parker, and Junhua Zheng, *Bounded t-structures, finitistic dimensions, and singularity categories of triangulated categories*, arXiv:2401.00130.
- Mikhail V. Bondarko, *Producing "new" semi-orthogonal decompositions in arithmetic geometry*, arXiv:2203.07315.
- Yongliang Sun and Yaohua Zhang, *Localization theorems for approximable triangulated categories*, arXiv:2402.04954.

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category. We define

Definition

Let $\mathcal{P}(\mathcal{S})$ be the collection of full subcategories $P \in \mathcal{S}$ satisfying $P[1] \subset P$. Then

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category. We define

Definition

Let $\mathcal{P}(\mathcal{S})$ be the collection of full subcategories $P \in \mathcal{S}$ satisfying $P[1] \subset P$. Then

1. We declare $P, \widetilde{P} \in \mathcal{P}(\mathcal{S})$ equivalent if there exists an integer $A > 0$ with

$$P[A] \subset \widetilde{P} \subset P[-A] \ .$$

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category. We define

Definition

Let $\mathcal{P}(\mathcal{S})$ be the collection of full subcategories $P \in \mathcal{S}$ satisfying $P[1] \subset P$. Then

1. We declare $P, \widetilde{P} \in \mathcal{P}(\mathcal{S})$ equivalent if there exists an integer $A > 0$ with
$$P[A] \subset \widetilde{P} \subset P[-A] .$$
2. If $[P]$ and $[\widetilde{P}]$ denote the equivalence classes of $P, \widetilde{P} \in \mathcal{P}(\mathcal{S})$, then we declare that $[P] \leq [\widetilde{P}]$ if there exists an integer $A > 0$ with $P[A] \subset \widetilde{P}$.

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category. We define

Definition

Let $\mathcal{Q}(\mathcal{S})$ be the collection of full subcategories $Q \in \mathcal{S}$ satisfying $Q \subset Q[1]$. Then

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category. We define

Definition

Let $\mathcal{Q}(\mathcal{S})$ be the collection of full subcategories $Q \in \mathcal{S}$ satisfying $Q \subset Q[1]$. Then

1. We declare $Q, \widetilde{Q} \in \mathcal{Q}(\mathcal{S})$ equivalent if there exists an integer $A > 0$ with

$$Q[-A] \subset \widetilde{Q} \subset Q[A] \ .$$

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category. We define

Definition

Let $\mathcal{Q}(\mathcal{S})$ be the collection of full subcategories $Q \in \mathcal{S}$ satisfying $Q \subset Q[1]$. Then

1. We declare $Q, \widetilde{Q} \in \mathcal{Q}(\mathcal{S})$ equivalent if there exists an integer $A > 0$ with
$$Q[-A] \subset \widetilde{Q} \subset Q[A] \ .$$
2. If $[Q]$ and $[\widetilde{Q}]$ denote the equivalence classes of $Q, \widetilde{Q} \in \mathcal{Q}(\mathcal{S})$, then we declare that $[Q] \leq [\widetilde{Q}]$ if there exists an integer $A > 0$ with $Q \subset \widetilde{Q}[A]$.

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category, and let $H \in \mathcal{S}$ be an object.

Definition

We define $P_H(\mathcal{S}) \in \mathcal{P}(\mathcal{S})$ and $Q_H(\mathcal{S}) \in \mathcal{Q}(\mathcal{S})$ by the formulas

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category, and let $H \in \mathcal{S}$ be an object.

Definition

We define $P_H(\mathcal{S}) \in \mathcal{P}(\mathcal{S})$ and $Q_H(\mathcal{S}) \in \mathcal{Q}(\mathcal{S})$ by the formulas

1. $$P_H(\mathcal{S}) = \bigcap_{i=0}^{\infty} H[-i]^{\perp}$$

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category, and let $H \in \mathcal{S}$ be an object.

Definition

We define $P_H(\mathcal{S}) \in \mathcal{P}(\mathcal{S})$ and $Q_H(\mathcal{S}) \in \mathcal{Q}(\mathcal{S})$ by the formulas

1. $$P_H(\mathcal{S}) = \bigcap_{i=0}^{\infty} H[-i]^{\perp} = H[0,\infty)^{\perp} \ .$$

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category, and let $H \in \mathcal{S}$ be an object.

