A PERSPECTIVE ON DERIVED NON-COMMUTATIVE GEOMETRY II

4. Homotopical epimorphisms

As we have seen so far it is impossible to extend the spectrum functor from commutative to non-commutative rings. But what goes wrong? More precisely, if we try to employ the same abstract definition in the non-commutative case, what happens?

Roughly speaking, everything that can go wrong, goes wrong.

Recall that a Zariski localization of commutative rings is a morphism $A \to B$ that is a flat epimorphism of finite presentation.

All these concepts make sense for non-commutative rings but this class of morphisms cannot be used to define a topology. Indeed, for a class of morphisms to define a topology

$$\{ \mathrm{Spec}(B_i) \to \mathrm{Spec}(A) \}$$

They need to be stable by fiber products, that correspond to push out of algebras. In the commutative case, pushouts of algebras are given by tensor products and the claim is that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "f'"] & B \otimes_A C \end{tikzcd}$$

If f is a Zariski localization, then f' is. But in the category of rings the pushouts are given by the free product of algebras and this functor does not preserve flatness.

There is another complication, the notion of finite presentation of a morphism depends on the category in which it is considered. More specifically, if $f: A \to B$ is a morphism of commutative ring it may happen that it is of finite presentation as a morphism of CRings but not as a presentation as a ...

morphism of Rings.

The only concept that does not give problems is that of epimorphism but we cannot use it to define a topology because there are epimorphisms of commutative rings that do not correspond to open immersions.

So, to simplify the subject let us work in a more complicated setting. Recently, derived geometry has been developed as an extension of classical geometry. Roughly speaking, we replace rings with simplicial rings up to quasi-isomorphism, and all the notions with corresponding derived/homotopical notions.

Since finite presentation is a bit problematic we get rid of it. So we would like to understand better flat epimorphisms of commutative rings.

Recall the following criterion.

<u>Lemma</u>

... the ... morphism

Lemma

Suppose that $\mathcal{C}$ is a category with pushouts, then a morphism $X \to Y$ is an epimorphism $\Leftrightarrow$ the canonical map $Y \amalg_X Y \to Y$ is an isomorphism.

In the category of commutative rings the pushout is given by the tensor product, thus $A \to B$ is an epimorphism iff

$$B \otimes_A B \xrightarrow{\cong} B.$$

We can ask for the same condition to hold in the derived setting, i.e. we can ask for

$$B \otimes^L_A B \xrightarrow{\cong} B$$

and call this a <u>homological epimorphism</u>.

Thm

A morphism $A \to B$ between commutative (Noetherian) rings is a flat epimorphism $\Leftrightarrow$ it is a homological epimorphism.

This result is important because it says that the class of morphisms used to define the Zariski topology of algebraic geometry essentially reduces to the class of homological epimorphisms.

Let us denote by sCRings the class of simplicial commutative rings, $HCRings = Ho(sCRings)$ its homotopy category with respect to the standard model structure, $HCAff = HCRings^{op}$ the category of derived affine schemes.
Similarly we denote $HRings = Ho(sRings)$ and $HAff = HRings^{op}$.

<u>Thm</u>
The following class of morphisms defines a topology on HCAff:
finite families $\{A \to B_i\}$ of homotopical epimorphisms such that the "restriction functors"

$$(-)\otimes^{L}_{A} B_i : HMod_A \to HMod_{B_i}$$

form a conservative family of functors.

<u>Remark</u>:
① If A is in degree zero then also B_i are, so the topology restricts to CRings.
② All covers for the Zariski topology are covers in this sense but there are more given by infinite intersections of Zariski opens.
③ The condition on conservativity of the restrictions is

③ The condition on conservativity of the restrictions is equivalent to $\coprod \mathrm{Spec}(B_i) \to \mathrm{Spec}(A)$ being surjective.

What if we now consider non-commutative simplicial rings? In this case the condition of being a homotopical epimorphism is given by

$$B *^{\mathbb{L}}_A B \xrightarrow{\cong} B$$

for a morphism of simplicial rings $A \to B$.

Thm (Chuang-Lazarev, 2019)
For a morphism of simplicial rings the following two conditions are equivalent

$$B *^{\mathbb{L}}_A B \xrightarrow{\cong} B \iff B \otimes^{\mathbb{L}}_A B \xrightarrow{\cong} B.$$

Remark:
① The meaning of this theorem is that if we try to define a topology on HRings using homotopical epimorphisms, then the resulting theory is compatible with the Zariski topology of algebraic geometry.

