An Introduction to Model Theory for Representation Theory

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Definition

A (first-order) **language** $\mathcal{L}$ consists of

- 3 mutually disjoint sets: $\mathcal{R}$ the set of **relation symbols**, $\mathcal{F}$ the set of **function symbols** and $\mathcal{C}$ the set of **constant symbols**; and
- an **arity function** $\lambda : \mathcal{R} \cup \mathcal{F} \to \mathbb{N}$.

For any $Q \in \mathcal{R} \cup \mathcal{F}$, we will refer to $\lambda(Q)$ as the **arity** of Q.

Definition

An $\mathcal{L}$-**structure** $\mathcal{A}$ is a non-empty set A called the **domain** of $\mathcal{A}$ together with

(i) a subset $R^{\mathcal{A}}$ of $A^{\lambda(R)}$ for each $R \in \mathcal{R}$;

(ii) a function $F^{\mathcal{A}}$ from $A^{\lambda(F)} \to A$ for each $F \in \mathcal{F}$; and

(iii) an element $c^{\mathcal{A}} \in A$ for each $c \in \mathcal{C}$.

For $Q \in \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$, we call $Q^{\mathcal{A}}$ the **interpretation** in $\mathcal{A}$.

Example

Let $\mathcal{L} := \{\circ\}$ where $\circ$ is a binary function symbol. An $\mathcal{L}$-structure is a magma.

More examples of languages

- The language of abelian groups is $\mathcal{L}_{Ab} := \{0, +, -\}$ where 0 is a constant symbol, $+$ is a binary function symbol and $-$ is unary function symbol.
- The language of rings is $\mathcal{L}_{rings} := \{0, 1, +, -, \cdot\}$ where 0 is a constant symbol, $+$ is a binary function symbol and $+$ and $\cdot$ are binary function symbols and $-$ is a unary function symbol.
- For R a ring, the language of R-modules is $\mathcal{L}_R := \{0, +, (\cdot r)_{r \in R}\}$ where 0 is a constant symbol, $+$ is a binary function symbol and for each $r \in R$, $\cdot r$ is a unary function symbol.
- The language of ordered sets if $\mathcal{L}_{\leq} := \{\leq\}$ where $\leq$ is a binary relation symbol.

Definition

The **alphabet** of a language $\mathcal{L}$ is the relation, functions and constant symbols of $\mathcal{L}$ together with a set of logical symbols which are part of every language consisting of:

Connectives: $\{\rightarrow, \wedge, \vee, \neg\}$

Quantifiers: $\forall$ and $\exists$

The equality symbols $=$

Brackets ")" and "("

Comma: ","

A set of variables denoted $\mathrm{Vbl} := \{v_i \mid i \in \mathbb{N}\} \cup \{u, v, w, x, y, z\}$

Examples of $\mathcal{L}$-formulae:
The $\mathcal{L}_{rings}$-formula

$$(\forall v_2 \ v_1 \cdot v_2 = v_2 \cdot v_1)$$

defines the centre of a ring.
The $\mathcal{L}_{rings}$-formula

$$(\forall v_1 (v_1 = 0 \vee (\exists v_2 \ v_1 \cdot v_2 = 1)))$$

expresses that every non-zero element is invertible.

Define $\mathrm{tm}_0(\mathcal{L})$ to be the set $\mathrm{Vbl} \cup \mathcal{C}$. For all $k \in \mathbb{N}$, let $\mathrm{tm}_{k+1}(\mathcal{L})$ be

$\mathrm{tm}_k(\mathcal{L}) \cup \{F(t_1, t_2, \ldots, t_n) \mid F \in \mathcal{F},\ \lambda(F) = n \text{ and } t_1, \ldots, t_n \in \mathrm{tm}_k(\mathcal{L})\}$.

