$\otimes$-structures in derived categories

(joint with Dolors Herbera and Giovanna Le Gros)

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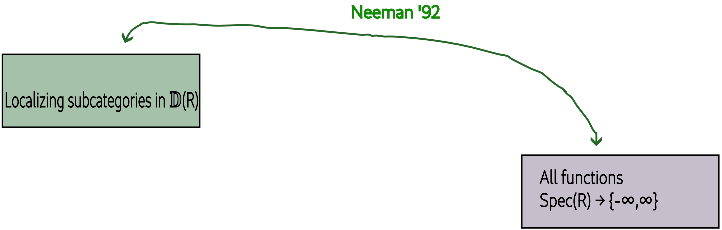
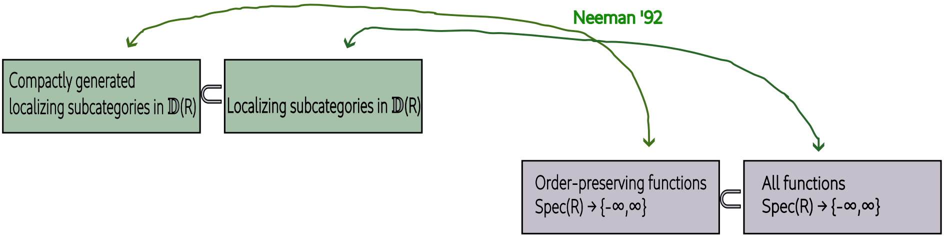
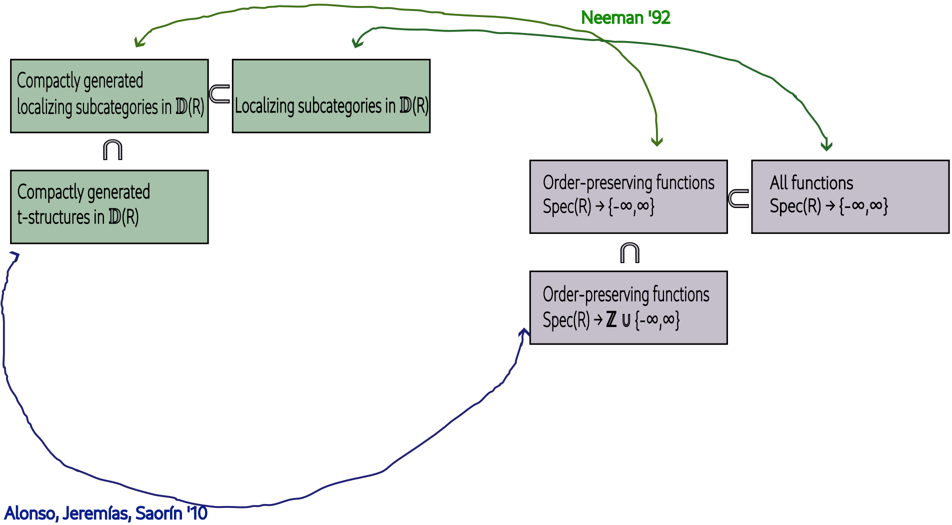
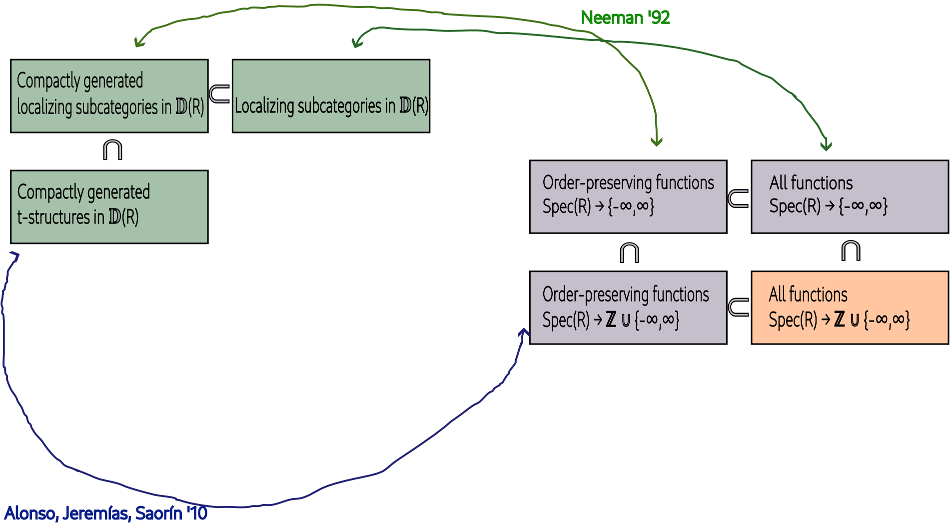