② This theorem can be generalized to any closed symmetric ∞-category.

③ Homotopical epimorphisms are obviously stable by homotopical pushouts, i.e. by $(-) *^{\mathbb{L}}_{A} B$. Therefore they are suitable to define a topology on dAff.

④ We have seen that if A is commutative and in degree 0 then B is in degree 0 as well. But this does not happen for non-commutative rings! Indeed, there are examples of homotopical epimorphisms $A \to B$ where B is not in degree 0. Roughly speaking, any localization of A at a subset $S \subseteq A$ that does not satisfy the Ore conditions gives such a localization.

5. The homotopical Zariski topology:

Let us finally define the Zariski topology on HRings.

<u>Definition</u> (Conjecturally equivalent to the actual definition)
A morphism $A \to B$ is called <u>Zariski localization</u> if it is a homotopical epimorphism (we do not impose finite presentation!).

A finite family of morphisms $\{A \to B_i\}$ define a cover on dAff if these are Zariski localizations and the family of dAff if they are Zariski localizations and the family of functors

$$\left\{ B_i \otimes^{\mathbb{L}}_A (-) : \mathrm{HMod}_A \longrightarrow \mathrm{HMod}_{B_i} \right\}$$

is faithful.

There is a lot to comment here.

First let us notice the following characterization of homotopical epimorphisms.

<u>Thm</u>

Let $f : A \to B$ be a morphism in HRings. Then the following are equivalent

1. f is a homotopical epimorphism,
2. The restriction of scalar functor $f_* : \mathrm{HMod}_B \to \mathrm{HMod}_A$ is fully faithful,
3. $\mathbb{L} f^*$ is a <u>smashing localization</u> of HMod_A, i.e. f_* admits both adjoints $\Leftrightarrow$ $\mathrm{Ker}(\mathbb{L} f^*)$ is a smashing subcategory of HMod_A.

The characterization of 3. is interesting because it permits to deduce the following.

<u>Cor</u>:

The family of smashing localizations is essentially small and form a poset by the relation of inclusion of subcategories.

So if we fix an object $A \in \mathrm{HRings}$, or equivalently $\mathrm{Spec}(A) \in \mathrm{dAff}$, and consider the family of homotopical epimorphisms starting at A we obtain a poset. Let us call it $\mathrm{Spec}^{NC}(A)$

The claim is the following.

<u>thm</u>

The Zariski Topology is well defined. Therefore it induces a Topology on $\mathrm{Spec}^{NC}(A)$ and every map $A \to B$ induces a continuous map $\mathrm{Spec}^{NC}(B) \longrightarrow \mathrm{Spec}^{NC}(A)$.

The proof requires some work. Let us look at examples instead.

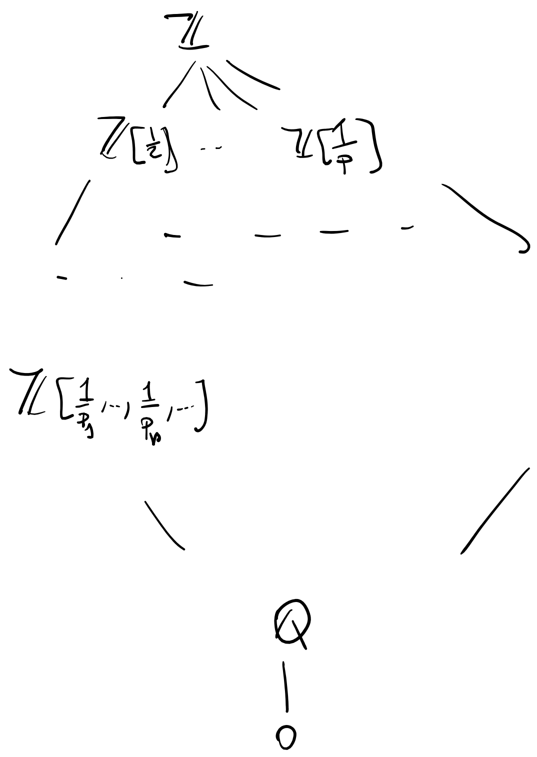
<u>Examples</u>

Examples

⓪ If k is a field then $\mathrm{Spec}^{NC}(k) = \mathrm{Spec}(k)$.

① If R is a DVR then $\mathrm{Spec}^{NC}(R) = \mathrm{Spec}(R)$.