We define the set of $\mathcal{L}$-**terms** to be

$$\mathrm{tm}(\mathcal{L}) := \bigcup_{k \in \mathbb{N}_0} \mathrm{tm}_k(\mathcal{L}).$$

Let $\mathrm{Fml}_0(\mathcal{L})$ be the set of strings in the alphabet of $\mathcal{L}$ of the form

$$t_1 = t_2 \quad \text{or} \quad R(t_1, \ldots, t_n)$$

where $t_1, \ldots, t_n$ are $\mathcal{L}$-terms and $R \in \mathcal{R}$ has arity n.

For each $k \in \mathbb{N}_0$, let

$$\begin{aligned}\mathrm{Fml}_{k+1}(\mathcal{L}) := \mathrm{Fml}_k(\mathcal{L}) &\cup \{(\varphi \to \psi), (\varphi \wedge \psi), (\varphi \vee \psi), \neg\varphi \mid \varphi, \psi \in \mathrm{Fml}_k(\mathcal{L})\} \\ &\cup \{(\forall x \varphi), (\exists x \varphi) \mid \varphi \in \mathrm{Fml}_k(\mathcal{L}) \text{ and } x \in \mathrm{Vbl}\}.\end{aligned}$$

We define the set of $\mathcal{L}$-**formulae** to be

$$\mathrm{Fml}(\mathcal{L}) := \bigcup_{i \in \mathbb{N}_0} \mathrm{Fml}_i(\mathcal{L}).$$

Interpreting formulae in $\mathcal{L}$-structures

Let θ be an $\mathcal{L}$-formula and

$$(\exists v_i \underline{\qquad}) \quad \text{or} \quad (\forall v_i \underline{\qquad})$$

a substring of θ.

Free variables: An instance of a variable is **free** if it is not a bound instance or a quantifier instance. The **free variables** of a formula θ are those variable which occur as free instances.

To indicate that an $\mathcal{L}$-formula θ has free variables contained in a set $\{x_1, \ldots, x_n\}$ we write $\theta(x_1, \ldots, x_n)$.

Let M be an $\mathcal{L}$-structure, $\theta(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and $(m_1, \ldots, m_n) \in M$. We say that $\theta(m_1, \ldots, m_n)$ holds in M if the expression obtained by replacing every free instance of x_i in θ by m_i is true in M.

A **sentence** is a formula without free variables.

The $\mathcal{L} = \{\circ\}$-formula

$$(\forall x(\forall y(\forall z(x \circ y) \circ z = x \circ (y \circ z))))$$

is a sentence.

Interpreting formulae in $\mathcal{L}$-structures

Let θ be an $\mathcal{L}$-formula and

$$(\exists v_i \ \underline{\hspace{2cm}}) \quad \text{or} \quad (\forall v_i \ \underline{\hspace{2cm}})$$

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A **sentence** is a formula without free variables.

Let $\theta(v_1)$ be the $\mathcal{L}_{rings}$-formula

$$(\forall v_2 \ v_1 \cdot v_2 = v_2 \cdot v_1)$$

is not a sentence.

If Σ is a set of sentences then we say an $\mathcal{L}$-structure $\mathcal{M}$ is a **model** of Σ if all sentences in Σ hold in $\mathcal{M}$.

If X is a class of $\mathcal{L}$-structures then the **theory of** X, written $\mathrm{Th}(X)$, is the set of all $\mathcal{L}$-sentences which hold in all members of X.

A class X of $\mathcal{L}$-structures is **axiomatisable** if there is a set of sentences Σ such that the members of X are exactly the models of Σ. Examples

We say $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are **elementary equivalent**, and write $\mathcal{M} \equiv N$, if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure with domain M. A subset $N \subseteq M$ is an **elementary** if for all formulas $\theta(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n) \in N$ of elements of N, $\theta(a_1, \ldots, a_n)$ holds in N if and only if $\theta(a_1, \ldots, a_n)$ holds in M.

The First Theorem of Model Theory

The Compactness Theorem

Let $\mathcal{L}$ be a language.

(i) A set Σ of $\mathcal{L}$-sentences has a model if and only if every finite subset of Σ has a model.

(ii) Let Σ be a set of formulas with free variables $(v_1, v_2, \ldots)$.