Neeman '92
Localizing subcategories in 𝔻(R)
All functions
Spec(R) → {-∞,∞}


Neeman '92
Compactly generated localizing subcategories in $\mathbb{D}(R)$
$\subset$
Localizing subcategories in $\mathbb{D}(R)$
Order-preserving functions Spec(R) → {-∞,∞}
$\subset$
All functions Spec(R) → {-∞,∞}


Neeman '92
Compactly generated localizing subcategories in 𝔻(R)
⊂
Localizing subcategories in 𝔻(R)
∩
Compactly generated t-structures in 𝔻(R)
Order-preserving functions Spec(R) → {-∞,∞}
⊂
All functions Spec(R) → {-∞,∞}
∩
Order-preserving functions Spec(R) → ℤ ∪ {-∞,∞}
Alonso, Jeremías, Saorín '10



Neeman '92
Compactly generated localizing subcategories in $\mathbb{D}(R)$
$\subset$
Localizing subcategories in $\mathbb{D}(R)$
$\cap$
Compactly generated t-structures in $\mathbb{D}(R)$
Order-preserving functions Spec(R) → {-∞,∞}
$\subset$
All functions Spec(R) → {-∞,∞}
$\cap$
$\cap$
Order-preserving functions Spec(R) → $\mathbb{Z}$ ∪ {-∞,∞}
$\subset$
All functions Spec(R) → $\mathbb{Z}$ ∪ {-∞,∞}
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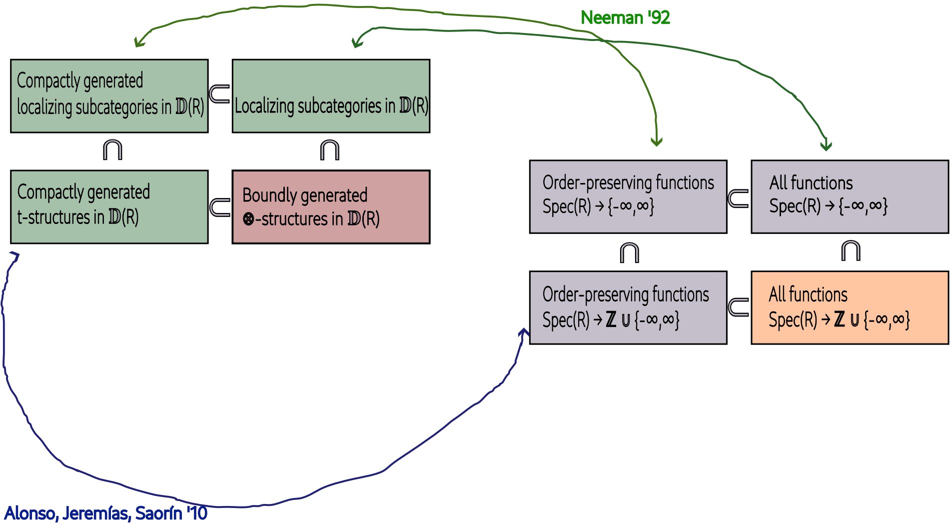
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Compactly generated t-structures in $\mathbb{D}(R)$
Boundly generated ⊗-structures in $\mathbb{D}(R)$
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All functions Spec(R) $\to \{-\infty,\infty\}$
Order-preserving functions Spec(R) $\to \mathbb{Z} \cup \{-\infty,\infty\}$
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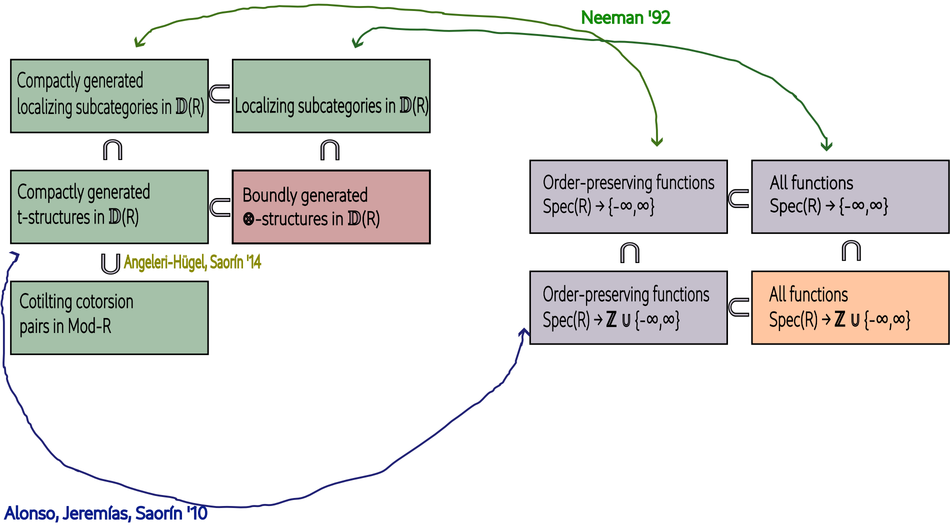
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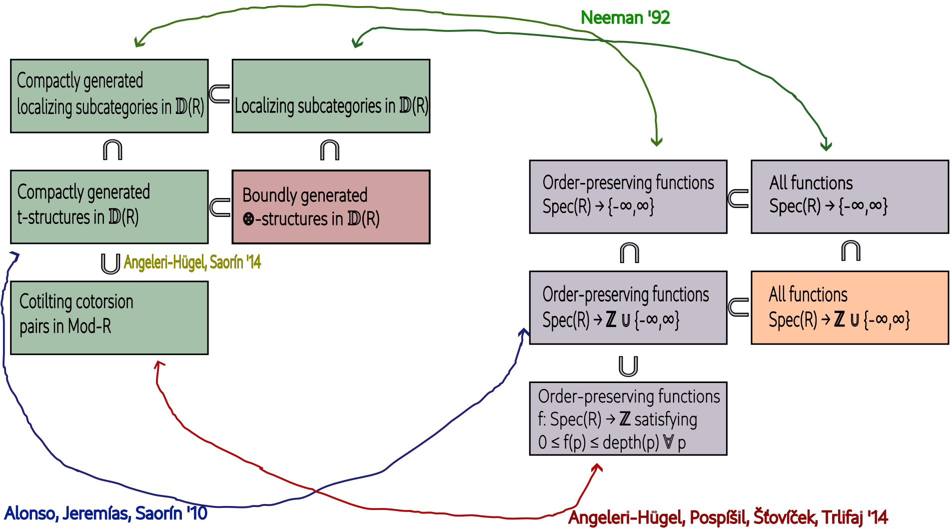