Definition

We define $P_H(\mathcal{S}) \in \mathcal{P}(\mathcal{S})$ and $Q_H(\mathcal{S}) \in \mathcal{Q}(\mathcal{S})$ by the formulas

1. $$P_H(\mathcal{S}) = \bigcap_{i=0}^{\infty} H[-i]^{\perp} = H[0,\infty)^{\perp} \ .$$

2. $$Q_H(\mathcal{S}) = \bigcap_{i=0}^{\infty} H[i]^{\perp}$$

Explaining the issue

Suppose $\mathcal{S}$ be a triangulated category, and let $H \in \mathcal{S}$ be an object.

Definition

We define $P_H(\mathcal{S}) \in \mathcal{P}(\mathcal{S})$ and $Q_H(\mathcal{S}) \in \mathcal{Q}(\mathcal{S})$ by the formulas

1. $$P_H(\mathcal{S}) = \bigcap_{i=0}^{\infty} H[-i]^{\perp} = H[0,\infty)^{\perp} \ .$$

2. $$Q_H(\mathcal{S}) = \bigcap_{i=0}^{\infty} H[i]^{\perp} = H(-\infty,0]^{\perp} \ .$$

Explaining the issue

Now suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$, and let G be a classical generator of $\mathcal{T}^c$

Explaining the issue

Now suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$, and let G be a classical generator of $\mathcal{T}^c$, which is the same thing as a compact generator of $\mathcal{T}$.

Explaining the issue

Now suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let G be a classical generator of $\mathcal{T}^c$, which is the same thing as a compact generator of $\mathcal{T}$.

Put $\mathcal{S} = \mathcal{T}^b_c$, and let $P_G = P_G(\mathcal{S})$ and $Q_G = Q_G(\mathcal{S})$.

Explaining the issue

Now suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let G be a classical generator of $\mathcal{T}^c$, which is the same thing as a compact generator of $\mathcal{T}$.

Put $\mathcal{S} = \mathcal{T}^b_c$, and let $P_G = P_G(\mathcal{S})$ and $Q_G = Q_G(\mathcal{S})$. If $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ is a t-structure in the preferred equivalence class then it can be proved that

Explaining the issue

Now suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$, and let G be a classical generator of $\mathcal{T}^c$, which is the same thing as a compact generator of $\mathcal{T}$.

Put $\mathcal{S} = \mathcal{T}_c^b$, and let $P_G = P_G(\mathcal{S})$ and $Q_G = Q_G(\mathcal{S})$. If $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ is a t-structure in the preferred equivalence class then it can be proved that

1. $[P_G] = [\mathcal{T}_c^b \cap \mathcal{T}^{\leq 0}]$.

Explaining the issue

Now suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let G be a classical generator of $\mathcal{T}^c$, which is the same thing as a compact generator of $\mathcal{T}$.

Put $\mathcal{S} = \mathcal{T}^b_c$, and let $P_G = P_G(\mathcal{S})$ and $Q_G = Q_G(\mathcal{S})$. If $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ is a t-structure in the preferred equivalence class then it can be proved that

1. $[P_G] = [\mathcal{T}^b_c \cap \mathcal{T}^{\leq 0}]$.
2. $[Q_G] = [\mathcal{T}^b_c \cap \mathcal{T}^{\geq 0}]$.

Explaining the issue

Suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$. Put $\mathcal{S} = \mathcal{T}_c^b$ and let $H \in \mathcal{S}$ be any object. We say that H satisfies the strong hypothesis if

Explaining the issue

Suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$. Put $\mathcal{S} = \mathcal{T}_c^b$ and let $H \in \mathcal{S}$ be any object. We say that H satisfies the strong hypothesis if

1. For every object $X \in \mathcal{S}$ there exists an integer $B > 0$, depending on X, with
$$X \in P_H[-B] \cap Q_H[B] \; .$$

Explaining the issue

Suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$. Put $\mathcal{S} = \mathcal{T}_c^b$ and let $H \in \mathcal{S}$ be any object. We say that H satisfies the strong hypothesis if

1. For every object $X \in \mathcal{S}$ there exists an integer $B > 0$, depending on X, with
$$X \in P_H[-B] \cap Q_H[B] \ .$$

Moreover there is an integer $A > 0$ such that the following hold:

Explaining the issue

Suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$. Put $\mathcal{S} = \mathcal{T}_c^b$ and let $H \in \mathcal{S}$ be any object. We say that H satisfies the strong hypothesis if

1. For every object $X \in \mathcal{S}$ there exists an integer $B > 0$, depending on X, with
$$X \in P_H[-B] \cap Q_H[B] \ .$$

Moreover there is an integer $A > 0$ such that the following hold:

1. $\mathrm{Hom}(P_H[A], Q_H) = 0$.