If for every finite subset Σ' of Σ, there exists an $\mathcal{L}$-structure M and a tuple of elements $\overline{m}$ such that $\theta(\overline{m})$ holds in M

then there exists an $\mathcal{L}$-structure M and a tuple of elements $\overline{m}$ such that $\theta(\overline{m})$ holds in M for all $\theta \in \Sigma$.

Model Theory of Modules

Let R be a ring and $\mathcal{L}_R := \{0, +, (\cdot r)_{r \in R}\}$ the language of R-modules.

Every $\mathcal{L}_R$-term is equivalent (relative to $\mathrm{Th}(\text{Mod-}R)$) to one of the form

$$\sum_{i=1}^{n} x_i \cdot r_i$$

where each x_i is a variable and each $r_i \in R$ for $1 \leq i \leq n$.

Every atomic formulae is equivalent (relative to $\mathrm{Th}(\text{Mod-}R)$) to one of the form

$$\sum_{i=1}^{n} x_i \cdot r_i = 0$$

(or $0 = 0$) where $x_1, \ldots, x_n$ are variables.

A (right) **pp-n-formula** (over R) is a formula $\varphi(\overline{x})$ of the form

$$\exists y_1, \ldots, y_m \bigwedge_{i=1}^{l} \sum_{j=1}^{n} x_j r_{ij} + \sum_{k=1}^{m} y_k s_{ik} = 0$$

where $r_{ij}, s_{ik} \in R$ and $\overline{x} = (x_1, \ldots, x_n)$.

For $M \in$ Mod-R, we write $\varphi(M)$ for the solution set of φ in M.

- If $m_1, m_2 \in \varphi(M)$ then $m_1 + m_2 \in \varphi(M)$.
- If $f : M \to L \in$ Mod-R and $\overline{m} \in \varphi(M)$ then $f(m) \in \varphi(L)$.
- Let $N_i \in$ Mod-R for $i \in I$. Then

$$\varphi(\oplus_i N_i) = \oplus_i \varphi(N_i).$$

A (right) **pp-n-formula** (over R) is a formula $\varphi(\overline{x})$ of the form

$$\exists y_1, \ldots, y_m \bigwedge_{i=1}^{l} \sum_{j=1}^{n} x_j r_{ij} + \sum_{k=1}^{m} y_k s_{ik} = 0$$

where $r_{ij}, s_{ik} \in R$ and $\overline{x} = (x_1, \ldots, x_n)$.

We write pp_R^n for the set of (right) pp-n-formulae over R where we identify pp-n-formulae φ, ψ if $\varphi(M) = \psi(M)$ for all $M \in \text{Mod-}R$.

pp_R^n is a bounded modular lattice when equipped with the order defined by

$$\psi \leq \varphi \text{ if and only if } \psi(M) \subseteq \varphi(M) \text{ for all } M \in \text{Mod-}R.$$

We write $\varphi + \psi$ for the join (l.u.b) and $\varphi \wedge \psi$ for the meet (g.l.b) in pp_R^n.

For all $M \in \text{Mod-}R$,

$$(\varphi + \psi)(M) = \varphi(M) + \psi(M) \text{ and } (\varphi \wedge \psi)(M) = \varphi(M) \cap \psi(M).$$

Modular: $a \leq b$ implies $a + (z \wedge b) = (a + z) \wedge b$.

Let $\varphi, \psi \in \mathrm{pp}_R^n$ with $\psi \leq \varphi$ and let $b \in \mathbb{N}$. We write

$$|\varphi/\psi| \geq b$$

for the $\mathcal{L}_R$-sentence which expresses in all R-modules M that

$$|\varphi(M)/\psi(M)| \geq b.$$

Suppose $n = 1$. Then we may take $|\varphi/\psi| \geq b$ to be

$$\exists z_1, \ldots, z_b \bigwedge_{i=1}^{b} \varphi(z_i) \wedge \bigwedge_{i<j} \neg\psi(z_i - z_j).$$

The Baur-Monk Theorem

Every formula in the language of R-modules is equivalent to a boolean combination of pp-formulae and sentences of the form

$$|\varphi/\psi| \geq b$$

where $b \in \mathbb{N}$ and φ, ψ are pp-1-formulae such that $\psi \leq \varphi$.