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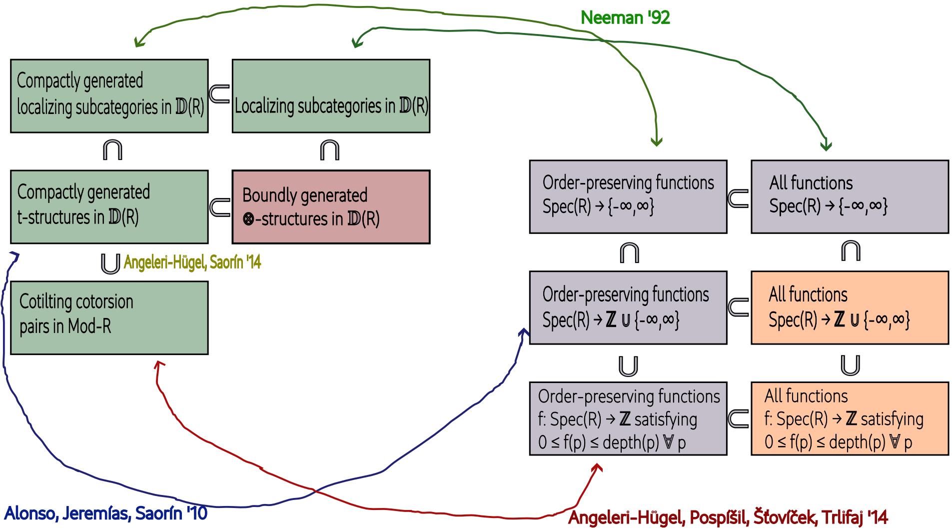
All functions Spec(R) → $\mathbb{Z}$ ∪ {-∞,∞}
Order-preserving functions f: Spec(R) → $\mathbb{Z}$ satisfying 0 ≤ f(p) ≤ depth(p) ∀ p
Alonso, Jeremías, Saorín '10
Angeleri-Hügel, Pospíšil, Šťovíček, Trlifaj '14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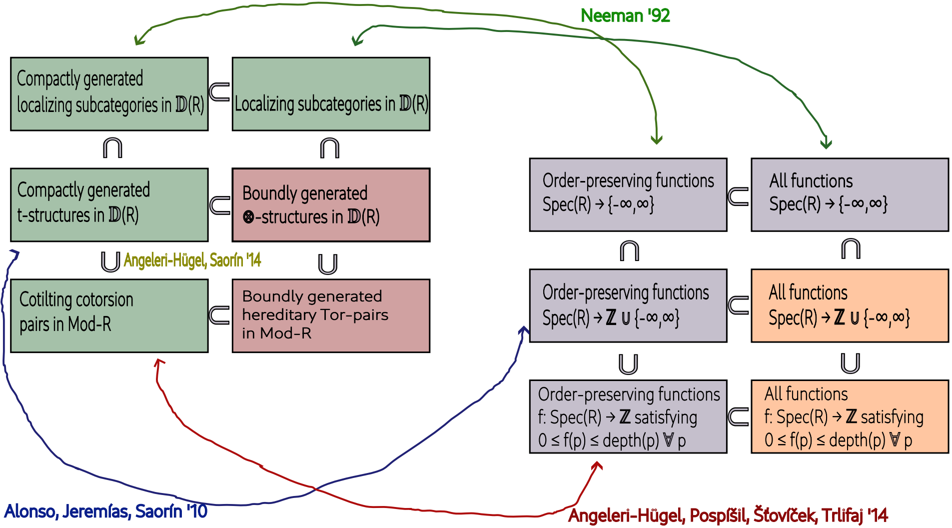
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Cotilting cotorsion pairs in Mod-R
Boundly generated hereditary Tor-pairs in Mod-R
Order-preserving functions Spec(R) → {-∞,∞}
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Let R be a (not necessarily commutative) ring. Let $\mathcal{D}(\text{Mod-}R)$ (resp. $\mathcal{D}(R\text{-Mod})$) denote the unbounded derived category of right (resp. left) R-modules.