Explaining the issue

Suppose $\mathcal{T}$ is a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$. Put $\mathcal{S} = \mathcal{T}_c^b$ and let $H \in \mathcal{S}$ be any object. We say that H satisfies the strong hypothesis if

1. For every object $X \in \mathcal{S}$ there exists an integer $B > 0$, depending on X, with
$$X \in P_H[-B] \cap Q_H[B] \ .$$

Moreover there is an integer $A > 0$ such that the following hold:

1. $\mathrm{Hom}(P_H[A], Q_H) = 0$.
2. For every object $F \in P_H$, and every integer $m > 0$, there exists a triangle $E_m \longrightarrow F \longrightarrow D_m$ with $E_m \in \langle H \rangle^{[1-m-A,A]}$ and with $D_m \in P_H[m]$.

Passing to the homotopy colimit of the triangles $E_m \longrightarrow F \longrightarrow D_m$ produces, in the category $\mathcal{T}_c^-$, a triangle $E \longrightarrow F \longrightarrow D$

Passing to the homotopy colimit of the triangles $E_m \longrightarrow F \longrightarrow D_m$ produces, in the category $\mathcal{T}_c^-$, a triangle $E \longrightarrow F \longrightarrow D$, with

$$D^\perp \qquad \text{containing} \qquad \mathcal{T}_c^b \ .$$

Passing to the homotopy colimit of the triangles $E_m \longrightarrow F \longrightarrow D_m$ produces, in the category $\mathcal{T}_c^-$, a triangle $E \longrightarrow F \longrightarrow D$, with

$$D^\perp \qquad \text{containing} \qquad \mathcal{T}_c^b \ .$$

For $\mathcal{T} = \mathbf{D}(R\text{–}\mathrm{Mod})$, a cochain complex D^* of finitely generated projective R–modules

Passing to the homotopy colimit of the triangles $E_m \longrightarrow F \longrightarrow D_m$ produces, in the category $\mathcal{T}_c^-$, a triangle $E \longrightarrow F \longrightarrow D$, with

$$D^\perp \qquad \text{containing} \qquad \mathcal{T}_c^b \ .$$

For $\mathcal{T} = \mathbf{D}(R\text{–Mod})$, a cochain complex D^* of finitely generated projective R–modules

$$\cdots \longrightarrow D^{-2} \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow 0$$

Passing to the homotopy colimit of the triangles $E_m \longrightarrow F \longrightarrow D_m$ produces, in the category $\mathcal{T}_c^-$, a triangle $E \longrightarrow F \longrightarrow D$, with

$$D^\perp \qquad \text{containing} \qquad \mathcal{T}_c^b \ .$$

For $\mathcal{T} = \mathbf{D}(R\text{–}\mathrm{Mod})$, a cochain complex D^* of finitely generated projective R–modules

$$\cdots \longrightarrow D^{-2} \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow 0$$

such that $D^\perp$ contains $\mathcal{T}_c^b$.

Passing to the homotopy colimit of the triangles $E_m \longrightarrow F \longrightarrow D_m$ produces, in the category $\mathcal{T}_c^-$, a triangle $E \longrightarrow F \longrightarrow D$, with

$$D^\perp \qquad \text{containing} \qquad \mathcal{T}_c^b \ .$$

For $\mathcal{T} = \mathbf{D}(R\text{–}\mathrm{Mod})$, a cochain complex D^* of finitely generated projective R–modules

$$\cdots \longrightarrow D^{-2} \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow 0$$

such that $D^\perp$ contains $\mathcal{T}_c^b$. And in particular $\mathrm{Hom}(D, R[n]) = 0$ for all $n \in \mathbb{Z}$.

Passing to the homotopy colimit of the triangles $E_m \longrightarrow F \longrightarrow D_m$ produces, in the category $\mathcal{T}_c^-$, a triangle $E \longrightarrow F \longrightarrow D$, with

$$D^\perp \qquad \text{containing} \qquad \mathcal{T}_c^b \ .$$

For $\mathcal{T} = \mathbf{D}(R\text{–}\mathrm{Mod})$, a cochain complex D^* of finitely generated projective R–modules

$$\cdots \longrightarrow D^{-2} \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow 0$$

such that $D^\perp$ contains $\mathcal{T}_c^b$. And in particular $\mathrm{Hom}(D, R[n]) = 0$ for all $n \in \mathbb{Z}$.