Corollary

Let $M, N \in \text{Mod-}R$. *Then*

$$M \equiv N \quad \textit{if and only if} \quad |\varphi(M)/\psi(M)| = |\varphi(N)/\psi(N)|,$$

when either is finite, for all $\varphi \geq \psi \in \mathrm{pp}^1_R$.

Examples

- Let N_i be a collection of R-modules for $i \in I$, then

$$\bigoplus_{i \in I} N_i \equiv \prod_{i \in I} N_i.$$

- If R is an algebra over an infinite field k then for all $M \in \text{Mod-}R$, $M^2 \equiv M$.
- As $\mathbb{Z}$-modules, $\mathbb{Z} \oplus \mathbb{Q} \equiv \mathbb{Z}$.

Corollary

A submodule $L \subseteq M$ *is elementary if and only if* $L \equiv M$ *and, for all pp-formulae* φ, $\varphi(L) = \varphi(M) \cap L$.

Purity

Definition

An embedding $f : M \hookrightarrow N$ is **pure** if for all pp-1-formulae φ,

$$\varphi(N) \cap f(M) = f(\varphi(M)).$$

An R-module M is **pure-injective** if every pure-embedding $M \hookrightarrow N$ splits.

Definition

An R-module M is **algebraically compact** if any system of (inhomogeneous) linear equations over R, in arbitrary many variables, which is finitely solvable in M, has a solution in M.

Equivalently, an R-module M is algebraically compact if for any $n \in \mathbb{N}$, the collection of sets of the form $\overline{a} + \varphi(M)$ where $\varphi \in \mathrm{pp}_R^n$ and $\overline{a}$ is an n-tuples in M, has the finite intersection property.

Theorem

An R-module is algebraically compact if and only if it is pure-injective.

Modules up to Elementary Equivalence

Write pinj_R for the set of indecomposable pure-injective R-modules.

Theorem (Ziegler)

For every R-module M, there exists $N_i \in \mathrm{pinj}_R$ such that M is elementary equivalent to $\bigoplus_{i\in I} N_i$.

Theorem

Let $N, M \in \mathrm{pinj}_R$. Then $N \equiv M$ if and only if for all $\varphi \geq \psi \in \mathrm{pp}_R^1$

$$|\varphi(N)/\psi(N)| > 1 \Leftrightarrow |\varphi(M)/\psi(M)| > 1.$$

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Reminders

A (right) **pp-n-formula** (over R) is a formula $\varphi(\overline{x})$ of the form

$$\exists y_1, \ldots, y_m \bigwedge_{i=1}^{l} \sum_{j=1}^{n} x_j r_{ij} + \sum_{k=1}^{m} y_k s_{ik} = 0$$

where $r_{ij}, s_{ik} \in R$ and $\overline{x} = (x_1, \ldots, x_n)$.

We write pp_R^n for the set of (right) pp-n-formulae over R. This set is a lattice when ordered by $\varphi \geq \psi$ if and only if $\varphi(M) \supseteq \psi(M)$ for all $M \in$ Mod-R.

Corollary of the Baur-Monk Theorem

Let $M, N \in$ Mod-R. Then

$$M \equiv N \quad \text{if and only if} \quad |\varphi(M)/\psi(M)| = |\varphi(N)/\psi(N)|,$$

when either is finite, for all $\varphi \geq \psi \in \mathrm{pp}_R^1$.

Pp-types

The **pp-type**, $\mathrm{pp}^M(\overline{m})$, of an n-tuple $\overline{m}$ in an R-module M is the set of all $\varphi \in \mathrm{pp}_R^n$ such that $\overline{m} \in \varphi(M)$.