- For $\mathcal{X} \subseteq \mathcal{D}(\text{Mod-}R)$, let

$$\mathcal{X}^{\top_{<0}} = \{Y \in \mathcal{D}(R\text{-Mod}) \mid X \otimes_R^{\mathbf{L}} Y \in \mathcal{D}^{\geq 0}\}$$

.

- For $\mathcal{Y} \subseteq \mathcal{D}(R\text{-Mod})$, let

$$^{\top_{<0}}\mathcal{Y} = \{X \in \mathcal{D}(\text{Mod-}R) \mid X \otimes_R^{\mathbf{L}} Y \in \mathcal{D}^{\geq 0}\}$$

.

Definition

A $\otimes$-structure over R is a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories $\mathcal{X} \subseteq \mathcal{D}(\text{Mod-}R)$ and $\mathcal{Y} \subseteq \mathcal{D}(R\text{-Mod})$ such that $\mathcal{Y} = \mathcal{X}^{\top_{<0}}$ and $\mathcal{X} = {}^{\top_{<0}}\mathcal{Y}$.

Basic facts

Definition

A $\otimes$-structure over R is a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories $\mathcal{X} \subseteq \mathcal{D}(\text{Mod-}R)$ and $\mathcal{Y} \subseteq \mathcal{D}(R\text{-Mod})$ such that $\mathcal{Y} = \mathcal{X}^{\top_{<0}}$ and $\mathcal{X} = {}^{\top_{<0}}\mathcal{Y}$.

- Both the classes $\mathcal{X}$ and $\mathcal{Y}$ are closed under Σ^{-1}, extensions, and homotopy directed colimits.
- The pair $(\mathcal{D}^{\geq 0}, \mathcal{K}^{\geq 0}(R\text{-dgFlat}))$ is the standard $\otimes$-structure.
- More generally, let $(\mathcal{F}, \mathcal{C})$ be a hereditary Tor-pair over R, that is, a pair of subcategories $\mathcal{F} \subseteq \text{Mod-}R$ and $\mathcal{C} \subseteq R\text{-Mod}$ maximal with respect to the orthogonality relation $\text{Tor}_i^R(\mathcal{F}, \mathcal{C}) = 0 \ \forall i > 0$. Then

$$(\mathcal{K}^{\geq 0}(R\text{-dgFlat}) \star \mathcal{F}, \mathcal{K}^{\geq 0}(R\text{-dgFlat}) \star \mathcal{C})$$

is a $\otimes$-structure.

Given a $\otimes$-structure $(\mathcal{X}, \mathcal{Y})$, there is a co-t-structure of the form $(\mathcal{X}, \mathcal{X}^{\perp})$ in $\mathcal{D}(\text{Mod-}R)$.

- We have $\mathcal{X} = {}^{\perp}(\mathcal{Y}^{+})$, where $(-)^{+} = \mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is the character duality.
- It follows by a purification argument that $(\mathcal{X}, \mathcal{X}^{\perp})$ is a co-t-structure cogenerated by a pure-injective object of $\mathcal{D}(\text{Mod-}R)$ [Laking-Vitória '20].
- This yields a bijection between the set (!) of $\otimes$-structures over R and the set of co-t-structures in $\mathcal{D}(\text{Mod-}R)$ cogenerated by dual objects.

$\otimes$-structures vs. t-structures

Let us call a $\otimes$-structure $(\mathcal{X}, \mathcal{Y})$ stable if $\mathcal{X}$ (equivalently, $\mathcal{Y}$) is closed under Σ.

Proposition

A thick subcategory $\mathcal{X} \subseteq \mathcal{D}(\text{Mod-}R)$ fits into a (stable) $\otimes$-structure $(\mathcal{X}, \mathcal{Y})$ if and only if $\mathcal{X} = \operatorname{Ker}(- \otimes_R^{\mathbf{L}} Y)$ for some $Y \in \mathcal{D}(R\text{-Mod})$. (I.e., $\mathcal{X}$ is a Bousfield class.)

Proposition

Let $(\mathcal{X}, \mathcal{Y})$ be a $\otimes$-structure over R. TFAE:

1. *$\mathcal{Y}$ is a coaisle of a t-structure in $\mathcal{D}(R\text{-Mod})$.*
2. *$\mathcal{Y}$ is closed under products.*

In this case, the t-structure $(\mathcal{U}, \mathcal{Y})$ in $\mathcal{D}(R\text{-Mod})$ is homotopically smashing, that is, $\mathcal{Y}$ is a definable subcategory [Angeleri-Hügel, Marks, Vitória '17].

Compactly generated t-structures

Let $(\mathcal{U}, \mathcal{Y})$ be a t-structure in $D(R\text{-Mod})$ generated by a subcategory $\mathcal{S}$ of compact objects.