If ${}^\perp\mathcal{T}_c^b \cap \mathcal{T}_c^- = \{0\}$, then there is a recipe giving $\mathcal{T}^c$ as a subcategory of $\mathcal{T}_c^b$.

The big finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ be a t-structure in the preferred equivalence class. Put

The big finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ be a t-structure in the preferred equivalence class. Put

$$\mathfrak{C}(\mathcal{T}) \quad = \quad \mathcal{T}^- \bigcap \left\{ \bigcup_{i=1}^{\infty} \left({}^{\perp}[\mathcal{T}^{\leq -i}] \cap [\mathcal{T}^{\leq -i}]^{\perp} \right) \right\} .$$

The big finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ be a t-structure in the preferred equivalence class. Put

$$\mathfrak{C}(\mathcal{T}) \quad = \quad \mathcal{T}^- \bigcap \left\{ \bigcup_{i=1}^{\infty} \left({}^\perp[\mathcal{T}^{\leq -i}] \cap [\mathcal{T}^{\leq -i}]^\perp \right) \right\} .$$

It is known that there exists an integer $B > 0$ with

$$\mathfrak{C}(\mathcal{T}) \cap {}^\perp[\mathcal{T}^{\leq -m}] \qquad \subset \qquad [\mathcal{T}^{\leq -m-B}]^\perp .$$

The big finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ be a t-structure in the preferred equivalence class. Put

$$\mathfrak{C}(\mathcal{T}) \quad = \quad \mathcal{T}^- \bigcap \left\{ \bigcup_{i=1}^{\infty} \left({}^\perp[\mathcal{T}^{\le -i}] \cap [\mathcal{T}^{\le -i}]^\perp \right) \right\} .$$

It is known that there exists an integer $B > 0$ with

$$\mathfrak{C}(\mathcal{T}) \cap {}^\perp[\mathcal{T}^{\le -m}] \qquad \subset \qquad [\mathcal{T}^{\le -m-B}]^\perp .$$

The "big" finitistic dimension conjecture says that there exists an integer $B > 0$ with

$$\mathfrak{C}(\mathcal{T}) \cap [\mathcal{T}^{\le -m}]^\perp \qquad \subset \qquad {}^\perp[\mathcal{T}^{\le -m-B}] .$$

The small finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ be a t-structure in the preferred equivalence class. Put

The small finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$, and let $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ be a t-structure in the preferred equivalence class. Put

$$\mathfrak{C}(\mathcal{T}) \quad = \quad \mathcal{T}_c^- \bigcap \left\{ \bigcup_{i=1}^{\infty} \left({}^{\perp}[\mathcal{T}^{\leq -i}] \cap [\mathcal{T}^{\leq -i}]^{\perp} \right) \right\} .$$

The small finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}^b_c$, and let $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ be a t-structure in the preferred equivalence class. Put

$$\mathfrak{C}(\mathcal{T}) \quad = \quad \mathcal{T}^-_c \bigcap \left\{ \bigcup_{i=1}^{\infty} \left({}^{\perp}[\mathcal{T}^{\leq -i}] \cap [\mathcal{T}^{\leq -i}]^{\perp} \right) \right\} .$$

It is known that there exists an integer $B > 0$ with

$$\mathfrak{C}(\mathcal{T}) \cap {}^{\perp}[\mathcal{T}^{\leq -m}] \qquad \subset \qquad [\mathcal{T}^{\leq -m-B}]^{\perp} .$$

The small finitistic dimension conjecture

Let $\mathcal{T}$ be a weakly approximable triangulated category with $\mathcal{T}^c \subset \mathcal{T}_c^b$, and let $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ be a t-structure in the preferred equivalence class. Put

$$\mathfrak{C}(\mathcal{T}) \quad = \quad \mathcal{T}_c^- \bigcap \left\{ \bigcup_{i=1}^{\infty} \left({}^\perp[\mathcal{T}^{\leq -i}] \cap [\mathcal{T}^{\leq -i}]^\perp \right) \right\} .$$

It is known that there exists an integer $B > 0$ with

$$\mathfrak{C}(\mathcal{T}) \cap {}^\perp[\mathcal{T}^{\leq -m}] \qquad \subset \qquad [\mathcal{T}^{\leq -m-B}]^\perp .$$

The "small" finitistic dimension conjecture says that there exists an integer $B > 0$ with

$$\mathfrak{C}(\mathcal{T}) \cap [\mathcal{T}^{\leq -m}]^\perp \qquad \subset \qquad {}^\perp[\mathcal{T}^{\leq -m-B}] .$$

Thank you!