Some aspects of (strong) generation in module categories

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A broad motivating question

Given a triangulated category, or subcategory of an abelian category, when can every object be built from a single object by repeatedly taking cones (respectively extensions in the abelian setting), retracts, and suspensions (in the triangulated setting) ?

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5. *We have an ascending chain of subcategories* $\langle \mathcal{C} \rangle_0 \subseteq \langle \mathcal{C} \rangle_1 \subseteq \langle \mathcal{C} \rangle_2 \ldots,$ *and their union* $\langle \mathcal{C} \rangle := \cup_{n \geq 0} \langle \mathcal{C} \rangle_n$ *is the smallest thick subcategory of* $\mathcal{T}$ *containing* $\mathcal{C}$.

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Some notions of generation

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- $\mathcal{T}$ is said to have a strong generator if there exists an object G in $\mathcal{T}$ such that $\mathcal{T} = \langle G \rangle_n$.
- Moreover, if $\mathcal{T}$ does admit a strong generator, the Rouquier dimension of $\mathcal{T}$ is defined to be

$$\dim \mathcal{T} := \inf\{n \in \mathbb{N} \mid \mathcal{T} = \langle G \rangle_{n+1} \text{ for some } G \text{ in } \mathcal{T}\}$$

If $\mathcal{T}$ does not have a strong generator, then we set $\dim \mathcal{T}$ to be ∞.

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Both of these quantities can potentially be infinite, and we have the obvious inequality: $\text{size}\, \mathcal{C} \leq \text{rank}\, \mathcal{C}$.

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- By definition, $\mathcal{C}$ has a strong generator if and only if $\text{size}\,\Omega^t\mathcal{C} < \infty$ for some $t \in \mathbb{N}$.
- $\mathcal{C}$ is also said to admit a classical generator if $\mathcal{C} = \text{thick}(G)$ for some G in $\mathcal{A}$.

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In general, finding upper bounds for Rouquier dimension can be quite difficult ...

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2. Since for any complex X in $\mathrm{D}^b(\operatorname{mod} R)$, there is an exact triangle $Z(X) \to X \to B(X)[1] \rightsquigarrow$, hence for any s, one also obtains

$$\dim \mathrm{D}^b(\operatorname{mod} R) \leq 2\,(s + \operatorname{size} \Omega^s(\operatorname{mod} R) + 1) - 1$$

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3. In particular, if $\operatorname{mod} R$ has a strong generator, then so does $\mathrm{D}^b(\operatorname{mod} R)$. The converse is not known.

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The proof goes via the notion of cohomology annihilator $\operatorname{ca}(R) := \cup_{n \geq 0} \cap_{M,N \in \operatorname{mod} R} \operatorname{ann} \operatorname{Ext}_R^{i \geq n}(M, N)$, and in-fact, in the above equivalent conditions, we can also add:

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"$\mathrm{ca}(R/\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$".

Comparing rank and size

Recall that size $\mathcal{C}$ is defined in terms of $\mathcal{C} \subseteq |G|_n$, whereas rank $\mathcal{C}$ is defined in terms of $\mathcal{C} = |G|_n$, so clearly size $\mathcal{C} \leq$ rank $\mathcal{C}$, and, trying to reverse this inequality in any sort of way can be quite hopeless ... however, we do have

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- Let R be Cohen–Macaulay, admitting a dualizing module. Then for all $n \geq 0$, it holds that

$$\operatorname{rank} \mathrm{CM}(R) \leq (n + \dim R + 1)(\operatorname{size} \Omega^n(\operatorname{mod} R) + 1) - 1$$

  where $\mathrm{CM}(R)$ is the category of maximal Cohen–Macaulay R-modules.

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  where $\mathrm{CM}(R)$ is the category of maximal Cohen–Macaulay R-modules. In particular, mod R has a strong generator if and only if $\mathrm{CM}(R)$ has finite rank. (Dey-Lank-Takahashi)

A little bit more about how generation is often controlled by closed integral subschemes

Theorem (Dey-Lank-Takahashi)

- *If* $\mathrm{Ass}(R) = \mathrm{Min}(R)$*, then for every* $n \geq 0$*, we have*
  $\operatorname{size} \Omega^{n+1}(\operatorname{mod} R) \leq \left(\sum_{\mathfrak{p} \in \mathrm{Min}(R)} \ell\ell(R_{\mathfrak{p}})\right)(n+1)\left(1 + \sup_{\mathfrak{p} \in \mathrm{Min}(R)} \operatorname{size} \Omega^{n}(\operatorname{mod}(R/\mathfrak{p}))\right) - 1$
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- *When* R *is Cohen–Macaulay, one-dimensional, and admits a dualizing module, then*

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Moreover, if any of these conditions are satisfied, then $\mathrm{D}^b_{coh}(Y)$ has a classical generator for any closed subscheme Y of X.

This generalizes, to schemes, the same result proved previously by Iyengar-Takahashi, in the affine case.

Thank you for listening :)