② $\mathrm{Spec}^{NC}(\mathbb{Z})$ is bigger than $\mathrm{Spec}(\mathbb{Z})$ because in the former we are considering all flat epimorphism of $\mathbb{Z}$ as a base for the topology. So the lattice is



The lattice is distributive and the cover relation is the same as the join relation in the lattice.

The result is a space that is the Stone-Cech compactification of $\mathbb{N}$ plus an extra specialization point.

③ For any commutative Noetherian ring you can draw a similar picture

③ For any commutative Noetherian ring you can draw a similar picture using generalization closed subsets of $\mathrm{Spec}(R)$.

④ Let us consider the simplest non-trivial ($\mathrm{Spec}^{nc}(M_n(k)) = \mathrm{Spec}(k)$) non-commutative example. The path algebra of the A_2 quiver$/k$.
Let us call this algebra A and remark that it is equivalent to the algebra of 2×2 lower triangular matrices.
It is known that as a left module

$$A = P_1 \oplus P_2$$

for two projective indecomposable and that there is a third non-projective indecomposable module S_2 and these form all the finite dimensional indecomposable left A-modules. It is known that these module correspond to smashing localizations of $H\mathrm{Mod}_A$, therefore and we have a lattice

$$\begin{tikzcd}
 & A & \\
\langle P_1 \rangle \arrow[ur, dash, "i_1"] & \langle P_2 \rangle \arrow[u, dash, "i_2"'] & \langle S_2 \rangle \arrow[ul, dash, "i_3"'] \\
 & 0 \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

of homotopical epimorphisms. This lattice is not distributive, therefore the Zariski Topology does not coincide with the join relation of the lattice.

Remark:

When we defined the Zariski Topology for commutative rings we asked for a family of Zariski localizations to be conservative and for such functors being conservative is equivalent to being faithful. But in general faithful $\Rightarrow$ conservative and faithfulness is strictly stronger. In the picture above any of the families (i_1, i_2), (i_1, i_3), (i_2, i_3), (i_1, i_2, i_3) is conservative but only the latter is faithful!

So, roughly speaking, faithful $\Leftrightarrow \coprod \mathrm{Spec}^{hc}(B_i) \to \mathrm{Spec}(A)$ is surjective, but conservative no! And the former is what we want from a cover.

Thus the Grothendieck Topology here has the following covers

$$\{ \mathrm{id}_0 \}, \{ \mathrm{id}_{P_1} \}, \{ \mathrm{id}_{P_2} \}, \{ \mathrm{id}_{S_2} \}, \{ (\bar{\iota}_1, \bar{\iota}_2, \bar{\iota}_3) \}, \{ \mathrm{id}_A \}.$$

It is possible to check that the associated topological space to this small site is the discrete topological space on three points. Thus $\mathrm{Spec}^{NC}(A) = \{ *, *, * \}$

(5) The previous example should give an understanding of what happens in many more non-commutative examples.

But this is not the end. We would like to put a structure sheaf on $\mathrm{Spec}^{NC}(A)$ to deduce Gelfand's duality. And this is where things get interesting.

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6. the structure sheaf

In geometry the dualities between spaces and algebras are not realized as simple dualities with topological spaces but the topological spaces are enriched with the structure of ringed or locally ringed spaces to carry extra information.

For example, the duality of algebraic geometry is

$$\{ \text{Commutative rings} \} \longleftrightarrow \left\{ \begin{array}{l} \text{locally ringed spaces of the} \\ \text{form } (\mathrm{Spec}(A), \mathcal{O}), \text{ with} \\ \text{morphisms of locally ringed spaces} \end{array} \right\}$$

Here $\mathcal{O}$ is the structure sheaf on $\mathrm{Spec}(A)$ determined by the requirement that for all Zariski localizations

$$V = (A \to B)$$

the value $\mathcal{O}(V) = B$. It is a classical result that this formula gives a sheaf.

Let us employ this idea to our example of the path algebra of the quiver A_2. It is possible to check that the three smashing localizations $\langle P_1 \rangle, \langle P_2 \rangle, \langle S_2 \rangle$ correspond to three homotopical epimorphisms $A \to k$, $A \to M_2(k)$, $A \to k$, that as left modules are respectively P_1, P_2^2, S_2.