- Let $\mathcal{S}^* = \{\mathbf{R}\mathrm{Hom}_R(S, R) \mid S \in \mathcal{S}\}$, a subcategory of compact objects of $\mathcal{D}(\text{Mod-}R)$.
- Then we have a $\otimes$-structure $(\mathcal{X}, \mathcal{Y})$ over R generated by $\mathcal{S}$ (meaning that $\mathcal{Y} = \mathcal{S}^{\top_{<0}}$).

Proposition

Let R be a ring such that every homotopically smashing t-structure in $\mathcal{D}(R\text{-Mod})$ is compactly generated. Then:

$$\left\{ \begin{array}{c} \otimes\textit{-structures } (\mathcal{X}, \mathcal{Y}) \\ \textit{with } \mathcal{Y} \textit{ product closed} \end{array} \right\} \overset{1\text{-}1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Compactly generated} \\ \textit{t-structures in } \mathcal{D}(R\text{-Mod}) \end{array} \right\}$$

The assumption holds e.g. if R is commutative noetherian [H-Nakamura '21] or left hereditary [Angeleri-Hügel, H '21].

Let us call a $\otimes$-structure $(\mathcal{X}, \mathcal{Y})$ compactly generated if there is a subcategory $\mathcal{S}$ of compact objects of $\mathcal{D}(\text{Mod-}R)$ such that $\mathcal{Y} = \mathcal{S}^{\top_{<0}}$.

Proposition (Šťovíček-Pospíšil '16, Angeleri-Hügel, H '21)

The following structures are in mutual bijection:

- *Compactly generated t-structures in* $\mathcal{D}(R\text{-Mod})$.
- *Compactly generated co-t-structures in* $\mathcal{D}(\text{Mod-}R)$.
- *Compactly generated* $\otimes$*-structures over* R.

Definition

Let us call a $\otimes$-structure $(\mathcal{X}, \mathcal{Y})$ boundly generated if $\mathcal{Y} = \mathcal{K}^{\top <0}$ for a subcategory $\mathcal{K}$ of $\mathcal{K}^{\mathrm{b}}(\text{Flat-}R)$.

- If there are integers $m < n$ such that $\mathcal{D}^{>n} \subseteq \mathcal{Y} \subseteq \mathcal{D}^{>m}$ ("intermediacy") then $(\mathcal{X}, \mathcal{Y})$ is boundly generated.
- Any compactly generated t-structure gives rise to a compactly, and thus boundly, generated $\otimes$-structure.

Corollary (of a Theorem of Neeman '92)

- *Any localizing subcategory $\mathcal{L}$ of $\mathcal{D}(R)$ is a Bousfield class.*
- *Each localizing subcategory is of the form $\mathcal{L} = \{k(\mathfrak{p}) \mid \mathfrak{p} \in P\}^{\top_{\mathbb{Z}}} = \{\mathbf{R}\Gamma_{V(\mathfrak{p})} R_{\mathfrak{p}} \mid \mathfrak{p} \in P\}^{\top_{\mathbb{Z}}}$ for a subset $P \subseteq \operatorname{Spec} R$.*
- *In particular, there is a boundly generated $\otimes$-structure $(\mathcal{X}, \mathcal{L})$.*

Tor-pairs

Proposition

If R is a regular commutative noetherian ring then every $\otimes$-structure is boundly generated.

Definition

Let us call a hereditary Tor-pair $(\mathcal{F}, \mathcal{C})$ boundly generated if there is a subcategory $\mathcal{S}$ of right R-modules of finite flat dimension such that $\mathcal{C} = \operatorname{Ker} \operatorname{Tor}^R_{>0}(\mathcal{S}, -)$.

A hereditary Tor-pair $(\mathcal{F}, \mathcal{C})$ is boundly generated if and only if the induced $\otimes$-structure is boundly generated.

Example

Let R be a commutative noetherian ring with a dualizing complex which is not Gorenstein and such that there is a non-trivial Gorenstein flat R-module. Then the hereditary Tor-pair $(\mathcal{GF}, \mathcal{C})$ is not boundly generated (from either side!).

Thank you for your attention!