Observations:

- $\mathrm{pp}^M(\overline{m})$ is non-empty.
- If $\psi \leq \varphi$ in pp_R^n and $\psi \in \mathrm{pp}^M(\overline{m})$ then $\varphi \in \mathrm{pp}^M(\overline{m})$.
- If $\varphi_1, \ldots, \varphi_n \in \mathrm{pp}^M(\overline{m})$ then $\bigwedge_{i=1}^n \varphi_i \in \mathrm{pp}^M(\overline{m})$.

Therefore $\mathrm{pp}^M(\overline{m})$ is a filter in pp_R^n.

Free realisations

Definition (Prest)

Let φ be a pp-n-formula. A **free realisation** of φ is a pair $(M, \overline{m})$ where M is a finitely presented R-module and $\overline{m}$ is an n-tuple of elements of M such that for all $\sigma \in \mathrm{pp}_R^n$, $\sigma \geq \varphi$ if and only if $\overline{m} \in \sigma(M)$.

Theorem

(i) *Let M be a finitely presented R-module and $\overline{m}$ an n-tuple from M. Then $\mathrm{pp}^M(\overline{m})$ is generated as a filter by some $\varphi \in \mathrm{pp}_R^n$ i.e. $(M, \overline{m})$ is a free realisation of φ.*

(ii) *Let $\varphi \in \mathrm{pp}_R^n$. There exists a finitely presented R-module and an n-tuple $\overline{m}$ such that $(M, \overline{m})$ is a free realisation of φ.*

The Compactness Theorem

Let R be a ring and let $\mathcal{L}_R := (0, +, (\cdot r)_{r \in R})$ be the language of R-modules.

(ii) Let Σ be a set of $\mathcal{L}_R$-formulas with free variables $(v_1, v_2, \ldots, v_n)$.

If for every finite subset Σ' of Σ, there exists an R-modules M and a tuple of elements $\overline{m} \in M$ such that $\theta(\overline{m})$ holds in M for all $\theta \in \Sigma'$

then there exists an R-module M and a tuple of elements $\overline{m} \in M$ such that $\theta(\overline{m})$ holds in M for all $\theta \in \Sigma$.

Application of the Compactness Theorem

Proposition

Any filter p in pp_R^n is the pp-type of an element of some R-module.

Proof (n=1).

Let $\Sigma := \{\varphi(x) \mid \varphi \in \mathrm{pp}_R^1 \text{ with } \varphi \in p\} \cup \{\neg\psi(x) \mid \psi \in \mathrm{pp}_R^1 \text{ with } \psi \notin p\}$.
By the Compactness Theorem, it is enough to show that for all

$$\varphi_1, \dots, \varphi_k \in p \text{ and } \psi_1, \dots, \psi_l \notin p,$$

there exists $M \in \text{Mod-}R$ and $m \in M$ such that

$$m \in \varphi_i(M) \text{ for all } 1 \le i \le k \text{ and } m \notin \psi_i(M) \text{ for all } 1 \le i \le l.$$

Since p is a filter, $\varphi := \varphi_1 \wedge \ldots \wedge \varphi_k \in p$ and $\psi_i \ngeq \varphi$ for $1 \le i \le l$.

Let (M, m) be a free realisation of φ. Then, for all $\sigma \in \mathrm{pp}_R^1$,

$$m \in \sigma(M) \text{ if and only if } \sigma \ge \varphi.$$

Therefore $m \in \varphi_i(M)$ for all $1 \le i \le k$ and $m \notin \psi_i(M)$ for $1 \le i \le l$. □

Modules up to Elementary Equivalence

Definition

An embedding $f : M \hookrightarrow N$ is **pure** if for all pp-1-formulae φ,

$$\varphi(N) \cap f(M) = f(\varphi(M)).$$

An R-module M is **pure-injective** if every pure-embedding $M \hookrightarrow N$ splits.

Write pinj_R for the set of indecomposable pure-injective R-modules.

Theorem (Ziegler)

For every R-module M, there exists $N_i \in \mathrm{pinj}_R$ such that M is elementary equivalent to $\bigoplus_{i \in I} N_i$.