Now, since $\mathrm{Spec}^{NC}(A)$ is the discrete topological space on three points, for a sheaf on it $\mathcal{F}$, we must have

$$\mathcal{F}(\mathrm{Spec}^{NC}(A)) = \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \oplus \mathcal{F}(U_3)$$

where U_1, U_2, U_3 are the three open subsets corresponding to each of the discrete point. If we apply this formula to the structure pre-sheaf defined as in algebraic geometry we get

$$\mathcal{O}(\mathrm{Spec}^{NC}(A)) \overset{?}{=} \mathcal{O}(U_1) \oplus \mathcal{O}(U_2) \oplus \mathcal{O}(U_3) =$$

$$\begin{array}{ccc} \mathcal{O}(\operatorname{Spec}(A)) & = & \mathcal{O}(U_1) \oplus \mathcal{O}(U_2) \oplus \mathcal{O}(U_3) = \\ \shortparallel & & P_1 \oplus P_2^2 \oplus S_2 \\ A & & \\ \shortparallel & & \\ P_1 \oplus P_2 & & \end{array}$$

So $\mathcal{O}$ is not a sheaf. Why? What goes wrong?

Let us think about what does it mean for a presheaf to be a sheaf. It means that if $\{U_i \to U\}$ is a cover than

$$\mathcal{F}(U) = \lim \Big(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_U U_j) \Big).$$

So, the sections $\mathcal{F}(U)$ can reconstructed from $\mathcal{F}(U_i)$ plus the data of how they agree on intersections. This kind of data is called <u>descent data</u>.

The notion of descent data can be reformulated in a more precise way as follows. As before let $\{U_i \to U\}$ be a cover and let

$$Sh(U) \underset{j_*}{\overset{j^*}{\rightleftarrows}} Sh(\textstyle\coprod U_i) = \prod Sh(U_i)$$

be the adjunction induced by the continuous map $j : \coprod U_i \to U$.

The composition $M = j_* \circ j^*$ is a monad on $Sh(U)$ and this

The composition $M = j_* \circ j^*$ is a monad on $Sh(U)$ and this adjunction is always monadic, i.e. $Mod_M = \prod Sh(U_i)$.

Now, the adjunction (j^*, j_*) also define a comonad $C = j^* \circ j_*$ on Mod_M and we can consider the category $CoMod_C$ of comodules of C. This is called the <u>descent category</u> and it is possible to prove that $CoMod_M \cong Sh(U)$ and the way you construct such a functor is precisely by computing the kernel of the two canonical maps

given by the adjunction

$$\lim \Big(\underset{\prod \mathcal{F}(U_i)}{\underset{\shortparallel}{C(\mathcal{F}(U))}} \rightrightarrows \underset{\prod \mathcal{F}(U_i \times_U U_j)}{\underset{\shortparallel}{C(C(\mathcal{F}(U)))}} \Big).$$

So the descent category is related to the computation of the Cech cohomology of the cover $\{U_i \to U\}$.

We can do a similar computation for the adjunctions

(i_1, i_2, i_3) of the non trivial cover of Mod_A, we thus obtain an adjunction

$$\begin{tikzcd} \mathrm{Mod}_A \arrow[loop above, "M = j_* \circ j^*"] \arrow[r, shift left, "j^*"] & \mathrm{Mod}_K \oplus \mathrm{Mod}_K \oplus \mathrm{Mod}_K \arrow[l, shift left, "j_*"] \end{tikzcd}$$

and construct the descent category associated to this adjunction, lets call it Desc_A.

<u>Prop</u>

The comparison functor $\mathrm{Mod}_A \to \mathrm{Desc}_A$ is fully faithful.

This is great! Because it means that we can localize objects of Mod_A via i_1, i_2, i_3 and then uniquely reconstruct them.

But still they do not form a sheaf on $\mathrm{Spec}^{nc}(A)$ because the restriction functors i_1, i_2, i_3 do not correspond to the restriction of functors from $\mathrm{Spec}^{nc}(A)$ to $\mathrm{Spec}^{nc}(K)$ as they do in algebraic geometry and then the first step of the Cech complex permits to reconstruct global step of the Cech complex permits to reconstruct global sections. Here we can still write

$$X = \lim \left(M(X) \rightrightarrows M(M(X)) \right)$$

canonical morphisms from adjuction

but this complex cannot be interpreted as the Cech complex of a sheaf on $\mathrm{Spec}^{NC}(A)$. But the pre-sheaf on $\mathrm{Spec}^{NC}(A)$ determined by any $X \in \mathrm{HMod}_A$ it's uniquely determined by its localizations at the cover and reconstructed by formulas that are not the usual ones of Cech cohomology.

Once a proper theory of such sheaves is developed then we can endow $\mathrm{Spec}^{NC}(A)$ with the correct structure of "ringed topological space" and deduce Gelfand's duality in the non-commutative setting.