Σ-pure-injectivity: characterisations and classifications from representation theory

Purity, Approximation Theory and Spectra (Cetraro)

(with Crawley-Boevey) Σ-pure-injective modules for string algebras and linear relations J. Alg. 513 (2018), 177-189

Characterisations of Σ-pure-injectivity in triangulated categories and applications to endocoperfect objects Fund. Math. 261 (2023), 133-155

Overview

Prelude on Σ-pure-injective modules

Annihilator subobjects in Grothendiek categories

Triangulated categories and languages

Homotopy categories for gentle algebras

Σ-pure-injective objects in triangulated categories

Prelude on Σ-pure-injective modules

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Setup:

Prelude on Σ-pure-injective modules

Setup: M a left module

Prelude on Σ-pure-injective modules

Setup: M a left module over a ring Λ

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Theorem [B-T–Crawley-Boevey, '18]: if Λ is a *string algebra*, and if M is Σ-pure-injective, then M is a coproduct of *string modules* and *band modules*

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Annihilator subobjects in Grothendiek categories

Setup: $\mathcal{A}$ Grothendieck and locally coherent
$\mathcal{A}^p$ subcategory of finitely presented objects
So $\mathcal{A}^p$ abelian, $\mathcal{A}$ generated by a set $\mathcal{G}$ of objects in $\mathcal{A}^p$

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$$\mathrm{ann}_S(A) = \{\alpha \in \mathrm{Hom}_{\mathcal{T}}(A, C) \mid f_C(1_C)\alpha = 0 \text{ for all } f \in S\}$$

Corollary [Dung–Garcia, '94; Krause, 00']: if $M \in \mathcal{T}$ then TFAE

(0) M is Σ-*pure-injective*, i.e. $M^{(I)}$ is pure-injective for any I

(1) $\boldsymbol{Y}(M)$ is Σ-injective, i.e. $\boldsymbol{Y}(M^{(I)})$ is injective for any I

(2) If $C \in \mathcal{T}^c$ then $\boldsymbol{Y}(C)$ has a.c.c. on $\boldsymbol{Y}(M)$-annihilators

Triangulated categories and purity: sorted language

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Triangulated categories and purity: solution sets

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Homotopy categories of complexes of projectives

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Setup:

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