Theorem

Let $N, M \in \mathrm{pinj}_R$. Then $N \equiv M$ if and only if for all $\varphi \geq \psi \in \mathrm{pp}_R^1$

$$|\varphi(N)/\psi(N)| > 1 \Leftrightarrow |\varphi(M)/\psi(M)| > 1.$$

The Ziegler Spectrum

The **Ziegler spectrum**, Zg_R, of R is the topological space with set of points pinj_R and basis of open sets

$$(\varphi/\psi) := \{N \in \mathrm{pinj}_R \mid |\varphi(N)/\psi(N)| > 1\}$$

where $\varphi \geq \psi \in \mathrm{pp}_R^1$.

Definable subcategories of Mod-R are in bijective correspondence with the closed subsets of the Ziegler spectrum via the map

$$\mathcal{D} \mapsto \mathcal{D} \cap \mathrm{Zg}_R.$$

Properties of Zg_R

- $N, M \in \mathrm{pinj}_R$ are topologically indistinguishable if and only if they are elementary equivalent.
- The sets (φ/ψ) are compact. In particular, $\mathrm{Zg}_R = (x = x/x = 0)$ is compact.
- Zg_R is often not T_0 and very rarely has the property that the intersection of two compact open sets is compact.

An example

The indecomposable pure-injective modules over $\mathbb{Z}$ are

- the finite modules $\mathbb{Z}/p^n\mathbb{Z}$ for p prime and $n \in \mathbb{N}$,
- the p-adic integers $\overline{\mathbb{Z}_{(p)}}$ for p prime,
- the p-Prüfer group $\mathbb{Z}_{p^\infty}$ for p prime, and
- $\mathbb{Q}$, the field of fractions of $\mathbb{Z}$.

A subset $\mathcal{C}$ of $\mathrm{Zg}_{\mathbb{Z}}$ is closed if and only if the following conditions hold:

- If $\mathcal{C}$ contains infinitely many finite modules then $\mathcal{C}$ contains $\mathbb{Q}$.
- If $\mathcal{C}$ contains infinitely many $\mathbb{Z}/p^n\mathbb{Z}$ for fixed prime p then $\mathcal{C}$ contains $\overline{\mathbb{Z}_{(p)}}$ and $\mathbb{Z}_{p^\infty}$.
- If $\mathcal{C}$ contains $\overline{\mathbb{Z}_{(p)}}$ or $\mathbb{Z}_{p^\infty}$ then $\mathcal{C}$ contains $\mathbb{Q}$.

Soberness and the Baire property

Definition

We say a topological space has the **Baire property** if every countable intersection of open and dense subsets is dense.

Theorem (Herzog - reinterpreted)

Every closed subset of Zg_R has the Baire property.

Definition

Let $\mathcal{T}$ be a topological space.

1. A subset S of $\mathcal{T}$ is irreducible if whenever $S \subseteq \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed subsets then $S \subseteq \mathcal{C}_1$ or $S \subseteq \mathcal{C}_2$.
2. $\mathcal{T}$ is sober if every non-empty irreducible closed set is the closure of a point.

Corollary (Herzog)

If Zg_R has a countable basis then Zg_R is sober. In particular, if R is countable then Zg_R is sober.

Remark

If $\mathcal{T}$ is a topological space with a countable basis of open sets such that every closed subset of $\mathcal{T}$ has the Baire property then $\mathcal{T}$ is sober.

Proof.

A topological space $\mathcal{V}$ is the closure of a point x if and only if x is a member of every non-empty open subset of x.

Let $\mathcal{V}$ be an irreducible topological space. Every non-empty open subset $\mathcal{U}$ of $\mathcal{V}$ is dense because if $\mathcal{U} \cap \mathcal{U}' = \emptyset$ for $\mathcal{U}' \subseteq \mathcal{V}$ open then

$$(\mathcal{V} \backslash \mathcal{U}) \cup (\mathcal{V} \backslash \mathcal{U}') = \mathcal{V}.$$

Therefore if $\mathcal{V}$ has the Baire property and a countable basis of open sets then $\mathcal{V}$ is the closure of a point. □

Soberness and Duality

We write ${}_R\mathrm{pp}^n$ for the lattice of left pp-n-formulae and ${}_R\mathrm{Zg}$ for the left Ziegler spectrum of R.

Duality for pp-formulae (Prest)
For each $n \in \mathbb{N}$, there is an order anti-isomorphism

$$D : \mathrm{pp}_R^n \to {}_R\mathrm{pp}^n.$$

For any topological space $\mathcal{T}$, the open subsets of $\mathcal{T}$ are a (complete) lattice, denoted $\mathcal{O}(\mathcal{T})$, under inclusion.

Duality for Ziegler Spectra (Herzog)
The map on basic open subsets of Zg_R, defined by

$$(\varphi/\psi) \mapsto (D\psi/D\varphi)\,,$$

induces a lattice isomorphism from $\mathcal{O}(\mathrm{Zg}_R)$ to $\mathcal{O}({}_R\mathrm{Zg})$.

This implies that if Zg_R and ${}_R\mathrm{Zg}$ are sober there is a homeomorphism

$$\mathrm{Zg}_R/T_0 \to {}_R\mathrm{Zg}/T_0.$$

Is the Ziegler Spectrum always sober?

We don't know.

Remark

If an irreducible closed set $\mathcal{C}$ contains a point x which is isolated in $\mathcal{C}$ then $\mathcal{C}$ is equal to the closure of x. Hence, if a topological space has Cantor-Bendixson rank then it is sober.

Theorem (Gregory-Puninski)

If R is a Prüfer ring then Zg_R is sober. If R is a (uni)serial ring then Zg_R is sober.

Remark/Theorem

If $\mathcal{A}$ is a tubular algebra then $\mathrm{Zg}_{\mathcal{A}}$ is sober.

A ring R is von Neumann regular if for all $a \in R$ there exists $x \in R$ such that $a = axa$.

Lemma

If R is a von Neumann regular ring then Zg_R is sober if and only if for all prime ideals P, there exists an irreducible right ideal I such that P is the largest two-sided ideal contained in I.

–Thank you–

An example

Let $\aleph_1$ be the set of all countable ordinals. The subsets $(\alpha, \aleph_1) \subseteq \aleph_1$ are the open sets of a topology. The closed sets are $[0, \alpha]$ for $\alpha \in \aleph_1$ and $\aleph_1$. They are all irreducible.

The sets $(\alpha, \aleph_1)$ are compact because

$$(\alpha, \aleph_1) \subseteq \bigcup_{\beta \in I} (\beta, \aleph_1)$$

if and only if $\beta \leq \alpha$ if and only if $(\alpha, \aleph_1) \subseteq (\beta, \aleph_1)$ for some $\beta \in I$.

Every closed subset of $\aleph_1$ has the Baire property but $\aleph_1$ is not the closure of a point because $\bigcap_{\alpha \in \aleph_1} (\alpha, \aleph_1) = \emptyset$.

Fun with Ziegler spectra of Artin algebras

Let $\mathcal{A}$ be an Artin algebra.

The indecomposable finite length modules are exactly the isolated points in $\mathrm{Zg}_{\mathcal{A}}$. The set of indecomposable finite length modules is dense in $\mathrm{Zg}_{\mathcal{A}}$.

Theorem

If $\mathcal{A}$ is not finite representation type then there is a infinite length indecomposable pure-injective $\mathcal{A}$-module.

Theorem (Herzog)

If there are infinitely many indecomposable modules of endolength n, then there is an infinite length indecomposable module of endolength $\leq n$.

Proof.

The set of indecomposable modules of endolength $\leq n$ is a closed (and hence compact) subset of $\mathrm{Zg}_{\mathcal{A}}$. Since this set contains infinitely many points, it must contain a non-isolated point. $\square$